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On Orthogonal Decomposition of $L^2(\Omega)$

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Abstract

In this article we show an orthogonal decomposition of the Hilbert space $L^2(\Omega)$ *as* $L^2(\Omega) = A^2(\Omega) \oplus \frac{d}{dx}\left(W_0^{1,2}(\Omega)\right)$, define orthogonal projections and see some of their properties. We display some decomposition of elementary functions as corollaries.

Notations.

Let $\Omega = [0, 1]$

$\oplus$: Set direct sum

$\uplus$: Unique direct sum of elements from mutually orthogonal sets

$\left(\frac{d}{dx}\right)_0^{-2}$: Inverse image of a second order derivative of a traceless function

$A^2(\Omega) = \ker \frac{d}{dx} \cap L^2(\Omega) = \{f : \int_\Omega f^2 dx < \infty \ni \left(\frac{d}{dx}\right) f = 0 \text{ on } \Omega\}$

$\| * \| := \| * \|_{L^2(\Omega)}$

We define the following function spaces

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(I) The Hilbert space of square integrable functions over Ω

$$L^2(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}, \text{ measurable and } \int_\Omega f^2 dx < \infty\}$$

(II) The Sobolev space

$$W^{1,2}(\Omega) = \{f \in L^2(\Omega) : f'_w \in L^2(\Omega)\}$$

where f'_w is a weak first order derivative of $f, i.e,$

$$\exists g \in L_{\text{loc}}(\Omega) : g = f'_w$$

with

$$\int_\Omega g\varphi dx = -\int_\Omega f\varphi dx, \forall \varphi \in C_0^\infty(\Omega)$$

and

(III) the traceless Sobolev space

$$W_0^{1,2}(\Omega) = \{f \in W^{1,2}(\Omega) : f(0) = f(1) = 0\}$$

The Hilbert space $L^2(\Omega)$ is an inner product space with inner product

$$\langle , \rangle_{L^2(\Omega)} : L^2(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$$

defined by

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)\, g(x)\, dx$$

and $W^{1,2}(\Omega)$ with an inner product

$$\langle f, g\rangle_{W^{1,2}(\Omega)} = \left(\langle f, g\rangle_{L^2(\Omega)} + \langle f'_w, g'_w\rangle_{L^2(\Omega)}\right)^{\frac{1}{2}}$$

where f'_w, g'_w are weak first order derivatives.

Definition 1. For

$$f \in L^2(\Omega), \quad \|f\|_{L^2(\Omega)} = \sqrt{\langle f, f\rangle_{L^2(\Omega)}}$$

and for

$$f \in W^{1,2}(\Omega), \quad \|f\|_{W^{1,2}(\Omega)} = \sqrt{\|f\|_{L^2(\Omega)} + \|f'_w\|_{L^2(\Omega)}}$$

With respect to the defined inner product above, we have the following orthogonal decomposition

Proposition 1. (*Orthogonal Decomposition)*

$$L^2(\Omega) = A^2(\Omega) \oplus \frac{d}{dx}\left(W_0^{1,2}(\Omega)\right)$$

Proof. We need to show:

(i) $A^2(\Omega) \oplus \frac{d}{dx}\left(W_0^{1,2}(\Omega)\right) = \{0\}$

(ii) $\forall f \in L^2(\Omega), \exists$ a unique $g \in A^2(\Omega)$ and $\exists$ a unique $h \in \frac{d}{dx}\left(W_0^{1,2}(\Omega)\right)$

such that

$$f = g \uplus h.$$

Indeed

(i) Let $f \in A^2(\Omega) \cap \frac{d}{dx}\left(W_0^{1,2}(\Omega)\right)$.

Then

$$f \in A^2(\Omega) \Longrightarrow \frac{d}{dx} f = 0$$

and so f is a constant. Also

$$f \in \frac{d}{dx}\left(W_0^{1,2}(\Omega)\right)$$

and hence

$$\exists h \in W_0^{1,2}(\Omega)$$

such that

$$f = h'$$

But then as f is a constant we have

$$h = cx + d$$

But

$$trh = 0 \quad \text{on} \quad \partial\Omega = \{0,1\}$$

and hence

$$h(0) = 0 \Longrightarrow d = 0$$

and

$$h(1) = 0 \Longrightarrow c = 0$$

Therefore

$$h \equiv 0 \quad \text{and hence} \quad f \equiv 0.$$

$$\therefore \qquad A^2(\Omega) \cap \frac{d}{dx}\left(W_0^{1,2}(\Omega)\right) = \{0\} \qquad (\alpha)$$

(ii) Let $f \in L^2(\Omega)$. Then consider

$$\psi = \left(\frac{d}{dx}\right)_0^{-2} \left(\frac{d}{dx}\right) f$$

which is in $W_0^{1,2}(\Omega)$ and let

$$g = f - \left(\frac{d}{dx}\right)\psi$$

Then

$$\begin{aligned} \frac{d}{dx} g &= \frac{d}{dx}\left(f - \left(\frac{d}{dx}\right)\psi\right) \\ &= \frac{d}{dx} f - \frac{d^2}{dx^2}\left(\left(\frac{d}{dx}\right)_0^{-2}\left(\frac{d}{dx}\right) f\right) \\ &= 0 \end{aligned}$$

Thus

$$g \in A^2(\Omega)$$

and hence with

$$\eta = \left(\frac{d}{dx}\right)\psi \in \frac{d}{dx}\left(W_0^{1,2}(\Omega)\right)$$

we have

$$f = g \uplus \eta \qquad (\beta)$$

From (α) and (β) the proposition follows.

Remark. The subset $A^2(\Omega)$ of $L^2(\Omega)$ is a closed set and its orthogonal complement

$$\frac{d}{dx}\left(W_0^{1,2}(\Omega)\right) = \left(A^2(\Omega)\right)^{\perp}$$

is closed as well.

In addition representation of elements in the Hilbert space $L^2(\Omega)$ is unique; i.e.,

$$\forall f \in L^2(\Omega), \exists \text{ a unique } g \in A^2(\Omega) \quad \text{and} \quad \text{a unique } h \in \frac{d}{dx}\left(W_0^{1,2}(\Omega)\right)$$

such that

$$f = g + h$$

which we denote it as

$$f = g \uplus h$$

Definition 2. Due to the orthogonal decomposition there are two orthogonal projections

$$P : L^2(\Omega) \longrightarrow A^2(\Omega)$$

and

$$Q : L^2(\Omega) \longrightarrow \frac{d}{dx}\left(W_0^{1,2}(\Omega)\right)$$

with

$$Q = I - P$$

where I is the identity operator.

Proposition 2. $\forall f \in L^2(\Omega)$ we have

$$\langle P(f), Q(f) \rangle = 0$$

Proof. Let $f \in L^2(\Omega)$. Then

$$Pf \in A^2(\Omega)$$

and so it is a constant and

$$Qf \in \frac{d}{dx}\left(W_0^{1,2}(\Omega)\right)$$

and hence

$$\exists \text{ a unique } h \in W_0^{1,2}(\Omega)$$

such that

$$Qf = h' \quad \text{with} \quad trh = 0$$

Therefore

$$\langle P(f), Q(f)\rangle = \langle P(f), h'\rangle = \int_{\Omega} P(f)h'dx$$

Then from integration by parts we have

$$\int_{\Omega} P(f)h'dx = -\int_{\Omega} P(f)'hdx = 0$$

since $P(f) \in \ker \frac{d}{dx}$ and we have no boundary integral that might have resulted from the application of integration by parts because of the traceless of h.

$$\therefore \qquad \langle P(f), Q(f)\rangle = 0$$

Proposition 3. We have the following properties

(i) $PQ = 0$

(ii) $P^2 = P$

(iii) $Q^2 = Q$

That is P and Q are *idempotent*

Proof. Let $f \in L^2(\Omega)$ and let

$$g = Pf \in A^2(\Omega)$$

Then $g \in L^2(\Omega)$ and let

$$\psi = \left(\frac{d}{dx}\right)_0^{-2}\left(\frac{d}{dx}g\right) = \left(\frac{d}{dx}\right)_0^{-2}(0)$$

Then $\psi = 0$ and setting

$$h = g - \underbrace{\frac{d}{dx}\psi}_{\overset{\|}{0}}$$

we have

$$g = h + \underbrace{\frac{d}{dx}\psi}_{\overset{\|}{0}}$$

with

$$Pg = h \quad \text{and} \quad Qg = 0$$

Therefore,

$$Pg = P^2 f = h = g = Pf$$

and

$$Qg = QPf = 0$$

Similarly let

$$\eta = Qf \in \frac{d}{dx}\left(W_0^{1,2}(\Omega)\right)$$

Proof. Let $f \in L^2(\Omega)$. Then we have the unique decomposition,

$$f = Pf + Qf$$

But then

$$\begin{aligned}\langle f, f\rangle &= \langle Pf + Qf, Pf + Qf\rangle \\ &= \langle Pf, Pf\rangle + \langle Qf, Qf\rangle\end{aligned}$$

That is

$$\|f\|^2 = \|Pf\|^2 + \|Qf\|^2$$

We will look at few examples whose validity is supported from *uniqueness* of representations in Hilbert spaces.

Corollary 1.

For $f(x) = x \in L^2(\Omega)$ we have

$$P(f) = \frac{1}{2} \quad \text{and} \quad Q(f) = x - \frac{1}{2}$$

and hence

$$f(x) = \frac{1}{2} \uplus \left(x - \frac{1}{2}\right)$$

Proof. Let

$$\psi = D_0^{-2}(Df) = \left(\frac{d}{dx}\right)_0^{-2}(1) = \frac{1}{2}x^2 - \frac{1}{2}x$$

with

$$\frac{d}{dx}\psi = x - \frac{1}{2}$$

and let

$$g = f - \frac{d}{dx}\psi = \frac{1}{2}$$

Then

$$\frac{d}{dx}(g) = \frac{d}{dx}\left(f - \frac{d}{dx}\psi\right) = 0$$

and hence

$$f = g + \frac{d}{dx}\psi$$

as a direct sum. That is

$$f = \frac{1}{2} \uplus \left(x - \frac{1}{2}\right)$$

Corollary 2. For $f(x) = x$

$$\langle P(f), Q(f)\rangle = 0$$

Proof. Indeed

$$\begin{aligned}\langle P(f), Q(f)\rangle &= \int_{\Omega} \frac{1}{2}\left(x - \frac{1}{2}\right) dx \\ &= \frac{1}{2}\left(\frac{x^2}{2} - \frac{x}{2}\right)_0^1 \\ &= 0\end{aligned}$$

Corollary 3. $\|x\|^2 = \|\frac{1}{2}\|^2 + \|x - \frac{1}{2}\|^2$

Corollary 4. For $f(x) = x^2$

$$P(f) = \frac{1}{3} \quad \text{and} \quad Q(f) = x^2 - \frac{1}{3}$$

Proof. Let

$$\psi = \left(\frac{d}{dx}\right)_0^{-2}\left(\frac{d}{dx}f\right) = \left(\frac{d}{dx}\right)_0^{-2}(2x)$$

$$\Longrightarrow \qquad \psi(x) = \frac{1}{3}x^3 - \frac{1}{3}x$$

and let

$$\begin{aligned}g &= f - \frac{d}{dx}\psi \\ &= x^2 - \left(x^2 - \frac{1}{3}\right) \\ &= \frac{1}{3}\end{aligned}$$

and so

$$g \in \ker \frac{d}{dx}$$

and so

$$f = g \uplus \frac{d}{dx}\psi = \frac{1}{3} \uplus \left(x^2 - \frac{1}{3}\right)$$

which signifies

$$P\,(f) = \frac{1}{3} \quad \text{and} \quad Q\,(f) = x^2 - \frac{1}{3}$$

with

$$\left\langle \frac{1}{3},\ \left(x^2 - \frac{1}{3}\right)\right\rangle = 0$$

Corollary 5. $\|x^2\|^2 = \|\frac{1}{3}\|^2 + \|\left(x^2 - \frac{1}{3}\right)\|^2$

Proposition 4. For the orthogonal projections P and Q we have the following results

(i) $x^n = \frac{1}{n+1} \uplus \left(x^n - \frac{1}{n+1}\right)$

i.e.

$$P(x^n) = \frac{1}{n+1}, \quad Q\,(x^n) = x^n - \frac{1}{n+1}$$

(ii) $e^x = (e-1) \uplus (e^x + 1 - e)$

i.e.,

$$P\,(e^x) = e - 1, \quad Q\,(e^x) = e^x + 1 - e$$

(iii) $P(\cos x) = \sin 1, \ \ Q\,(\cos x) = \cos x - \sin 1$

so that

$$\cos x = \sin 1 \uplus (\cos x - \sin 1)$$

(iv) $P(\sin x) = 1 - \cos 1, \ \ Q\,(\sin x) = \sin x + \cos 1 - 1$

so that

$$\sin x = (1 - \cos 1) \uplus (\sin x + \cos 1 - 1)$$

Proof of (iii). Let

$$\psi = \left(\frac{d}{dx}\right)_0^{-2} \left(\frac{d}{dx}\cos x\right) = \sin x - (\sin 1)\, x$$

$$\Longrightarrow \qquad \frac{d}{dx}\psi(x) = \cos x - \sin 1$$

Then set

$$g = f - \frac{d}{dx}\psi = \sin 1 \in \ker \frac{d}{dx}$$

Thus

$$\cos x = \sin 1 \uplus (\cos x - \sin 1)$$

and hence

$$P(\cos x) = \sin 1 \quad \text{and} \quad Q(\cos x) = \cos x - \sin 1$$

Corollary 6.

(i) $\|x^n\|^2 = \|\frac{1}{n+1}\|^2 + \|x^n - \frac{1}{n+1}\|^2$

(ii) $\|e^x\|^2 = \|e-1\|^2 + \|e^x + 1 - e\|^2$

(iii) $\|\cos x\|^2 = \|\sin 1\|^2 + \|\cos x - \sin 1\|^2$

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