International Journal of Mathematics and
Computer Science, **10**(2015), no. 1, 39–49

# Connection Formulae among Special Polynomials

**Juan Carlos López Carreño[1], Rosalba Mendoza Suárez[1], Jairo Alonso Mendoza[2]**

[1]Departamento de Matemáticas
Universidad de Pamplona
Pamplona, Norte de Santander, Colombia

[2]Departamento de Física y Geología
Universidad de Pamplona
Pamplona, Norte de Santander, Colombia. Grupo de investigacion INTEGRAR

email: jclopez@unipamplona.edu.co, rosalbame@unipamplona.edu.co,
jairoam@unipamplona.edu.co

(Received February 28, 2015, March 25, 2015)

**Abstract**

In this article we solve the connection problem of the Hermite polynomials with the classical continuous orthogonal polynomials belonging to the Askey scheme, using the hypergeometric functions method in the joint work of Fields and Ismail.

## 1 Introduction

The connection problem is to find the coefficients $c_{nk}$ in the expansion of a polynomial $P_n(x)$ in terms of an arbitrary sequence of orthogonal polynomials $\{Q_k(x)\}$,

$$P_n(x) = \sum_{k=0}^{n} c_{nk} Q_k(x). \tag{1.1}$$

**Key words and phrases:** Orthogonal polynomials, Laguerre polynomials, connection problems, hypergeometric functions.
**AMS (MOS) Subject Classifications**: 33D45, 33C45.
**ISSN** 1814-0432, 2015, http://ijmcs.future-in-tech.net

A wide variety of methods have been devised for computing the connection coefficients $c_{nk}$, either in closed form or by means of recurrence relations, usually in $k$. Lewanowicz [1] has shown that the connection problem (1.1) can sometimes be solved by taking advantage of known results from the theory of generalized hypergeometric functions, derived by Fields and Wimp [2]. The hypergeometric functions method has been devised to solve the connection problem involving classical orthogonal polynomials [1, 3, 4, 5, 6, 7, 8]. These methods have also been used by Sánchez-Ruiz [7] to obtain the connection formulae involving squares of Gegenbauer Polynomials. In [8], the authors seeking to solve (1.1) for a much wider class of polynomials, defined by terminating hypergeometric series, obtain connection formulae for Wilson and Racah polynomials with special parameter values. They also solve the connection problem for the families of generalized Jacobi and Laguerre polynomials defined by Sister Celine. In [5] the authors consider the expansion of arbitrary power series in various classes of polynomial sets. In particular, they obtain the connection formulae which generalize the expansions formulae of Verma, Fields and Wimp (See Lemma 2.1, Lemma 2.2).

## 2 Notation and Preliminary Results

The generalized hypergeometric function is defined by

$$ {}_pF_q\left( \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}, \tag{2.2} $$

where $(a)_n$ represents the Pochhammer symbol:

$$ (a)_n := a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} = \frac{(-1)^n\,\Gamma(1-a)}{\Gamma(1-a-n)}, $$

$$ (a)_0 = 1, $$

where $a_i \in C$, $1 \leq i \leq p$, $b_j \in C$, $1 \leq j \leq q$, with $b_j \notin N_0$. Throughout this article, the letters $p, q, r, s, t, u$ and $n$ stand for nonnegative integers. We call $x$ the argument of the function, and $a_j$, $b_j$ the parameters. To shorten the notation for the left-hand side of (2.2), we will write it as

$$ {}_pF_q\left( \begin{array}{c} [a_p] \\ [b_q] \end{array} \middle| x \right) = \sum_{k=0}^{\infty} \frac{[a_p]_k\, x^k}{[b_q]_k\, k!}\,, \tag{2.3} $$

where $[a_p]$ and $[b_q]$ represent the sets $\{a_1, a_2, ..., a_p\}$ and $\{b_1, b_2, ..., b_q\}$ respectively. We use the abbreviated notation

$$[a_p]_k = \prod_{i=1}^{p}(a_i)_k, \qquad [b_q]_k = \prod_{j=1}^{q}(b_j)_k. \tag{2.4}$$

To prove the theorems in section 3, we use known results from the theory of generalized hypergeometric functions.
Lemma 2.1 (See [5], Formulae 1.3 and 3.2)

$$\begin{aligned}\sum_{m=0}^{\infty} a_m b_m \frac{(zw)^m}{m!} = & \sum_{n=0}^{\infty} \frac{(-z)^n}{n!(\gamma+n)_n} \sum_{r=0}^{\infty} \frac{(\mu)_{n+r}(\theta)_{n+r}}{r!(\gamma+2n+1)_r} b_{n+r}\ z^r \\ & \times \sum_{s=0}^{n} \frac{(-n)_s}{s!} \frac{(n+\gamma)_s}{(\mu)_s(\theta)_s} a_s w^s.\end{aligned} \tag{2.5}$$

$$\sum_{m=0}^{\infty} a_m b_m (zw)^m = \sum_{n=0}^{\infty} \frac{(c)_n(-z)^n}{n!} \sum_{j=0}^{\infty} \frac{(n+c)_j}{j!} b_{n+j}\ z^j \sum_{k=0}^{n} \frac{(-n)_k}{(c)_k} a_k w^k. \tag{2.6}$$

Lemma 2.2 (See [2], [4, Vol. II, p. 7])

$$\begin{aligned}{}_{p+r}F_{q+s}\left(\begin{matrix}[a_p],[c_r]\\ [b_q],[d_s]\end{matrix}\middle| z\omega\right) = & \sum_{n=0}^{\infty} \frac{(a_p)_n(c)_n(-z)^n}{(b_q)_n n!}\ {}_{p+1}F_q\left(\begin{matrix}[n+\alpha],[n+a_p]\\ [n+b_q]\end{matrix}\middle| z\right) \\ & \times {}_{r+1}F_{s+1}\left(\begin{matrix}-n,[c_r]\\ c,[d_s]\end{matrix}\middle| \omega\right).\end{aligned} \tag{2.7}$$

$$\begin{aligned}{}_{p+r+1}F_{q+s}\left(\begin{matrix}-n,[a_p],[c_r]\\ [b_q],[d_s]\end{matrix}\middle| zw\right) = & \sum_{k=0}^{n} \binom{n}{k} \frac{[a_p]_k\,[\alpha_t]_k\,z^k}{[b_q]_k[\beta_u]_k} \\ \times\ {}_{p+t+1}F_{q+u}\left(\begin{matrix}k-n,[k+a_p],[k+\alpha_t]\\ [k+b_q],[k+\beta_u]\end{matrix}\middle| z\right) & {}_{r+u+1}F_{s+t}\left(\begin{matrix}-k,[c_r],[\beta_u]\\ [d_s],[\alpha_t]\end{matrix}\middle| w\right).\end{aligned} \tag{2.8}$$

Lemma 2.3 From the generalized hypergeometric definition, we obtain:

$$\begin{aligned}{}_pF_q\left(\begin{matrix}[a_p]\\ [b_q]\end{matrix}\middle| z\right) = & {}_{2p}F_{2q+1}\left(\begin{matrix}\left[\frac{a_p}{2}\right],\left[\frac{a_p+1}{2}\right]\\ \frac{1}{2},\left[\frac{b_q}{2}\right],\left[\frac{b_q+1}{2}\right]\end{matrix}\middle| 4^{p-q-1}z^2\right) + \frac{(\prod_{i=1}^{p} a_i)z}{(\prod_{i=1}^{q} b_i)} \\ & {}_{2p}F_{2q+1}\left(\begin{matrix}\left[\frac{a_p+1}{2}\right],\left[\frac{a_p+2}{2}\right]\\ \frac{3}{2},\left[\frac{b_q+1}{2}\right],\left[\frac{b_q+2}{2}\right]\end{matrix}\middle| 4^{p-q-1}z^2\right).\end{aligned} \tag{2.9}$$

# 3 Results

In this section, we use the lemmas 2.1, 2.2, 2.3 and the different hypergeometric series representation the classical continuous orthogonal polynomials to solve the connection problem among them.

## 3.1 Theorem (Connection formula for Laguerre Polynomials in a series of Hermite Polynomials)

$$L_n(x) = \sum_{k=0}^{n} \frac{(-n)_k}{2^k (k)!(k)!} \, {}_2F_2\left( \begin{matrix} \frac{k-n}{2}, \frac{k+1-n}{2} \\ \frac{k+1}{2}, \frac{k+2}{2}, \end{matrix} \middle| \frac{1}{4} \right) H_k(x).$$

**Proof:** The hypergeometric representations of Laguerre and Hermite polynomials are given by:

$$L_n(x) = \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{x^k}{k!} = {}_1F_1\left( \begin{matrix} -n \\ 1 \end{matrix} \middle| x \right), \tag{3.10}$$

$$H_{2m}(x) = (-1)^m 2^{2m} \left( \frac{1}{2} \right)_m {}_1F_1\left( \begin{matrix} -m \\ \frac{1}{2} \end{matrix} \middle| x^2 \right), \tag{3.11}$$

$$H_{2m+1}(x) = (-1)^m 2^{2m+1} \left( \frac{3}{2} \right)_m x \, {}_1F_1\left( \begin{matrix} -m \\ \frac{3}{2} \end{matrix} \middle| x^2 \right). \tag{3.12}$$

Using formula (2.9) of Lemma 2.3 with the following identification

$$p = 1, \; q = 1, \; \{a_1\} = \{-n\}, \; \{b_1\} = \{1\}, \; z = x$$

we obtain

$${}_1F_1\left( \begin{matrix} -n \\ 1 \end{matrix} \middle| x \right) = {}_2F_3\left( \begin{matrix} -\frac{n}{2}, \frac{-n+1}{2} \\ \frac{1}{2}, \frac{1}{2}, 1 \end{matrix} \middle| \frac{1}{4}x^2 \right) + x(-n) \, {}_2F_3\left( \begin{matrix} \frac{-n+1}{2}, \frac{-n+2}{2} \\ \frac{3}{2}, 1, \frac{3}{2} \end{matrix} \middle| \frac{1}{4}x^2 \right). \tag{3.13}$$

Letting

$$A_1(x) = {}_2F_3\left( \begin{matrix} -m, \frac{-2m+1}{2} \\ \frac{1}{2}, \frac{1}{2}, 1 \end{matrix} \middle| \frac{1}{4}x^2 \right), \tag{3.14}$$

$$A_2(x) = -2mx \, {}_2F_3\left( \begin{matrix} -(m-1), \frac{-2m+1}{2} \\ \frac{3}{2}, 1, \frac{3}{2} \end{matrix} \middle| \frac{1}{4}x^2 \right). \tag{3.15}$$

and $n = 2m$, (3.13) can be written as:

$$L_n(x) = A_1(x) + A_2(x). \tag{3.16}$$

Using Fields-Wimp formula (2.7) we connect $A_1(x)$ given in (3.14) with the Hermite polynomials of even degree, (3.11):

$$[d_s] = \emptyset,\ s = 0,\ [c_r] = \emptyset,\ r = 0,\ c = \frac{1}{2}\,[a_p] = \left\{-m, \frac{-2m+1}{2}\right\}$$

$$p = 2,\ [b_q] = \{\frac{1}{2}, \frac{1}{2}, 1\},\ q = 3,\ w = x^2,\ z = \frac{1}{4},$$

we obtain

$$\begin{aligned} {}_2F_3\left( \begin{matrix} -m, \frac{-2m+1}{2} \\ \frac{1}{2}, \frac{1}{2}, 1 \end{matrix} \middle| \frac{x^2}{4} \right) = \; & \sum_{k=0}^{m} \frac{(-m)_k \left(\frac{-2m+1}{2}\right)_k \left(\frac{1}{2}\right)_k \left(\frac{-1}{4}\right)^k}{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k (1)_k k!} \\ & {}_3F_3\left( \begin{matrix} k+\frac{1}{2}, k-m, k+\frac{1-2m}{2} \\ k+\frac{1}{2}, k+\frac{1}{2}, k+1 \end{matrix} \middle| \frac{1}{4} \right) {}_1F_1\left( \begin{matrix} -k \\ \frac{1}{2} \end{matrix} \middle| x^2 \right). \end{aligned} \tag{3.17}$$

The left-hand side of (3.17) is $A_1(x)$ given in (3.14) by reorganizing the terms of the right-hand side of (3.17) the Hermite polynomials of even degree arise (3.11) and using the formula for duplicating the Pochhammer symbol it follows that:

$$A_1(x) = \sum_{k=0}^{m} \frac{(-2m)_{2k}}{2^{2k}(2k)!(2k)!}\, {}_2F_2\left( \begin{matrix} \frac{2k-2m}{2}, \frac{2k+1-2m}{2} \\ \frac{2k+1}{2}, \frac{2k+2}{2}, \end{matrix} \middle| \frac{1}{4} \right) H_{2k}(x). \tag{3.18}$$

Likewise, polynomials $A_2(x)$ of (3.15) are connected with Hermite polynomials of odd degree, (3.12):

$$[d_s] = \emptyset,\ s = 0,\ [c_r] = \emptyset,\ r = 0, c = \frac{3}{2}, [a_p] = \left\{-\frac{(2m-2)}{2}, \frac{-2m+1}{2}\right\}, p = 2$$

$$[b_q] = \left\{\frac{3}{2}, \frac{3}{2}, 1\right\}, q = 3, w = x^2,\ z = \frac{1}{4},$$

$$A_2(x) = \sum_{k=0}^{m-1} \frac{(-2m)_{2k+1}}{2^{2k+1}(2k+1)!(2k+1)!}\, {}_2F_2\left( \begin{matrix} \frac{2k+1-2m}{2}, \frac{2k+1-2m+1}{2} \\ \frac{2k+1+2}{2}, \frac{2k+1+1}{2}, \end{matrix} \middle| \frac{1}{4} \right) H_{2k+1}(x). \tag{3.19}$$

Recalling $n = 2m$, and $L_n(x) = A_1(x) + A_2(x)$, from (3.18) and (3.19) we conclude that

$$L_n(x) = \sum_{k=0}^{n} \frac{(-n)_k}{2^k(k)!(k)!}\, {}_2F_2\left( \begin{matrix} \frac{k-n}{2}, \frac{k+1-n}{2} \\ \frac{k+1}{2}, \frac{k+2}{2}, \end{matrix} \middle| \frac{1}{4} \right) H_k(x).$$

## 3.2 Theorem (Connection formula for Hermite polynomials in a series of Laguerre Polynomials)

$$H_N(x) = N!2^N \sum_{m=0}^{N} {}_2F_2\left( \begin{matrix} -\frac{1}{2}(N-m), -\frac{1}{2}(N-m-1) \\ -\frac{1}{2}(N), -\frac{1}{2}(N-1) \end{matrix} \middle| -\frac{1}{4} \right) \frac{(-N)_m}{(m)!} L_m(x),$$

**Proof:** Hermite polynomials can be expressed as:

$$H_N(x) = N! \sum_{k=0}^{[\frac{N}{2}]} \frac{(-1)^k}{k!(N-2k)!} (2x)^{N-2k},$$

Setting $N = 2p$:

$$H_{2p}(x) = (2p)! \sum_{k=0}^{p} \frac{(-1)^k}{k!(2p-2k)!} (2x)^{2p-2k}. \tag{3.20}$$

Letting $m := 2p - 2k$, for $k = 0, 1, 2, ..., p$ one realizes that $m$ goes through the set $\{2p, 2p-2, ..., 0\}$ In (3.20) we make the substitution $m = 2p - 2k$, to obtain:

$$H_{2p}(x) = (2p)! \sum_{m \in \{2p, 2p-2, ..., 0\}} \left\{ \frac{(-1)^{\frac{2p-m}{2}} 2^m x^m}{m!(\frac{2p-m}{2})!} \right\} \tag{3.21}$$

With the following identification $a_k = \frac{1}{k!}$, $c = 1$, $w = x$, $a_m = \frac{1}{m!}$ $z = 1$ and

$$b_m = \begin{cases} \frac{(-1)^{\frac{2p-m}{2}} 2^m (2p)!}{(\frac{2p-m}{2})!}, m \in \{2p, 2p-2, ..., 0\} \\ 0, \quad \text{otherwise} \end{cases} \tag{3.22}$$

The left-hand side of formula (2.6) of Lemma 2.1 is the expression for Hermite polynomials given in (3.21). By reorganizing the terms of the right-hand side of (2.6)

$$H_{2p}(x) = \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^{\infty} \frac{(n+1)_j}{j!} b_{n+j} L_n(x)$$

Keeping in mind the definition given for the coefficients $b_m$, $H_{2p}(x)$ we can write:

$$\begin{aligned} H_{2p}(x) = \; & \left\{ \sum_{k=0}^{p} \sum_{s=0}^{p-k} (-1)^{2k} \frac{(2k+1)_{2s}}{(2s)!} b_{2k+2s} L_{2k}(x) \right\} \\ & + \left\{ \sum_{k=0}^{p-1} \sum_{s=0}^{p-k-1} (-1)^{2k+1} \frac{(2k+1+1)_{2s+1}}{(2s+1)!} b_{2k+1+2s+1} L_{2k+1}(x) \right\}. \end{aligned}$$

Let

$$H_{2p}(x) := I + II.$$

Making use of the expression of $b_m$ given in (3.27), we have

$$I = \sum_{k=0}^{p} \sum_{s=0}^{p-k} (-1)^{2k} \frac{(2k+1)_{2s}}{(2s)!} \frac{(-1)^{p-k-s} 2^{2k+2s}(2p)!}{(p-k-s)!} L_{2k}(x).$$

$$II = \sum_{k=0}^{p-1} \sum_{s=0}^{p-k-1} (-1)^{2k+1} \frac{(2k+2)_{2s+1}}{(2s+1)!} \frac{(-1)^{p-k-s-1} 2^{2k+2s+2}(2p)!}{(p-k-s-1)!} L_{2k+1}(x).$$

By distributing some terms, the expression of $I$ can be written as:

$$I = (2p)! 2^{2p} \sum_{k=0}^{p} \left\{ \sum_{s=0}^{p-k} \frac{(2k+1)_{2s}(-1)^{p-k-s} 2^{2k+2s}}{(2s)!(p-k-s)!} \frac{(2k)!}{2^{2p}(-2p)_{2k}} \right\} \frac{(-2p)_{2k}}{(2k)!} L_{2k}(x). \tag{3.23}$$

Using some properties of the Pochhammer symbol, the interior sum of (3.23), over $s$, can be written as:

$$\sum_{s=0}^{p-k} \frac{(k-p+\frac{1}{2})_{p-k-s}}{(\frac{1}{2}-p)_{p-k-s}} \frac{(k+s)!(p-k)!(-1)^{p-k-s} 2^{2k+2s}}{s!p!(p-k-s)! 2^{2p}} \frac{2^{2s}(-\frac{1}{4})^s}{(-1)^s}. \tag{3.24}$$

With the substitution

$$2k + 2s = 2p - 2t,$$

the above expression can be written as:

$$\sum_{t=0}^{p-k} \frac{(-(p-k))_t (k-p+\frac{1}{2})_t (-1)^t}{(-p)_t t! (\frac{1}{2}-p)_t 2^{2t}}.$$

As $N = 2p$, (3.23) is transformed into:

$$I = N! 2^N \sum_{k=0}^{\frac{N}{2}} {}_2F_2\left( \begin{matrix} -\frac{1}{2}(N-2k), -\frac{1}{2}(N-2k-1) \\ -\frac{1}{2}(N), -\frac{1}{2}(N-1) \end{matrix} \,\middle|\, -\frac{1}{4} \right) \frac{(-N)_{2k}}{(2k)!} L_{2k}(x). \tag{3.25}$$

Likewise, $II$ turns out to be

$$II = N!2^N \sum_{k=0}^{\frac{N}{2}-1} {}_2F_2\left( \begin{matrix} -\frac{1}{2}(N-2k-1), -\frac{1}{2}(N-2k-2) \\ -\frac{1}{2}(N), -\frac{1}{2}(N-1) \end{matrix} \middle| -\frac{1}{4} \right) \frac{(-N)_{2k+1}}{(2k+1)!} L_{2k+1}(x). \tag{3.26}$$

Recall that

$$H_N(x) = I + II.$$

Hence, (3.25) and (3.26) allow us to write the connection between Hermite $H_N(x)$ with Laguerre polynomials:

$$H_N(x) = N!2^N \sum_{k=0}^{\frac{N}{2}} {}_2F_2\left( \begin{matrix} -\frac{1}{2}(N-2k), -\frac{1}{2}(N-2k-1) \\ -\frac{1}{2}(N), -\frac{1}{2}(N-1) \end{matrix} \middle| -\frac{1}{4} \right) \frac{(-N)_{2k}}{(2k)!} L_{2k}(x)$$

$$+N!2^N \sum_{k=0}^{\frac{N}{2}-1} {}_2F_2\left( \begin{matrix} -\frac{1}{2}(N-2k-1), -\frac{1}{2}(N-2k-2) \\ -\frac{1}{2}(N), -\frac{1}{2}(N-1) \end{matrix} \middle| -\frac{1}{4} \right) \frac{(-N)_{2k+1}}{(2k+1)!} L_{2k+1}(x)$$

$$H_N(x) = N!2^N \sum_{m=0}^{N} {}_2F_2\left( \begin{matrix} -\frac{1}{2}(N-m), -\frac{1}{2}(N-m-1) \\ -\frac{1}{2}(N), -\frac{1}{2}(N-1) \end{matrix} \middle| -\frac{1}{4} \right) \frac{(-N)_m}{(m+)!} L_m(x),$$

***Note:*** Likewise the procedure can be applied when $N$ is odd.

## 3.3 Theorem (Connection formula for Hermite polynomials in a series of Shifted Jacobi Polynomials)

$$H_n(x) = \frac{(-n)_m 4^n (2m+\lambda)(\alpha+1)_n}{(\alpha+1)_m (\lambda+m)_{n+1}} {}_4F_2\left( \begin{matrix} \Delta(2, m-n), \Delta(2, -\lambda-n-m) \\ \Delta(2, -\alpha-n) \end{matrix} \middle| \frac{1}{4} \right) \times P_m^{(\alpha,\beta)}(1-x)$$

where, $\lambda = \alpha+\beta+1$, y $\Delta(r;\varphi) = \frac{(\varphi+j-1)}{r}$, $j = 1, \cdots, r$.
**Proof:** The shifted Jacobi polynomials can be written as:

$$R_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(\beta+1)_n}{n!} {}_2F_1\left( \begin{matrix} -n, n+\alpha+\beta+1 \\ \beta+1 \end{matrix} \middle| x \right)$$

$$R_n^{(\alpha,\beta)}(x)=\frac{(-1)^n(\beta+1)_n}{n!}\sum_{s=0}^{n}\frac{(-n)_s(n+\alpha+\beta+1)_s}{(\beta+1)_s}\frac{x^s}{s!}$$

With the substitution $\gamma=\alpha+\beta+1$, $\theta=\beta+1$, $\mu=1$, $a_s=s!$,$w=x$, $z=1$ and

$$b_m=\begin{cases}\frac{(-1)^{\frac{2p-m}{2}}2^m(2p)!}{m!(\frac{2p-m}{2})!}, & m\in\{2p,2p-2,...,0\}\\ 0 & \text{otherwise}\end{cases}\tag{3.27}$$

the left-hand side of formula (2.5) of Lemma 2.1, is the expression for Hermite polynomials given in (3.21). On the other hand, on the right-hand side (2.5) we have the shifted Jacobi polynomials. Therefore (2.5) is transformed into

$$H_{2p}(x)=\sum_{n=0}^{\infty}\frac{1}{(\beta+1)_n(\alpha+\beta+1)_n}\sum_{r=0}^{\infty}\frac{(n+r)!(\beta+1)_{n+r}}{r!(\alpha+\beta+2n+2)_r}b_{n+r}R_n^{(\alpha,\beta)}(x).$$

The result is also obtained by a similar procedure to the one followed in the proof of Theorem 3.2.

## 3.4 Theorem (Connection formula for Shifted Jacobi Polynomials in a series of Hermite polynomials)

$$R_n^{(\alpha,\beta)}(x)=\sum_{j=0}^{n}\frac{(-1)^n(-1)^j(\beta+1)_n(n+\alpha+\beta+1)_j}{(n-j)!2^j\,(j)!\,(\beta+1)_j}$$
$$\times\sum_{m=0}^{N}{}_4F_2\left(\begin{matrix}\frac{j-n}{2},\frac{j+1-n}{2},\frac{j+n+\alpha+\beta+1}{2},\frac{j+n+\alpha+\beta+2}{2}\\ \frac{j+\beta+1}{2},\frac{j+\beta+2}{2}\end{matrix}\middle|1\right)H_j(x).$$

**Proof:** Using formula(2.9) of Lemma 2.3 with the following identification

$$p=2\quad q=1\quad a_1=\{-n\}\quad a_2=\{n+\alpha+\beta+1\}\quad b_1=\{\beta+1\}z=x$$

it is obtained

$$R_n^{(\alpha,\beta)}(x)=(-1)^n\frac{(\beta+1)_n}{n!}\,{}_4F_3\left(\begin{matrix}-\frac{n}{2},\frac{n+\alpha+\beta+1}{2},\frac{1-n}{2},\frac{n+\alpha+\beta+2}{2}\\ \frac{1}{2},\frac{\beta+1}{2},\frac{\beta+2}{2}\end{matrix}\middle|x^2\right)$$
$$+(-1)^n\frac{(\beta+1)_n}{n!}\frac{x(-n)(n+\alpha+\beta+1)}{\beta+1}\,{}_4F_3\left(\begin{matrix}\frac{1-n}{2},\frac{n+\alpha+\beta+2}{2},\frac{2-n}{2},\frac{n+\alpha+\beta+3}{2}\\ \frac{3}{2},\frac{\beta+2}{2},\frac{\beta+3}{2}\end{matrix}\middle|x^2\right)\tag{3.28}$$

Letting

$$A_1(x) = (-1)^n \frac{(\beta+1)_n}{n!} \, {}_4F_3\left( \begin{matrix} -\frac{n}{2}, \frac{n+\alpha+\beta+1}{2}, \frac{1-n}{2}, \frac{n+\alpha+\beta+2}{2} \\ \frac{1}{2}, \frac{\beta+1}{2}, \frac{\beta+2}{2} \end{matrix} \middle| x^2 \right)$$

$$A_2(x) = (-1)^n \frac{(\beta+1)_n}{n!} \frac{x(-n)(n+\alpha+\beta+1)}{\beta+1} \, {}_4F_3\left( \begin{matrix} \frac{1-n}{2}, \frac{n+\alpha+\beta+2}{2}, \frac{2-n}{2}, \frac{n+\alpha+\beta+3}{2} \\ \frac{3}{2}, \frac{\beta+2}{2}, \frac{\beta+3}{2} \end{matrix} \middle| x^2 \right),$$

(3.28) can be written as

$$R_n^{(\alpha,\beta)}(x) = A_1(x) + A_2(x).$$

Following a similar procedure to that used in Theorem 3.1 one ends up with the expected result.

# 4 Summary and future work

In this work, we presented the connection between the Laguerre Polynomials and a series of the Hermite polynomials, between the Hermite polynomials and the Laguerre polynomials, between the Hermite polynomials and the Shifted Jacobi polynomials, between the Shifted Jacobi polynomials and the Hermite polynomials. In a future work, we expect to give the connection between the Bessel polynomials and the Hermite polynomials.

# References

[1] S. Lewanowicz, The hypergeometric functions approach to the connection problem for the classical orthogonal polynomials, Preprint, 1997.

[2] J. L. Fields, J. Wimp, Expansions of hypergeometric functions in hypergeometric functions, Math. Comp. **15,** 1961, 390–395.

[3] R. Askey, Orthogonal Polynomials and Special Functions, Regional Conference Series in Applied Mathematics, **21,** SIAM, Philadelphia, 1975.

[4] Y. L. Luke, The special functions and their approximations, Academic Press, New York, 1969.

[5] Jerry L. Fields, Mourad E. H. Ismail, Polynomial expansion, Math. Comp. **131,** 1975, 894-902.

[6] Jerry L. Fields, J. Wimp, Expansions of hypergeometric functions in hypergeometric functions, Math. Comp., **15,** 1961, 390-395.

[7] J. Sánchez-Ruiz, Linearization and connection formulae involving squares of Gegenbauer polynomials, Appl. Math. Lett., **14,** 2001, 261–267.

[8] J. Sánchez-Ruiz, J. S. Dehesa, Some connection and linearization problems for polynomials in and beyond the Askey scheme, J. Comput. Appl. Math., **133,** 2001, 579–591.

[9] J. Sánchez-Ruiz, P. Lopéz-Artés, J. S. Dehesa, Expansions in series of varing Laguerre polynomials and some applications to molecular potentials, J. Comput. Appl. Math., **153,** 2003, 411-421.