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Luus-Jaakola Optimization Procedure for Ramsey Number Lower Bounds

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Abstract

Ramsey numbers have been widely studied for decades, but the exact values for all but a handful are still unknown. In recent years, optimization algorithms have proven useful in calculating lower bounds for certain Ramsey numbers. In this paper, we define an optimization algorithm based on the Luus-Jaakola procedure to calculate Ramsey number lower bounds. We demonstrate the effectiveness of the algorithm for that task, show the effect of parameter selection and determine optimal parameters for the algorithm, and provide results from the implementation of the algorithm.

1 Introduction

How many people must be in a room to ensure that there are either 3 mutual friends or 3 mutual strangers? What about ensuring either 3 mutual friends or 4 mutual strangers? These are questions that can be answered through the use of Ramsey theory; in particular, they are the Ramsey numbers $R(3,3)$ and $R(3,4)$. The values of these numbers, and thus the answers to the questions, are 6 and 9 people, respectively.

The Ramsey numbers are a set of numbers with a fairly straightforward definition, yet their exact values are notoriously difficult to determine. In

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terms of graph theory, a Ramsey number $r = R(x_1, x_2, ..., x_n)$ is defined as the smallest number r such that any edge coloring of a complete graph of order r with n colors must contain a complete subgraph of color i on x_i vertices, for some i.

There are some general proofs for lower and upper bounds for the Ramsey numbers, but these bounds are often widely separated. Establishing exact values for these numbers has proven extremely difficult, as evidenced by the fact that today, over 80 years after Ramsey theory was introduced, only a handful of non-trivial Ramsey number values are known. These values are:

$R(3,3) = 6$	$R(3,4) = 9$	$R(3,5) = 14$
$R(3,6) = 18$	$R(3,7) = 23$	$R(3,8) = 28$
$R(3,9) = 36$	$R(4,4) = 18$	$R(4,5) = 25$
$R(3,3,3) = 17$		

Other work has been done on tightening the general bounds for specific other Ramsey numbers or classes thereof, but currently these 10 values are the only known exact values.

While most of both the bounds and actual values for the Ramsey numbers have come from mathematical proofs or constructions, more recently it has become increasingly feasible to use computer algorithms to attempt to generate colorings that establish new lower bounds. Since the definition of a Ramsey number is the smallest number such that a complete graph of that order must contain one of the subgraphs in question, lower bounds can be proven simply by providing an edge coloring of a complete graph of the desired order that does not contain any of these subgraphs. The algorithm presented below is of this type. It utilizes an optimization-based strategy to construct a desired edge coloring, in order to provide lower bounds for the Ramsey numbers in question.

In this paper, we make the following original contributions:

- We define a parametrized optimization algorithm based on the Luus-Jaakola procedure.
- We present a study of the effects of parameter values on the algorithm.
- We present the results of tests using tuned parameters with the algorithm.

2 General Optimization-Based Approach

To approach this problem as an optimization problem, given a Ramsey number:

$$R(x_1, x_2, ..., x_n)$$

and an order m (for which we wish to construct an edge coloring of K_m with n colors that contains no complete subgraphs of order x_i in color i), we consider each possible edge coloring of K_m with n colors to be a point in the search space of the problem. The objective function, then, can be defined as a count of the undesired subgraphs, which we then seek to minimize to 0. As discussed in [3], this count can be weighted in order to increase the performance of the optimization algorithm, since the size of the subgraph in question affects its level of "undesirability" within the edge coloring. The "Results" section includes an examination of appropriate weighting and its effect on the performance of the algorithm presented below.

An optimization-based approach to the construction of these Ramsey number edge colorings has been attempted in a few contexts (e.g. [1, 7, 2]), most successfully in the work of Geoffrey Exoo, who has used variants of simulated annealing and tabu search algorithms to construct colorings that provided new lower bounds for several Ramsey numbers [2, 3]. The algorithm presented below shares the same fundamental goal, but with a different optimization approach.

As with many optimization algorithms, a significant challenge is that of getting stuck in local minima. This is particularly true for the type of optimization approach described, as many edge colorings cannot be locally modified – that is, modified with a small number of edge color changes – to decrease the number of undesired subgraphs. The Luus-Jaakola optimization procedure is often well-suited to problems with many local minima [4], which was part of the motivation for its selection for this problem.

One additional consideration related to this problem domain is the fact that the search space is discrete. Furthermore, since for this problem we consider each individual edge color as a dimension, the range of possible values within each dimension is very small – the number of colors for the Ramsey number under consideration, typically 2 or 3. This limits the set of potential optimization approaches, since some algorithms, particularly some gradient-based and related algorithms, do not perform well in discrete spaces of this type. Our implementation of the Luus-Jaakola procedure does not suffer from this limitation, as we show in the following section.

3 Luus-Jaakola Procedure

The Luus-Jaakola procedure [4] is often presented as a general optimization heuristic, rather than an algorithm that converges to an optimal solution. However, it has been shown to provide a useful optimization approach in several contexts (e.g. [5, 6, 8]), and to outperform genetic algorithms for a number of optimization problems [9].

The basic Luus-Jaakola procedure is as follows. Given a function $f : R^n \rightarrow R$ to minimize, where b_l and b_u are the lower and upper bounds of the search space, respectively, iteratively examine the element $x \in R^n$:

$x = rand(b_l, b_u)$
$d = b_u - b_l$
$q = 0.95$
while termination condition not met
$a = rand(-d, d)$
$y = x + a$
if $f(y) < f(x)$ **then**
$x = y$
else
d = q*d
end if
endwhile

Algorithm 1 Luus-Jaakola Procedure.

The procedure iteratively examines a random position from a narrowing window of the search space and, if the new position yields a better objective function value, moves to the new position. The narrowing window and examination of a random position within it at each step helps to avoid the problem of getting stuck in a one localized place in the search space until late in the algorithm, when the local minima problem should be less of an issue and, if the algorithm is performing well, the position should instead be approaching the global minimum.

For our implementation of this procedure for the Ramsey number edge coloring problem specifically, we consider a point within the search space to be a specific edge coloring, and the distance between two points to be the number of edges that have different colors between the two represented colorings. Choosing a random vector within the interval $(-d, d)$ and adding it to the current position, then, involves randomly changing the color of up to $round(d)$ edges (since there is no inherent meaning in the order of colors for a Ramsey problem, we make no distinction between positive and negative changes for the color of a particular edge). One further modification at the top level of the algorithm involves restarting the procedure if $round(d) = 0$ and no optimal solution has been found, since at that point no further movements within the search space would be made.

We consider both q and the initial value for d as parameters for this algorithm, and examine the effects of both in the results section below. Additionally, as mentioned earlier, the definition of the objective function for this problem can include a weighting for different undesired subgraphs. For example, in examining possible colorings for $R(3, 4)$, a K_3 subgraph of color 1 is worse for the construction algorithm than a K_4 subgraph of color 2, and therefore can be weighted higher in the objective function to improve the performance of the optimization algorithm. This weighting factor is also examined below.

4 Results

Overall, the algorithm performed very well. It was able to construct colorings to prove the highest lower bounds for $R(3, 3)$, $R(3, 4)$, $R(3, 5)$, $R(3, 6)$, $R(3, 7)$, $R(3, 8)$, $R(4, 4)$, and $R(4, 5)$, 8 of the 10 known non-trivial Ramsey numbers.

The algorithm's performance was found to be sensitive to both the q parameter and the initial d parameter. A full design of experiments was performed for the highest lower bound of several different Ramsey numbers, wherein all combinations of various values for the q and d parameters were run 1,000 times, and the average number of objective function evaluations – the metric normally used to compare performance of optimization algorithms [10] – was recorded for each.

4.1 Parameter q

In the typical presentation of the Luus-Jaakola procedure, the default value for the parameter q, if it is even presented as a parameter, is 0.95. This value has been shown to work well for a variety of optimization tasks; however, for this particular problem, higher values for q yielded better performance in every tested case. Figure 1, with values from the algorithm constructing graphs of order 13 for the Ramsey number $R(3, 5)$, shows results for several values of d (fairly low values; the reason for this will be discussed in the section on parameter d), with q ranging from 0.9 to 0.99. As can be seen, the performance of the algorithm is much better at the higher end of this range for all values of d.

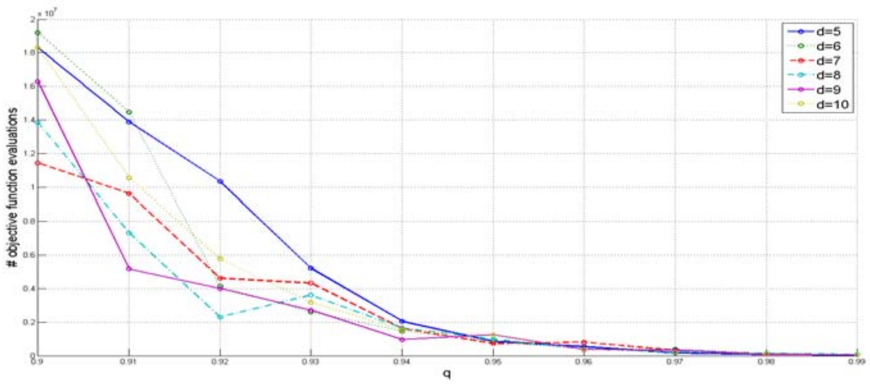

Figure 1: $R(3,5)$ results, $q = (0.9, 0.99)$

Figure 2 shows results for the same values of d, with higher values for q, from 0.98 to 0.994. Even in this higher range, the trend continues of higher q values producing better algorithm performance.

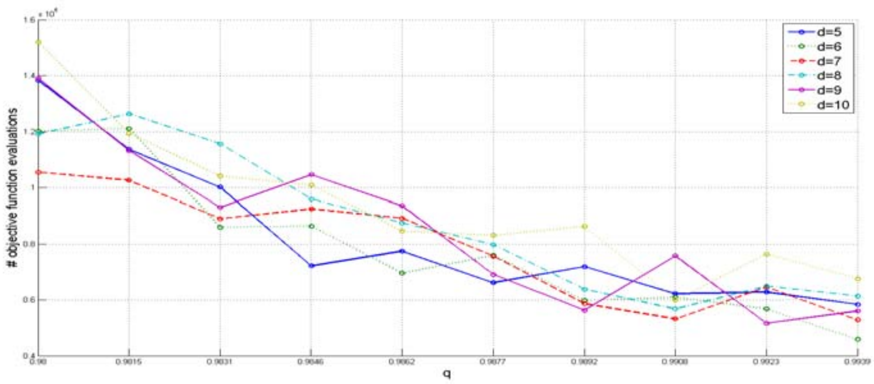

Figure 2: $R(3,5)$ results, $q = (0.98, 0.994)$

However, the same trend does not continue indefinitely as q approaches 1. Figure 3 shows results for the same values of d, now with q ranging from 0.998 to 0.9998. Here the previous trend is reversed, and the performance of the algorithm gets worse as the value of q increases. The average best value found for q for $R(3,5)$ was around 0.9982. In fact, the average best value for q was very close to this value for all Ramsey numbers tested, with a slight shift toward higher values for q with larger graph sizes.

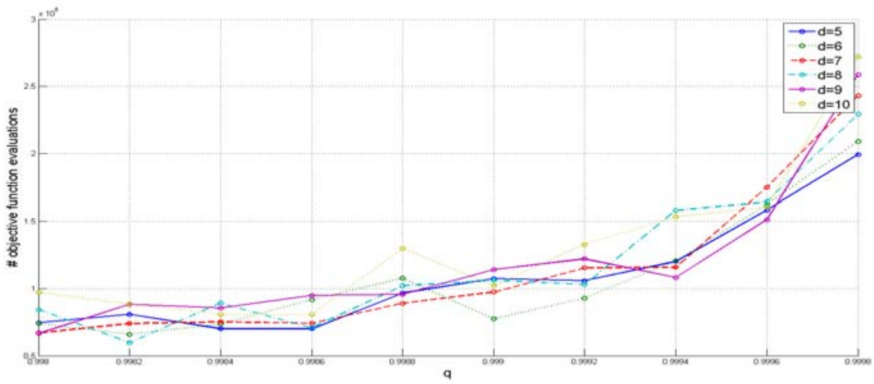

Figure 3: $R(3,5)$ results, $q = (0.998, 0.9998)$

4.2 Parameter d

The results of the designs of experiments with respect to parameter d were somewhat mixed, but the general trend was that lower values for d, above $d = 1$, yielded better results. Figure 4 shows the results of a wide range of d values, from $d = 1$ to $d = 64$, again for a graph of order 13 for the Ramsey number $R(3,5)$, with the best found value for q from the previous section, $q = 0.9982$. The general trend can easily be seen, that the number of average objective function evaluations rises as the value for parameter d rises.

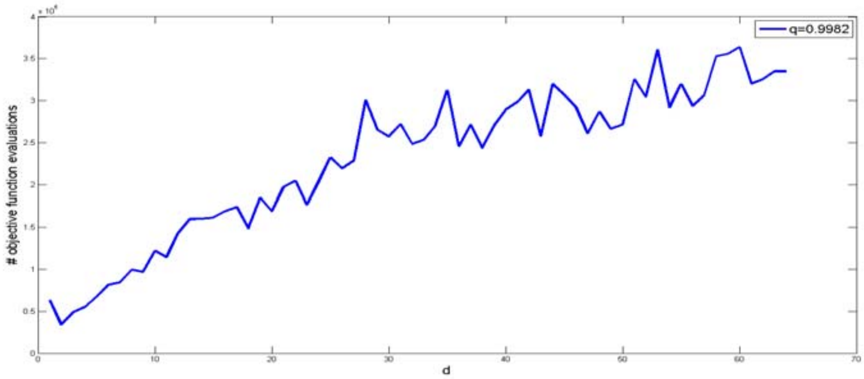

Figure 4: $R(3,5)$ results, $d = (1, 64)$

This same general trend holds true for each of the values for parameter q run in the design of experiments. Figure 5 illustrates this trend, showing three values for the parameter q over a range of values for d. The same trend appears for each, while again showing the advantages of the higher values for parameter q.

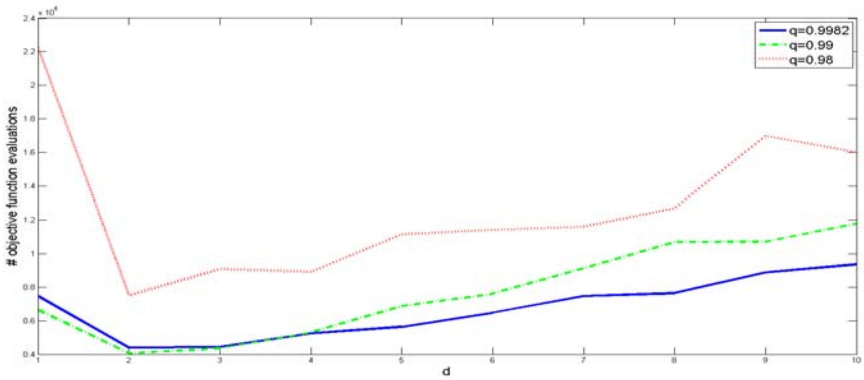

Figure 5: $R(3,5)$ results, $d = (1, 64)$

4.3 Objective Function Weights

For Ramsey numbers for which the sizes of the subgraphs are not equal ("off-diagonal" Ramsey numbers), a natural question arises of whether, under the optimization approach, to treat all subgraphs the same, or to weight them based on their size. For instance, in our example problem of $R(3,5)$, a subgraph of order 3 with 3 edges and a subgraph of order 5 with 10 edges both produce a violation, but they are not equally "bad" in terms of moving toward an optimal solution. This is because only one edge of each violating subgraph needs to be recolored in order to remove the violation. In the case of the subgraph of order 3, there are only 3 possible changes that can be made. In the case of the subgraph of order 5, there are 10 possible changes, and thus it is far more likely that an edge of that subgraph could be recolored successfully. For that reason, smaller subgraphs can be considered "worse" under the optimization approach.

Exoo [3] addresses this briefly, and offers the empirical formula $(R_1/R_2)^{1.5}$. For the case of our example usage of $R(3,5)$, this would be $(5/3)^{1.5} \approx 2.152$, so a subgraph of order 3 would be weighted 2.152 times as much as one of order 5, using this formula.

We tested the impact of this weighting by again using $R(3,5)$ and searching for a graph of order 13, and varying the factor by which R_3 subgraphs were weighted, again averaging the results across a number of runs. Figure 6 shows the results of these tests, with the weighting factor graphed against the average number of objective function evaluations required to obtain a valid coloring.

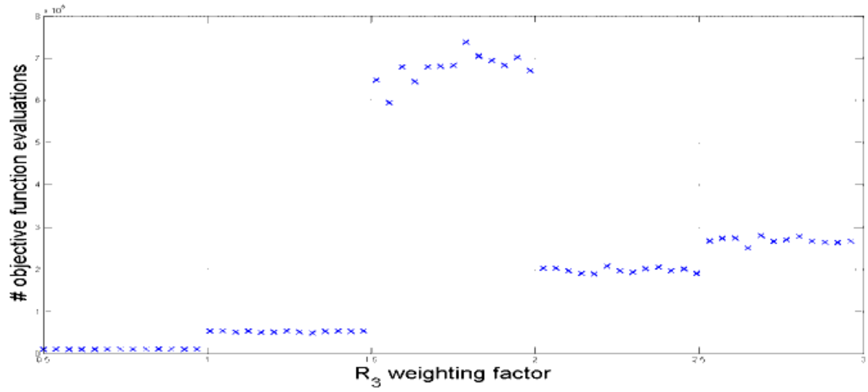

Figure 6: $R(3,5)$ subgraph weighting results

The results of these tests were surprising in a number of ways. Perhaps most noticeably, a "clumping" effect appears, in which results for nearby weighting factors are similar, not in a smoothly increasing or decreasing way, but in a series of distinct quanta. In addition, the best values for this series of tests are found below 1, meaning that the algorithm converges more quickly when the R_3 subgraph is weighted lower than the R_5 subgraph. These results contradicted our intuition in both ways, and may indicate that this approach is more sensitive to the subgraph weighting than initially thought. A more thorough examination of this factor is outside the scope of this paper, but is planned for future work.

4.4 Interpretation of Results

The high-level result with reference to the q and d parameters is that this Luus-Jaakola-based approach works best with large values of q (close to 1) and small values of d (close to 2). In terms of the algorithm itself, this means that even initially, only small changes to the graph coloring should be made at each timestep, and that the shrinking of this window should be very slow. To some extent, the results for the parameter d are out of step with the basic premise of the Luus-Jaakola procedure, since the overarching idea involves an initially wide search window that shrinks over time, whereas for this problem, it seems that even at the beginning, the search window should be relatively small. However, by parametrizing the algorithm in this way, sufficient flexibility is designed in to handle this case.

As discussed above, the results for the experiments involving weighting the subgraphs within the optimization algorithm were somewhat surprising and motivate further research. However, even with equal weighting of all subgraphs for the objective function, the Luus-Jaakola procedure proved effective as an approach to this problem.

5 Conclusion and Future Work

Based on the results presented in this paper, it is clear that the Luus-Jaaokla optimization procedure, with appropriate parameter selection, can be used effectively for finding lower bounds of Ramsey numbers. This provides further evidence that the optimization approach to this type of problem is a promising one, and that the problem can be construed in a number of ways to allow for the use of various optimization algorithms. Naturally, as computational power increases, these methods will be ever more applicable and appropriate for ever larger and more complex search spaces and problem domains.

For future work, we plan to more thoroughly investigate the effect of subgraph weighting on the optimization approach to this problem, both for 2-color graphs and higher numbers of colors. In addition, since these results have demonstrated the viability of the presented algorithm, we plan to apply the algorithm to Ramsey numbers with currently unknown values, to attempt to produce improved lower bounds for some of these numbers and increase the base of knowledge for the values of Ramsey numbers in general.

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