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Unitary Multiperfect Numbers in Certain Quadratic Rings

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Abstract

A unitary divisor c of a positive integer n is a positive divisor of n that is relatively prime to $\frac{n}{c}$. For any integer k, the function σ_k^* is a multiplicative arithmetic function defined so that $\sigma_k^*(n)$ is the sum of the k^{th} powers of the unitary divisors of n. We provide analogues of the functions σ_k^* in imaginary quadratic rings that are unique factorization domains. We then explore properties of what we call n-powerfully unitarily t-perfect numbers, analogues of the unitary multiperfect numbers that have been defined and studied in the integers. We end with a list of several opportunities for further research.

1 Introduction

We convene to let $\mathbb{N}$ and $\mathbb{P}$ denote the set of positive integers and the set of (integer) prime numbers, respectively.

The arithmetic functions σ_k are defined, for every integer k, by $\sigma_k(n) = \displaystyle\sum_{\substack{c|n \\ c>0}} c^k$. The unitary divisor functions σ_k^* are defined by $\sigma_k^*(n) = \displaystyle\sum_{\substack{0<c|n \\ \gcd(c,\frac{n}{c})=1}} c^k$. In other words, $\sigma_k^*(n)$ is the sum of the k^{th} powers of the

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unitary divisors of n, where a unitary divisor of n is simply a positive divisor c of n such that c and $\frac{n}{c}$ are relatively prime. The author has invented and investigated analogues of the divisor functions in imaginary quadratic integer rings that are unique factorization domains [2]. Here, we seek to investigate analogues of the unitary divisor functions in these rings.

For any square-free integer d, let $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ be the quadratic integer ring given by

$$\mathcal{O}_{\mathbb{Q}(\sqrt{d})} = \begin{cases} \mathbb{Z}[\frac{1+\sqrt{d}}{2}], & \text{if } d \equiv 1 \pmod 4; \\ \mathbb{Z}[\sqrt{d}], & \text{if } d \equiv 2, 3 \pmod 4). \end{cases}$$

Throughout the remainder of this paper, we will work in the rings $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ for different specific or arbitrary values of d. We will use the symbol "$|$" to mean "divides" in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ in which we are working. Whenever we are working in a ring other than $\mathbb{Z}$, we will make sure to emphasize when we wish to state that one integer divides another in $\mathbb{Z}$. For example, if we are working in $\mathbb{Z}[i]$, the ring of Gaussian integers, we might say that $1+i|1+3i$ and that $2|6$ in $\mathbb{Z}$. We will also refer to primes in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ as "primes," whereas we will refer to (positive) primes in $\mathbb{Z}$ as "integer primes." For an integer prime p and a nonzero integer n, we will let $\upsilon_p(n)$ denote the largest integer k such that $p^k|n$ in $\mathbb{Z}$. For a prime π and a nonzero number $x \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, we will let $\rho_\pi(x)$ denote the largest integer k such that $\pi^k|x$. Furthermore, we will henceforth focus exclusively on values of d for which $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is a unique factorization domain and $d < 0$. In other words, $d \in K$, where we will define K to be the set $\{-163, -67, -43, -19, -11, -7, -3, -2, -1\}$. The set K is known to be the complete set of negative values of d for which $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is a unique factorization domain [3].

For an element $a+b\sqrt{d} \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $a, b \in \mathbb{Q}$, we define the conjugate by $\overline{a+b\sqrt{d}} = a-b\sqrt{d}$. The norm and absolute value of an element z are defined, respectively, by $N(z) = z\overline{z}$ and $|z| = \sqrt{N(z)}$. We assume familiarity with the properties of these object, which are treated in Keith Conrad's online notes [1]. For $x, y \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, we say that x and y are associated, denoted $x \sim y$, if and only if $x = uy$ for some unit u in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. Furthermore, we will make repeated use of the following well-known facts.

Fact 1.1. *Let $d \in K$. If p is an integer prime, then exactly one of the following is true.*

- *p is also a prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. In this case, we say that p is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.*

- $p \sim \pi^2$ *and* $\pi \sim \overline{\pi}$ *for some prime* $\pi \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. *In this case, we say* p *ramifies (or* p *is ramified) in* $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.
- $p = \pi\overline{\pi}$ *and* $\pi \not\sim \overline{\pi}$ *for some prime* $\pi \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. *In this case, we say* p *splits (or* p *is split) in* $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.

Fact 1.2. *Let* $d \in K$. *If* $\pi \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ *is a prime, then exactly one of the following is true.*

- $\pi \sim q$ *and* $N(\pi) = q^2$ *for some inert integer prime* q.
- $\pi \sim \overline{\pi}$ *and* $N(\pi) = p$ *for some ramified integer prime* p.
- $\pi \not\sim \overline{\pi}$ *and* $N(\pi) = N(\overline{\pi}) = p$ *for some split integer prime* p.

Fact 1.3. *Let* p *be an odd integer prime. Then* p *ramifies in* $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ *if and only if* $p|d$ *in* $\mathbb{Z}$. *If* $p \nmid d$ *in* $\mathbb{Z}$, *then* p *splits in* $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ *if and only if* d *is a quadratic residue modulo* p. *Note that this implies that* p *is inert in* $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ *if and only if* $p \nmid d$ *in* $\mathbb{Z}$ *and* d *is a quadratic nonresidue modulo* p. *Also, the integer prime* 2 *ramifies in* $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ *and* $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$, *splits in* $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, *and is inert in* $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ *for all* $d \in K\backslash\{-1,-2,-7\}$.

Fact 1.4. *Let* $\mathcal{O}^*_{\mathbb{Q}(\sqrt{d})}$ *be the set of units in the ring* $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. *Then* $\mathcal{O}^*_{\mathbb{Q}(\sqrt{-1})} = \{\pm 1, \pm i\}$, $\mathcal{O}^*_{\mathbb{Q}(\sqrt{-3})} = \left\{\pm 1, \pm\frac{1+\sqrt{-3}}{2}, \pm\frac{1-\sqrt{-3}}{2}\right\}$, *and* $\mathcal{O}^*_{\mathbb{Q}(\sqrt{d})} = \{\pm 1\}$ *whenever* $d \in K\backslash\{-1,-3\}$.

For a nonzero complex number z, let $\arg(z)$ denote the argument, or angle, of z. We convene to write $\arg(z) \in [0, 2\pi)$ for all nonzero $z \in \mathbb{C}$. For each $d \in K$, we define the set $A(d)$ by

$$A(d) = \begin{cases} \{z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} : 0 \le \arg(z) < \frac{\pi}{2}\}, & \text{if } d = -1; \\ \{z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} : 0 \le \arg(z) < \frac{\pi}{3}\}, & \text{if } d = -3; \\ \{z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} : 0 \le \arg(z) < \pi\}, & \text{otherwise.} \end{cases}$$

Thus, every nonzero element of $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ can be written uniquely as a unit times a product of primes in $A(d)$. Also, every $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$ is associated to a unique element of $A(d)$. For nonzero elements $x, z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, we will write $x \Diamond z$ if and only if $x \in A(d)$, $x|z$, and x is relatively prime to $\frac{z}{x}$ (meaning x and $\frac{z}{x}$ have no nonunit common divisors).

Definition 1.1. *Let $d \in K$, and let $n \in \mathbb{Z}$. Define the function $\delta_n^*\colon \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} \to [1,\infty)$ by*

$$\delta_n^*(z) = \sum_{x \Diamond z} |x|^n.$$

Remark 1.2. *We note that, for each x in the summation in the above definition, we may cavalierly replace x with one of its associates. This is because associated numbers have the same absolute value. In other words, the only reason for the criterion $x \in A(d)$ in the summation that appears in Definition 1.1 (which is implied by the relation $x \Diamond z$) is to forbid us from counting associated divisors as distinct terms in the summation, but we may choose to use any of the associated divisors as long as we only choose one. This should not be confused with how we count conjugate divisors (we treat $2+i$ and $2-i$ as distinct divisors of 5 in $\mathbb{Z}[i]$ because $2+i \not\sim 2-i$). Also, note that the functions δ_n^* depend on the ring in which we are working (this is also true of the function I_n^*, which we will soon define).*

We will say that a function $f\colon \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} \to \mathbb{R}$ is multiplicative if $f(xy) = f(x)f(y)$ whenever x and y are relatively prime.

Theorem 1.3. *Let us work in a ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in K$. For any $n \in \mathbb{Z}$, the function δ_n^* is multiplicative.*

Proof. Let z_1 and z_2 be relatively prime elements of $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$ for some $d \in K$. First note that if $x \Diamond z_1 z_2$, then $x \sim x_1 x_2$, where $x_1 \Diamond z_1$ and $x_2 \Diamond z_2$. Furthermore, the numbers x_1 and x_2 are unique because of the requirement $x_1, x_2 \in A(d)$ inherent in the relations $x_1 \Diamond z_1$ and $x_2 \Diamond z_2$. On the other hand, if $x_1 \Diamond z_1$ and $x_2 \Diamond z_2$, then $x_1 x_2$ is associated to a unique number x such that $x \Diamond z_1 z_2$. Therefore,

$$\delta_n^*(z_1 z_2) = \sum_{x \Diamond z_1 z_2} |x|^n = \sum_{\substack{x_1 \Diamond z_1 \\ x_2 \Diamond z_2}} |x_1 x_2|^n = \sum_{x_1 \Diamond z_1} |x_1|^n \sum_{x_2 \Diamond z_2} |x_2|^n = \delta_n^*(z_1)\delta_n^*(z_2).$$

□

Definition 1.4. *For $d \in K$, define the function $I_n^*\colon \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} \to [1,\infty)$, for each $n \in \mathbb{Z}$, by $I_n^*(z) = \dfrac{\delta_n^*(z)}{|z|^n}$. For a positive integer $t \geq 2$, we say that a number $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$ is n-powerfully unitarily t-perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ if $I_n^*(z) = t$, and, if $t = 2$, we simply say that z is n-powerfully unitarily perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. If $n = 1$, we will omit the adjective "1-powerfully."*

Theorem 1.5. *Let $k, n \in \mathbb{N}$, $d \in K$, and $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \backslash \{0\}$. Then, if we are working in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, the following statements are true.*

(a) The range of I_n^* is a subset of the interval $[1, \infty)$, and $I_n^*(z) = 1$ if and only if z is a unit in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.

(b) I_n^* is multiplicative.

(c) $I_n^*(z) = \delta_{-n}^*(z)$.

Proof. Part (a) is fairly trivial. To prove part (b), let z_1 and z_2 be relatively prime elements of $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. We may use Theorem 1.3 to write

$$I_n^*(z_1 z_2) = \frac{\delta_n^*(z_1 z_2)}{|z_1 z_2|^n} = \frac{\delta_n^*(z_1)\delta_n^*(z_2)}{|z_1|^n |z_2|^n} = I_n^*(z_1) I_n^*(z_2).$$

To prove part (c), it suffices, due to the truth of part (b), to show that $I_n^*(\pi^\alpha) = \delta_{-n}^*(\pi^\alpha)$ for an arbitrary prime π and positive integer α. We have

$$I_n^*(\pi^\alpha) = \frac{\delta_n^*(\pi^\alpha)}{|\pi^\alpha|^n} = |\pi|^{-\alpha n} \sum_{x \lozenge \pi^\alpha} |x|^n = |\pi|^{-\alpha n}(1 + |\pi^\alpha|^n)$$

$$= 1 + |\pi^\alpha|^{-n} = \sum_{x \lozenge \pi^\alpha} |x|^{-n} = \delta_{-n}^*(\pi^\alpha).$$

□

Remark 1.6. *Let $d \in K$, and let $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \backslash \{0\}$ satisfy $z \sim \prod_{j=1}^{r} \pi_j^{\alpha_j}$, where, for all distinct $j, \ell \in \{1, 2, \ldots, r\}$, π_j is a prime, α_j is a positive integer, and $\pi_j \not\sim \pi_\ell$. Combining parts (b) and (c) of Theorem 1.5, we see that, for any positive integer n, we may calculate $I_n^*(z)$ as $I_n^*(z) = \prod_{j=1}^{r} (1 + |\pi_j|^{-\alpha_j n})$.*

As an example, let us calculate $I_2^*(30)$ in $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. We have

$$30 \sim (1+i)^2 \cdot 3(2+i)(2-i),$$

so

$$I_2^*(30) = I_2^*\left((1+i)^2\right) I_2^*(3) I_2^*(2+i) I_2^*(2-i)$$

$$= \left(1 + \frac{1}{N(1+i)^2}\right)\left(1 + \frac{1}{N(3)}\right)\left(1 + \frac{1}{N(2+i)}\right)\left(1 + \frac{1}{N(2-i)}\right)$$

$$= \frac{5}{4} \cdot \frac{10}{9} \cdot \frac{6}{5} \cdot \frac{6}{5} = 2.$$

Thus, 30 is 2-powerfully unitarily perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$.

Now that we have established the foundations that we will need, we may study the properties of some n-powerfully unitarily t-perfect numbers.

2 Investigating n-powerfully Unitarily t-perfect Numbers

Theorem 2.1. *Let $d \in K$, and let $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \backslash \{0\}$. For any integer $n \geq 4$, $I_n^*(z) < 2$. Furthermore, if $I_3^*(z)$ is rational, then $I_3^*(z) < 2$.*

Proof. Let $\Psi(z)$ be the set of all primes in $A(d)$ that divide z, and let Φ be the set of all primes in $A(d)$. Then, for any integer $n \geq 3$,

$$I_n^*(z) = \prod_{\pi \in \Psi(z)} (1 + |\pi|^{-\rho_\pi(z)n}) < \prod_{\pi \in \Psi(z)} (1 + |\pi|^{-n}) < \prod_{\pi \in \Phi} (1 + |\pi|^{-n})$$

$$= \prod_{\substack{\pi \in \Phi \\ |\pi| \in \mathbb{N}}} (1 + |\pi|^{-n}) \prod_{\substack{\pi \in \Phi \\ |\pi| \notin \mathbb{N} \\ \pi \sim \overline{\pi}}} (1 + |\pi|^{-n}) \prod_{\substack{\pi \in \Phi \\ |\pi| \notin \mathbb{N} \\ \pi \not\sim \overline{\pi}}} (1 + |\pi|^{-n})$$

$$= \prod_{\substack{q \in \mathbb{P} \\ q \text{ is inert}}} (1 + q^{-n}) \prod_{\substack{p \in \mathbb{P} \\ p \text{ ramifies}}} (1 + \sqrt{p}^{-n}) \prod_{\substack{p \in \mathbb{P} \\ p \text{ splits}}} (1 + \sqrt{p}^{-n})^2.$$

If $n \geq 5$, then we have

$$I_n^*(z) < \prod_{\substack{q \in \mathbb{P} \\ q \text{ is inert}}} (1 + q^{-n}) \prod_{\substack{p \in \mathbb{P} \\ p \text{ ramifies}}} (1 + \sqrt{p}^{-n}) \prod_{\substack{p \in \mathbb{P} \\ p \text{ splits}}} (1 + \sqrt{p}^{-n})^2$$

$$< \prod_{\substack{q \in \mathbb{P} \\ q \text{ is inert}}} (1 + \sqrt{q}^{-n})^2 \prod_{\substack{p \in \mathbb{P} \\ p \text{ ramifies}}} (1 + \sqrt{p}^{-n})^2 \prod_{\substack{p \in \mathbb{P} \\ p \text{ splits}}} (1 + \sqrt{p}^{-n})^2$$

$$= \prod_{p \in \mathbb{P}} (1 + \sqrt{p}^{-n})^2 \leq \prod_{p \in \mathbb{P}} \left(1 + \sqrt{p}^{-5}\right)^2 = \prod_{p \in \mathbb{P}} \left(\frac{1 - p^{-5}}{1 - \sqrt{p}^{-5}}\right)^2$$

$$= \left(\frac{\zeta(5/2)}{\zeta(5)}\right)^2 < 2,$$

where ζ denotes the Riemann zeta function.

Next, suppose $n = 4$. Let us assume that $d \neq -7$ so that 2 does not split in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. Then

$$I_4^*(z) < \prod_{\substack{q \in \mathbb{P} \\ q \text{ is inert}}} \left(1 + q^{-4}\right) \prod_{\substack{p \in \mathbb{P} \\ p \text{ ramifies}}} \left(1 + \sqrt{p}^{-4}\right) \prod_{\substack{p \in \mathbb{P} \\ p \text{ splits}}} \left(1 + \sqrt{p}^{-4}\right)^2$$

$$< \prod_{\substack{p \in \mathbb{P} \\ p \text{ does} \\ \text{not split}}} \left(1 + \sqrt{p}^{-4}\right) \prod_{\substack{p \in \mathbb{P} \\ p \text{ splits}}} \left(1 + \sqrt{p}^{-4}\right)^2$$

$$= \left(1 + \sqrt{2}^{-4}\right) \prod_{\substack{p \in \mathbb{P} \setminus \{2\} \\ p \text{ does} \\ \text{not split}}} \left(1 + \sqrt{p}^{-4}\right) \prod_{\substack{p \in \mathbb{P} \\ p \text{ splits}}} \left(1 + \sqrt{p}^{-4}\right)^2$$

$$< \left(1 + \sqrt{2}^{-4}\right) \prod_{p \in \mathbb{P} \setminus \{2\}} \left(1 + \sqrt{p}^{-4}\right)^2 = \left(1 + \sqrt{2}^{-4}\right)^{-1} \prod_{p \in \mathbb{P}} \left(1 + \sqrt{p}^{-4}\right)^2$$

$$= \frac{4}{5} \prod_{p \in \mathbb{P}} \left(1 + p^{-2}\right)^2 = \frac{4}{5} \prod_{p \in \mathbb{P}} \left(\frac{1 - p^{-4}}{1 - p^{-2}}\right)^2 = \frac{4}{5} \left(\frac{\zeta(2)}{\zeta(4)}\right)^2 < 2.$$

Now, assume that $d = -7$ so that 3 is inert. We then have

$$I_4^*(z) < \prod_{\substack{q \in \mathbb{P} \\ q \text{ is inert}}} \left(1 + q^{-4}\right) \prod_{\substack{p \in \mathbb{P} \\ p \text{ ramifies}}} \left(1 + \sqrt{p}^{-4}\right) \prod_{\substack{p \in \mathbb{P} \\ p \text{ splits}}} \left(1 + \sqrt{p}^{-4}\right)^2$$

$$< \left(1 + 3^{-4}\right) \prod_{p \in \mathbb{P} \setminus \{3\}} \left(1 + \sqrt{p}^{-4}\right)^2 = \frac{1 + 3^{-4}}{\left(1 + \sqrt{3}^{-4}\right)^2} \prod_{p \in \mathbb{P}} \left(1 + \sqrt{p}^{-4}\right)^2$$

$$= \frac{41}{50} \prod_{p \in \mathbb{P}} \left(\frac{1 - p^{-4}}{1 - p^{-2}}\right)^2 = \frac{41}{50} \left(\frac{\zeta(2)}{\zeta(4)}\right)^2 < 2.$$

Finally, suppose $n = 3$ and $I_3^*(z)$ is rational. If π is a prime and $|\pi| = \sqrt{p}$ for some integer prime p, then it is easy to see that $\rho_\pi(z)$ must be even in order for $I_3^*(z)$ to be rational. Therefore,

$$I_3^*(z) = \prod_{\pi \in \Psi(z)} \left(1 + |\pi|^{-3\rho_\pi(z)}\right)$$

$$< \prod_{\substack{q\in\mathbb{P} \\ q \text{ is inert}}} (1+q^{-3}) \prod_{\substack{p\in\mathbb{P} \\ p \text{ ramifies}}} \left(1+\sqrt{p}^{-6}\right) \prod_{\substack{p\in\mathbb{P} \\ p \text{ splits}}} \left(1+\sqrt{p}^{-6}\right)^2$$

$$< \prod_{p\in\mathbb{P}} (1+p^{-3})^2 = \prod_{p\in\mathbb{P}} \left(\frac{1-p^{-6}}{1-p^{-3}}\right)^2 = \left(\frac{\zeta(3)}{\zeta(6)}\right)^2 < 2.$$

□

Corollary 2.2. *If $n \geq 3$ and $t \geq 2$ are integers, then there are no n-powerfully unitarily t-perfect numbers in any ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in K$.*

Theorem 2.3. *Let us work in a ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in K$. Suppose $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ satisfies $I_n^*(z) = t$ for some $n \in \{1,2\}$ and $t \in \mathbb{N}\backslash\{1\}$. Then $N(z)$ is even.*

Proof. Assume, for the sake of finding a contradiction, that $N(z)$ is odd. Write $z \sim \prod_{i=1}^{r} \pi_j^{\alpha_j}$, where, for all distinct $j, \ell \in \{1,2,\dots,r\}$, π_j is a prime, α_j is a positive integer, and $\pi_j \not\sim \pi_\ell$. Suppose that $n = 1$. If $N(\pi_j)$ is an integer prime for some $j \in \{1,2,\dots,r\}$, then it is easy to see that α_j must be even in order for $I_1^*(z) = \prod_{i=1}^{r}(1+|\pi_j|^{-\alpha_j})$ to be an integer (or even a rational number). This means that $|\pi_j|^{\alpha_j}$ is an integer for each $j \in \{1,2,\dots,r\}$, so $\delta_1^*(z) = \prod_{i=1}^{r}(1+|\pi_j|^{\alpha_j})$ and $|z| = \prod_{i=1}^{r}|\pi_j|^{\alpha_j}$ are positive integers. Furthermore, $|\pi_j|^{\alpha_j}$ must be odd for each $j \in \{1,2,\dots,r\}$, so $2^r|\delta_1^*(z)$ in $\mathbb{Z}$. As $\delta_1^*(z) = t|z|$ and $|z|$ is odd, we see that $2^r|t$ in $\mathbb{Z}$. However, $t = I_1^*(z) = \prod_{i=1}^{r}(1+|\pi_j|^{-\alpha_j}) \leq \prod_{i=1}^{r}\left(1+\frac{1}{3}\right) = \left(\frac{4}{3}\right)^r$, which a contradiction.

Now, suppose $n = 2$. Then $\delta_2^*(z)$ and $N(z)$ are positive integers. Because $\delta_2^*(z) = \prod_{i=1}^{r}(1+N(\pi_j)^{\alpha_j})$ and $N(\pi_j)^{\alpha_j}$ is odd for each $j \in \{1,2,\dots,r\}$, we see that $2^r|\delta_2^*(z)$ in $\mathbb{Z}$. Again, $2^r|t$ in $\mathbb{Z}$, which is a contradiction because $t = I_2^*(z) = \prod_{i=1}^{r}(1+N(\pi_j)^{-\alpha_j}) \leq \prod_{i=1}^{r}\left(1+\frac{1}{3}\right) = \left(\frac{4}{3}\right)^r$. □

The rings $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ and $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ are two of the most heavily-studied quadratic rings, so it is not surprising that they prove to be particularly interesting for our purposes. We proceed to prove a theorem about 2-powerfully t-perfect numbers in each of these rings.

Theorem 2.4. *Suppose z is 2-powerfully unitarily t-perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ for some integer $t \geq 2$. Then we may write $z = (1+i)^{\gamma}x$, where $x \in \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ and $N(x)$ is odd. Also, x has $\gamma + \upsilon_2(t)$ nonassociated prime divisors.*

Proof. Let us write $x \sim \prod_{j=1}^{r} \pi_j^{\alpha_j}$, where, for all distinct $j, \ell \in \{1, 2, \ldots, r\}$, π_j is a prime, α_j is a positive integer, and $\pi_j \not\sim \pi_\ell$. From Fact 1.3, we know that an integer prime is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ if and only if it is congruent to 3 modulo 4. Therefore, if we choose any $j \in \{1, 2, \ldots, r\}$, then either $N(\pi_j) = q^2$ for some integer prime q that is congruent to 3 modulo 4 or $N(\pi_j) = p$ for some integer prime p that is congruent to 1 modulo 4. Either way, $N(\pi_j) \equiv 1 \pmod 4$, so $\upsilon_2(\delta_2^*(x)) = \upsilon_2\left(\prod_{i=1}^{r}(1 + N(\pi_j)^{\alpha_j})\right) = r$. Then the desired result follows from the equation $(2^{\gamma}+1)\delta_2^*(x) = 2^{\gamma}tN(x)$ and the fact that $\upsilon_2(N(x)) = \upsilon_2(2^{\gamma}+1) = 0$. $\square$

Theorem 2.5. *Let us work in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$. If a and b are relatively prime positive integers and $3|a$ in $\mathbb{Z}$, then $\frac{a}{b}$ is not in the range of the function I_2^*.*

Proof. For the sake of finding a contradiction, suppose $I_2^*(z) = \frac{a}{b}$ for some $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$. Then $b\delta_2^*(z) = aN(z)$, which implies that $3|\delta_2^*(z)$ in $\mathbb{Z}$. This means that there must be some prime π_0 such that $N(\pi_0)^{\rho_{\pi_0}(z)} \equiv 2 \pmod 3$. Fact 1.3 tells us that an integer prime is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ if and only if it is congruent to 2 modulo 3. If $N(\pi_0) = q^2$ for some inert integer prime q, then $N(\pi_0)^{\rho_{\pi_0}(z)} = q^{2\rho_{\pi_0}(z)} \equiv 1 \pmod 3$, which is a contradiction. Clearly $\pi_0 \not\sim 3$, so $N(\pi_0)$ must be a split integer prime. However, this means that $N(\pi_0) \equiv 1 \pmod 3$, so $N(\pi_0)^{\rho_{\pi_0}(z)} \equiv 1 \pmod 3$, which is a contradiction. $\square$

Corollary 2.6. *If t is a positive integer multiple of 3, then there are no 2-powerfully unitarily t-perfect numbers in $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$.*

Now, let us work in rings $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in K\backslash\{-7\}$ so that 2 does not split. Then there is a unique prime $\xi(d) \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \cap A(d)$ of minimal even norm. Namely,

$$\xi(d) = \begin{cases} 1+i, & \text{if } d = -1; \\ \sqrt{-2}, & \text{if } d = -2; \\ 2, & \text{if } d \in K\backslash\{-1, -2, -7\}. \end{cases}$$

Suppose z is a 2-powerfully unitarily t-perfect number in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ for $d \in K\backslash\{-7\}$ and $t \in \mathbb{N}\backslash\{1\}$. By Theorem 2.1, we see that we may write $z \sim (\xi(d))^{\mu} x_0$, where $\mu \in \mathbb{N}$, $x_0 \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, and $2 \nmid N(x_0)$ in $\mathbb{Z}$. Furthermore, if $d \in \{-1, -2\}$, then we have $2^{\mu} + 1 = \delta_2^*((\xi(d))^{\mu}) | \delta_2^*(z) = tN(z)$. Hence, if we assume that $3 \nmid N(z)$ in $\mathbb{Z}$, then μ must be even. Therefore, under the assumption that $3 \nmid N(z)$ in $\mathbb{Z}$, we may write

$$\gamma = \begin{cases} \frac{1}{2}\mu, & \text{if } d \in \{-1, -2\}; \\ \mu, & \text{if } d \in K\backslash\{-1, -2, -7\} \end{cases}$$

so that $z \sim 2^{\gamma} x_0$. Then $z = 2^{\gamma} x$, where x is an associate of x_0.

When M. V. Subbarao and L. J. Warren studied unitary perfect numbers, which are positive integers n that satisfy $\sigma^*(n) = 2n$, they noticed that all known unitary perfect numbers are multiples of 3. They then gave four conditions that any unitary perfect numbers not divisible by 3 would need to satisfy [4]. Using the information discussed in the preceding paragraph, we will find analogues of the conditions that Subbarao and Warren established.

Theorem 2.7. *Let $d \in K\backslash\{-7\}$. Suppose z is 2-powerfully perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ and $3 \nmid N(z)$ in $\mathbb{Z}$. Then we may write $z = 2^{\gamma} x$, where $\gamma \in \mathbb{N}$, $x \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, and $N(x)$ is odd. For any prime π, we have $N(\pi)^{\rho_\pi(x)} \equiv 1 \pmod 6$. Furthermore, there exists a prime divisor π_0 of x such that $N(\pi_0) \equiv 5 \pmod 6$, and x has an even number of nonassociated prime factors.*

Proof. We already established that we may write $z = 2^{\gamma} x$ for $\gamma \in \mathbb{N}$. As $\delta_2^*(2^{\gamma}) = 2^{2\gamma} + 1$ and $N(2^{\gamma}) = 2^{2\gamma}$, we see that $(2^{2\gamma} + 1)\delta_2^*(x) = 2^{2\gamma+1} N(x)$. Now, let π be a prime. We wish to show that $N(\pi)^{\rho_\pi(x)} \equiv 1 \pmod 6$. The result is clear if $\rho_\pi(x) = 0$, and if $\rho_\pi(x) > 0$, the result is still quite trivial when we consider that $1 + N(\pi)^{\rho_\pi(x)} | \delta_2^*(x)$ in $\mathbb{Z}$. The fact that there exists some prime divisor π_0 of x such that $N(\pi_0) \equiv 5 \pmod 6$ follows from the fact that $2^{2\gamma} + 1 \equiv 5 \pmod 6$. Finally, to show that x has an even number of nonassociated prime divisors, we use the fact that $N(\pi)^{\rho_\pi(x)} \equiv 1 \pmod 6$ for all primes π. This implies that $N(x) \equiv 1 \pmod 3$. As $2^{2\gamma+1} \equiv 2^{2\gamma} + 1 \equiv 2 \pmod 3$, we see that $\delta_2^*(x) \equiv 1 \pmod 3$. Let us write $x \sim \prod_{j=1}^{r} \pi_j^{\alpha_j}$, where, for all distinct $j, \ell \in \{1, 2, \ldots, r\}$, π_j is a prime, α_j is a positive integer, and $\pi_j \not\sim \pi_\ell$. Then $\delta_2^*(x) = \prod_{j=1}^{r} (1 + N(\pi_j)^{\alpha_j}) \equiv \prod_{j=1}^{r} (2) \pmod 3$, so r must be even. □

We pause to mention that we may easily establish results analogous to those given in Theorem 2.7 in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$. In this ring, 2 splits as $2 = \varepsilon\overline{\varepsilon}$, where $\varepsilon = \dfrac{1+\sqrt{-7}}{2}$. Suppose that z is 2-powerfully unitarily perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$ and that $3 \nmid N(z)$ in $\mathbb{Z}$. Then we may write $z = \varepsilon^{\gamma_1}\overline{\varepsilon}^{\gamma_2}x$, where $x \in \mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$ and $N(x)$ is odd. If $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$, then $(2^{\gamma_1}+1)(2^{\gamma_2}+1)\delta_2^*(x) = 2^{\gamma_1+\gamma_2+1}N(x)$. On the other hand, if $\gamma_1 = 0$ or $\gamma_2 = 0$ (γ_1 and γ_2 cannot both be 0 by Theorem 2.1), then we may write $\gamma = \gamma_1 + \gamma_2$ to get $(2^\gamma + 1)\delta_2^*(x) = 2^{\gamma+1}N(x)$. Because $3 \nmid N(x)$ in $\mathbb{Z}$, we know that γ_1 and γ_2 must be even and that $N(\pi)^{\rho_\pi(x)} \equiv 1 \pmod 6$ for all primes π. Furthermore, because $2^{\gamma_1}+1 \equiv 2^{\gamma_2}+1 \equiv 5 \pmod 6$, we see that x must have some prime divisor whose norm is congruent to 5 modulo 6. Finally, if $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$, then $(2^{\gamma_1}+1)(2^{\gamma_2}+1) \equiv 1 \pmod 3$ and $2^{\gamma_1+\gamma_2+1}N(x) \equiv 2 \pmod 3$, so x must have an odd number of nonassociated prime divisors. If $\gamma_1 = 0$ or $\gamma_2 = 0$, then x must have an even number of nonassociated prime divisors because $2^\gamma + 1 \equiv 2^{\gamma+1}N(x) \equiv 2 \pmod 3$.

We end with a note about unitarily t-perfect numbers. If $d \in K$ and $t \geq 2$ is an integer, then we can find a unitarily t-perfect number in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ for every unitary t-perfect number in $\mathbb{Z}$. We formalize and generalize this notion in the following theorem.

Theorem 2.8. *Let $b > 1$ be a rational number, and let $d \in K$. Let $U(b) = \{n \in \mathbb{N} \colon \sigma^*(n) = bn\}$, and let $V_d(b) = \{z \in A(d) \colon I_1^*(z) = b\}$. Then there exists an injective function $g \colon U(b) \to V_d(b)$.*

Proof. If p is an integer prime that does not split in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, let $g(p) = p$. If p is an integer prime that splits in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ as $p = \pi\overline{\pi}$, where $\pi \in A(d)$, let $g(p)$ be the associate of π^2 in $A(d)$. Now, for any positive integer $n \in U(b)$ with canonical prime factorization $n = \prod_{j=1}^{r} p_j^{\alpha_j}$, let $g(n)$ be the associate of $\prod_{j=1}^{r} g(p_j)^{\alpha_j}$ that lies in $A(d)$. It is easy to see, using the fact that $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is a unique factorization domain, that g is an injection. To show that the range of g is a subset of $V_d(b)$, note that $|g(p)| = p$ for all primes p. Therefore, with n as before, we have

$$I_1^*(g(n)) = I_1^*\left(\prod_{j=1}^{r} g(p_j)^{\alpha_j}\right) = \prod_{j=1}^{r} \left(1 + |g(p_j)|^{-\alpha_j}\right)$$

$$= \prod_{j=1}^{r} \left(1 + p_j^{-\alpha_j}\right) = \frac{\sigma^*(n)}{n} = b.$$

□

3 Ideas for Further Research

With Theorem 1.5 as evidence, we see that the functions δ_n^* and I_n^* have some fairly nice properties that we may exploit for further research. We pose some ideas here.

First, we note that we could generalize the ideas presented in this paper to other quadratic rings. However, if we choose to continue working with imaginary quadratic rings that are unique factorization domains, we could still look at analogues of many other objects defined in the integers. For example, one might wish to investigate analogues of superperfect numbers and unitary superperfect numbers. One could also look at analogues of biunitary or even infinitary divisor functions in quadratic rings.

There are also plenty of questions left open related to the ideas discussed in this paper. For example, the author has made no attempt to actually find n-powerfully unitarily t-perfect numbers, so it is likely that many could be quite easy to discover. One question of particular interest is the following. For a given $d \in K$, what are the rational numbers $b > 1$ for which the function $g\colon U(b) \to V_d(b)$ defined in the proof of Theorem 2.8 is bijective?

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