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Theoretical Friends of Finite Proximity

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Abstract

Using a definition given by Jeffrey Ward [4], we look into which positive integers have a theoretical friend of finite proximity (TFOFP). We show that 1 and odd primes do not have TFOFPs. We show that prime powers satisfying some conditions have TFOFPs. Furthermore, we prove that certain integers have a TFOFP if and only if they have a friend.

1 Abundancy Index and Friends

Let $\mathbb{Z}^+$ denote the set of positive integers and $\mathbb{N}$ the set of nonnegative integers. For $n \in \mathbb{Z}^+$, the sum of the positive divisors of n is a denoted $\sigma(n)$. The ratio $\frac{\sigma(n)}{n}$ is known as the *abundancy ratio* or *abundancy index* of n and is denoted $I(n)$. A *perfect number* is a positive integer n satisfying $I(n) = 2$.

Ward [4] lists the following properties of the abundancy index, where m and n are positive integers and all primes are positive.

Property 1.1. $I(n) \geq 1$ with equality if and only if $n = 1$.

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Property 1.2. If m divides n, then $I(m) \leq I(n)$ with equality if and only if $m = n$.

Property 1.3. I is weakly multiplicative, i.e., if m and n are relatively prime, then $I(mn) = I(m)I(n)$. This follows from the fact that σ is weakly multiplicative.

Property 1.4. If $p_1, \ldots, p_k$ are distinct primes and $e_1, \ldots, e_k$ are positive integers, then

$$I\left(\prod_{j=1}^{k} p_j^{e_j}\right) = \prod_{j=1}^{k} I(p_j^{e_j}) = \prod_{j=1}^{k} \frac{\sum_{i=0}^{e_j} p_j^i}{p_j^{e_j}} = \prod_{j=1}^{k}\left(\sum_{i=0}^{e_j} p_j^{-i}\right) = \prod_{j=1}^{k} \frac{p_j^{e_j+1}-1}{p_j^{e_j}(p_j-1)}$$

since I is weakly multiplicative and $\sigma(p_j^{e_j}) = \sum_{i=0}^{e_j} p_j^i = \frac{p^{e_j+1}-1}{p_j-1}$ for each $1 \leq j \leq k$.

Property 1.5. Suppose that $p_1, \ldots, p_k$ are distinct primes, $q_1, \ldots, q_k$ are distinct primes, $e_1, \ldots, e_k$ are positive integers, and $p_j \leq q_j$ for $1 \leq j \leq k$. Then

$$I\left(\prod_{j=1}^{k} p_j^{e_j}\right) \geq I\left(\prod_{j=1}^{k} q_j^{e_j}\right)$$

with equality if and only if $p_j = q_j$ for every $1 \leq j \leq k$. This follows from Property 1.4 and the observation that if $e \geq 1$, then $\frac{p^{e+1}-1}{p^e(p-1)}$ is a decreasing function of p on $(1, \infty)$.

Property 1.6. If the distinct prime factors of n are $p_1, \ldots, p_k$, then $I(n) < \prod_{j=1}^{k} \frac{p_j}{p_j-1}$. This can be seen by applying Property 1.4 and observing that if $p > 1$,

$$\frac{p^{e+1}-1}{p^e(p-1)} = \frac{p - \frac{1}{p^e}}{p-1}$$

is an increasing function of e, and

$$\lim_{e\to\infty} \frac{p - \frac{1}{p^e}}{p-1} = \frac{p}{p-1}.$$

Positive integers m and n are *friends* if and only if $m \neq n$ and $I(m) = I(n)$. If the positive integer m has a friend, then m is a *friendly number.* If the positive integer m does not have any friends, then m is a *solitary number.*

As a consequence of Property 1.2, if m divides n, then m and n cannot be friends. The following lemma states a sufficient condition for an integer to be solitary.

Lemma 1.1. *Let m be a positive integer. If $\gcd(m, \sigma(m)) = 1$, then m is solitary.*

Proof. Toward contradiction, assume m is not solitary number. Then m has a friend, say n, so that $I(m) = I(n)$. By the definition of abundancy index, this means $\frac{\sigma(m)}{m} = \frac{\sigma(n)}{n}$, which is equivalent to $n\sigma(m) = m\sigma(n)$. It follows that m divides $n\sigma(m)$. Since $\gcd(m, \sigma(m)) = 1$, m must divide n. Therefore, as noted above, m and n cannot be friends by Property 1.2. □

Corollary 1.2. *No prime power has a friend. That is, if p is a prime and $k \in \mathbb{Z}^+$, then p^k is solitary.*

Proof. Let p be a prime and $k \in \mathbb{Z}^+$. Then

$$\sigma(p^k) = 1 + p + \cdots + p^k = \sum_{j=0}^{k} p^j = 1 + \sum_{j=1}^{k} p^j = 1 + p \sum_{j=1}^{k} p^{j-1}.$$

Therefore, p does not divide $\sigma(p^k) = 1 + p \sum_{j=1}^{k} p^{j-1}$. Thus, since all of the divisors of p^k that are not 1 are divisible by p, $\gcd(p^k, \sigma(p^k)) = 1$. The corollary follows from Lemma 1.1. □

The corollary tells us that there are infinitely many solitary numbers. However, the natural density of solitary numbers is unknown. Anderson and Hickerson [1] conjectured that the natural density of friendly numbers is 1, implying that the natural density of solitary numbers is 0. As far as the authors know, no proof of this conjecture has been published, but it is known that the density of friendly numbers is positive. If m and n are friends and k is a positive integer relatively prime to both m and n, then, by the weak multiplicativity of I (Property 1.3), mk and nk are friends. Applying this, it follows that for any given friendly number m that the set of friendly numbers that are multiples of m has a positive natural density. Therefore, the natural density of friendly numbers is positive, but the exact natural density of friendly numbers is unknown.

On the other hand, it is known that the range of abundancy indices is dense in the interval $[1, \infty)$ [2]. Also, the set of rational numbers that are not abundancy indices is dense in $[1, \infty)$ [5, 3]. The following is a lemma used by Weiner [5] to prove that the set of rational numbers that are not abundancy indices is dense in $[1, \infty)$.

Lemma 1.3 (Weiner [5])**.** *If k is relatively prime to m, and $m < k < \sigma(m)$, then k/m is not the abundancy ratio of any integer.*

2 Theoretical Friends

In [4], Ward formulated some conditions for 10 to have a friend. Ward [4] also defined a theoretical friend of proximity t of a positive integer m. We use the following modification of Ward's definition of a theoretical friend of proximity t of a positive integer m: A theoretical friend of proximity t of a positive integer m is a sequence $s = \{s_k\} = \{s_k\}_{k=1}^{\infty}$ of positive integers such that

1. $\lim_{k \to \infty} I(s_k) = I(m)$,
2. $|P_s| = t$, where $P_s = \{p \mid p \text{ is a positive prime and } p \text{ divides } s_k \text{ for some } k\}$, and
3. $s_k \neq m$ for all k. [1]

We say that s is a theoretical friend of finite proximity (TFOFP) of a positive integer m if and only if s is a theoretical friend of m of proximity t for some nonnegative integer t.

[1]Ward excluded this condition from his definition, but the question "Does every positive integer have a theoretical friend of finite proximity?" becomes trivial without this condition. Without condition 3, the constant sequence $m, m, m, \ldots$ is a theoretical friend of m of finite proximity. Thus, without condition 3, every positive integer would have a theoretical friend of finite proximity.

Example 2.1. Let $\{p_1, p_2, p_3, \ldots\}$ be the set of primes such that $p_1 < p_2 < p_3 < \cdots$. Let $s = \{p_k\}$. Since $p_k \to \infty$ as $k \to \infty$,

$$\lim_{k\to\infty} I(p_k) = \lim_{k\to\infty} \left(1 + \frac{1}{p_k}\right) = 1 = I(1).$$

Therefore, one can say that s is a theoretical friend of 1. Since p_k divides itself for each $k \geq 1$, P_s is the set of all primes. Thus, s is not a TFOFP of 1.

Example 2.2. Let m be a perfect number so that $I(m) = 2$. Let $s = \{2^k\}$. Observe that

$$\lim_{k\to\infty} I(2^k) = 2 = I(m),$$

so s is a theoretical friend of proximity 1 of any perfect number.

Example 2.3. Let $s = \{3^k\}$. Since

$$\lim_{k\to\infty} I(3^k) = \frac{3}{2} = I(2),$$

s is a theoretical friend of proximity 1 of 2.

Example 2.4. This example was given by Ward in [4]. Let $s = \{3^k \cdot 5\}$. Since

$$\lim_{k\to\infty} I(3^k \cdot 5) = \frac{3}{2} \cdot \frac{6}{5} = \frac{9}{5} = I(10),$$

s is a theoretical friend of proximity 2 of 10.

In the above examples, each theoretical friend was nonconstant. However, our definition of theoretical friends does not forbid constant sequences. If m and n are friends, then the sequence $m, m, m, \ldots$ is a TFOFP of n and the sequence $n, n, n, \ldots$ is a TFOFP of m.

Because the set of abundancy indices is dense in $[1, \infty)$, every positive integer m has a theoretical friend $\{s_k\}$ such that $\lim_{k\to\infty} I(s_k) = I(m)$. However, it was unknown whether every positive integer had a TFOFP. In [4], Ward poses the following question: Does every positive integer have a theoretical friend of finite proximity? The following theorem tells us that the answer to Ward's question is no.

Theorem 2.1. *The integer* 1 *does not have a TFOFP.*

Proof. Toward contradiction, assume that 1 has a theoretical friend $s = \{s_k\}$ of proximity t for some $t \in \mathbb{N}$. Note that $t > 0$; otherwise $s_k = 1$ for all k, contradicting $s_k \neq 1$ for all k. This means for each k,

$$s_k = \prod_{j=1}^{t} p_j^{e_{k,j}},$$

where $\{p_1, p_2, \ldots, p_t\} = P_s$, $p_1 < p_2 < \cdots < p_t$, and each $e_{k,j} \in \mathbb{N}$.

Since each $s_k > 1$, we know that for any k there exists some j that $e_{k,j} > 0$. For such a j for an arbitrary k, we see by Property 1.2 that

$$1 + \frac{1}{p_t} \leq 1 + \frac{1}{p_j} = I(p_j) \leq I(p_j^{e_{k,j}}) \leq I(s_k).$$

Thus, $\lim_{k\to\infty} I(s_k) \geq 1 + \frac{1}{p_t} > 1 = I(1)$, so s is not a theoretical friend of 1, contradicting our initial assumption. □

Now that we have proved that there is a positive integer (namely the integer 1) that does not have a TFOFP, we introduce the definition of a super solitary number. A *super solitary number* is a positive integer that does not have a TFOFP. We now ask a question related to Ward's question: Are there infinitely many super solitary numbers? We have to introduce some theorems before we can answer this question.

The following theorem tells us that if m has a TFOFP then it must have at least one TFOFP of a certain form. In all that follows, a product over the empty set is understood to be 1.

Theorem 2.2. *If the positive integer m has a TFOFP, then m has a TFOFP $\{s_k\}$ of the form*

$$s_k = \prod_{r\in R} r^{e(r)} \prod_{q\in Q} q^k,$$

where R and Q are finite sets of primes, $R \cap Q = \emptyset$, and $e(r) \in \mathbb{Z}^+$ for each $r \in R$.

Proof. Let $x = \{x_k\}$ be a TFOFP of m such that

$$x_k = \prod_{j=1}^{t} p_j^{e_j(k)},$$

where $t \in \mathbb{N}$, the p_j's are distinct primes, and each $e_j(k) \in \mathbb{N}$.

If $t = 0$, then each $x_k = 1$. Take $R = Q = \emptyset$ so that each $s_k = 1$.

Now suppose that $t \geq 1$. First look at $\{e_1(k)\}_{k=1}^{\infty}$. If $\{e_1(k)\}_{k=1}^{\infty}$ is bounded, by the Pigeonhole Principle, there exists a subsequence $x' = \{x'_k\}$ of x such that $\{e'_1(k)\}_{k=1}^{\infty}$ is a constant sequence, where $e'_1(k)$ is the exponent of p_1 in the prime factorization of x'_k. If $e'_1(1) \neq 0$, add p_1 to R and let $e(p_1) = e'_1(1)$.

If $\{e_1(k)\}_{k=1}^{\infty}$ is unbounded, there exists a subsequence $x' = \{x'_k\}$ of x such that $\{e'_1(k)\}_{k=1}^{\infty}$ is a strictly increasing sequence, where $e'_1(k)$ is defined the same as in the bounded case. Place p_1 in Q.

If $t \geq 2$, look at $\{e'_2(k)\}_{k=1}^{\infty}$, where $e'_2(k)$ is the exponent of p_2 in the prime factorization of x'_k. If $\{e'_2(k)\}_{k=1}^{\infty}$ is bounded, by the Pigeonhole Principle, there exists a subsequence $x'' = \{x''_k\}$ of x' such that $\{e''_2(k)\}_{k=1}^{\infty}$ is a constant sequence, where $e''_2(k)$ is the exponent of p_2 in the prime factorization of x''_k. If $e''_2(1) \neq 0$, add p_2 to R and let $e(p_2) = e''_2(1)$.

If $\{e'_2(k)\}_{k=1}^{\infty}$ is unbounded, there exists a subsequence $x'' = \{x''_k\}$ of x' such that $\{e''_2(k)\}_{k=1}^{\infty}$ is a strictly increasing sequence, where $e''_2(k)$ is defined the same as in the bounded case. Place p_2 in Q.

Continuing (if necessary) in this manner for $p_3, \ldots, p_t$, by taking subsequences of subsequences, we arrive at a subsequence $y = \{y_k\}$ of x such that

$$y_k = \prod_{r \in R} r^{e(r)} \prod_{q \in Q} q^{f(q,k)},$$

where each $f(q, k) \in \mathbb{N}$ and $f(q, k) \to \infty$ as $k \to \infty$ for each $q \in Q$. Because y is a subsequence of the TFOFP x of m, y is a TFOFP of m. By Property 1.3 and the remarks in the verification of Property 1.6,

$$I(m) = \lim_{k\to\infty} I(y_k) = \lim_{k\to\infty} \prod_{r\in R} I\left(r^{e(r)}\right) \prod_{q\in Q} I\left(q^{f(q,k)}\right) = \prod_{r\in R} I\left(r^{e(r)}\right) \prod_{q\in Q} \frac{q}{q-1}.$$

If $Q = \emptyset$, let $\{s_k\} = \{y_k\}$ be the TFOFP of m stated in the theorem. If $Q \neq \emptyset$, let $\{s_k\}$ be the sequence defined by $s_k = \prod_{r\in R} r^{e(r)} \prod_{q\in Q} q^k$. By Property 1.3 and the remarks in the verification of Property 1.6,

$$\begin{aligned}\lim_{k\to\infty} I(s_k) &= \lim_{k\to\infty} \prod_{r\in R} I\left(r^{e(r)}\right) \prod_{q\in Q} I\left(q^k\right)\\ &= \prod_{r\in R} I\left(r^{e(r)}\right) \prod_{q\in Q} \frac{q}{q-1} = \lim_{k\to\infty} I(y_k) = I(m).\end{aligned}$$

By Property 1.2, $I(s_k)$ strictly increases to $I(m)$ as k increases, so s_k cannot equal m for any k. Since R and Q are finite sets of primes, we have shown that $\{s_k\}$ is a TFOFP of m. □

The next lemma follows from Property 1.4 and the last part of the proof of Theorem 2.2.

Lemma 2.3. *Let $\{s_k\}$ be a sequence such that*

$$s_k = \prod_{r\in R} r^{e(r)} \prod_{q\in Q} q^k,$$

where R and Q are finite sets of primes, $R \cap Q = \emptyset$, and $e(r) \in \mathbb{Z}^+$ for each $r \in R$. Then

$$\lim_{k\to\infty} I(s_k) = \prod_{r\in R} \frac{\sigma\left(r^{e(r)}\right)}{r^{e(r)}} \prod_{q\in Q} \frac{q}{q-1} = \prod_{r\in R} \frac{\sum_{i=0}^{e(r)} r^i}{r^{e(r)}} \prod_{q\in Q} \frac{q}{q-1}.$$

Lemma 2.4. *Suppose there exist finite sets R and Q of primes such that*

$$I(m) = \prod_{r\in R} \frac{\sigma\left(r^{e(r)}\right)}{r^{e(r)}} \prod_{q\in Q} \frac{q}{q-1} = \prod_{r\in R} \frac{\sum_{i=0}^{e(r)} r^i}{r^{e(r)}} \prod_{q\in Q} \frac{q}{q-1},$$

$R \cap Q = \emptyset$, and $e(r) \in \mathbb{Z}^+$ for each $r \in R$. If $m \neq \prod_{r\in R} r^{e(r)}$, then m has a TFOFP, namely $\{s_k\}$ such that $s_k = \prod_{r\in R} r^{e(r)} \prod_{q\in Q} q^k$.

Proof. Let $s_k = \prod_{r\in R} r^{e(r)} \prod_{q\in Q} q^k$. By Lemma 2.3, we have

$$\lim_{k\to\infty} I(s_k) = \prod_{r\in R} \frac{\sigma\left(r^{e(r)}\right)}{r^{e(r)}} \prod_{q\in Q} \frac{q}{q-1} = I(m).$$

Note that $|P_s|$, where $s = \{s_k\}$, is at most $|R| + |Q|$, so $|P_s|$ is finite.

Suppose s is not a TFOFP of m. Since $\lim_{k\to\infty} I(s_k) = I(m)$ and $|P_s|$ is finite, this means there exists a j such that

$$m = s_j = \prod_{r\in R} r^{e(r)} \prod_{q\in Q} q^j.$$

Let $k > j$. Because s_j divides s_k, $I(s_j) \leq I(s_k)$ by Property 1.2. Since $I(m) = I(s_j) \leq I(s_k)$ and $I(m) = \lim_{k\to\infty} I(s_k)$, we have $I(s_j) = I(m) = I(s_k)$. By Property 1.2, this is only possible if

$$\prod_{r\in R} r^{e(r)} \prod_{q\in Q} q^j = s_j = s_k = \prod_{r\in R} r^{e(r)} \prod_{q\in Q} q^k,$$

which implies $1 = \prod_{q\in Q} q^{k-j}$. Because $k > j$, this can only occur if $Q = \emptyset$. Thus, $m = s_j = s_k = \prod_{r\in R} r^{e(r)}$. □

We now provide an alternative proof of Theorem 2.1.

Alternative proof of Theorem 2.1. [2] Toward contradiction, assume that 1 has a TFOFP. By Theorem 2.2, 1 has a TFOFP $\{s_k\}$ of the form

$$s_k = \prod_{r\in R} r^{e(r)} \prod_{q\in Q} q^k,$$

where R and Q are finite sets of primes, $R \cap Q = \emptyset$, and $e(r) \in \mathbb{Z}^+$ for each $r \in R$. Note that $R \cup Q \neq \emptyset$, otherwise if $R \cup Q = \emptyset$ then each $s_k = 1$, contradicting $\{s_k\}$ is a TFOFP of 1. Because $\{s_k\}$ is a TFOFP of 1,

$$1 = I(1) = \lim_{k\to\infty} I(s_k) = \prod_{r\in R} \frac{\sum_{i=0}^{e(r)} r^i}{r^{e(r)}} \prod_{q\in Q} \frac{q}{q-1},$$

[2] Jackson Morrow independently developed this proof in 2013. However, he assumed Theorem 2.2 without proving it. The authors of this paper have included some details omitted in his proof.

by Lemma 2.3. Since every term in the product on the right hand side is greater than 1, the equality implies $R \cup Q = \emptyset$, which produces a contradiction. $\square$

Lemma 2.5. *Let a and b be positive integers so that* $\gcd(a, b) = 1$*. Suppose*

$$\frac{a}{b} = \prod_{r \in R} \frac{\sum_{i=0}^{e(r)} r^i}{r^{e(r)}} \prod_{q \in Q} \frac{q}{q-1}, \tag{2.1}$$

where R and Q are finite sets of primes, $R \cap Q = \emptyset$, and $e(r) \in \mathbb{Z}^+$ for each $r \in R$. If $Q \neq \emptyset$, then $\hat{q} = \max(Q)$ divides a.

Proof. Equation (2.1) implies

$$a \prod_{r \in R} r^{e(r)} \prod_{q \in Q} (q-1) = b \prod_{r \in R} \left(\sum_{i=0}^{e(r)} r^i \right) \prod_{q \in Q} q.$$

Because $\hat{q} \in Q$, $\hat{q}$ divides $a \prod_{r \in R} r^{e(r)} \prod_{q \in Q} (q-1)$. For any $q \in Q$, $q - 1 < q \leq \max(Q) = \hat{q}$. This means $\hat{q}$ cannot divide $\prod_{q \in Q} (q-1)$. Since $R \cap Q = \emptyset$ and R and Q are sets of primes, $\hat{q} \in Q$ does not divide $\prod_{r \in R} r^{e(r)}$. Therefore, $\hat{q}$ divides a. $\square$

Theorem 2.6. *Any odd prime p is super solitary.*

Proof. Toward contradiction, let p be an odd prime and let it have a TFOFP. By Theorem 2.2, this implies that p has a TFOFP $\{s_k\}$ of the form

$$s_k = \prod_{r \in R} r^{e(r)} \prod_{q \in Q} q^k,$$

where R and Q are finite sets of primes, $R \cap Q = \emptyset$, and $e(r) \in \mathbb{Z}^+$ for each $r \in R$. By Lemma 2.3, this implies that

$$I(p) = \frac{p+1}{p} = \prod_{r \in R} \frac{\sum_{i=0}^{e(r)} r^i}{r^{e(r)}} \prod_{q \in Q} \frac{q}{q-1} = \lim_{k \to \infty} I(s_k) \tag{2.2}$$

Since $I(p)$ is in reduced form, its denominator must divide the denominator of $\lim_{k\to\infty} I(s_k)$. That is, p must divide $\prod_{r\in R} r^{e(r)} \prod_{q\in Q}(q-1)$.

Since all odd primes are solitary by Corollary 1.2, we know that $Q \neq \emptyset$, because if $Q = \emptyset$ then $\prod_{r\in R} r^{e(r)}$ would be a friend of p. By Lemma 2.5, we know that $\hat{q} = \max(Q)$ divides $p+1$. Furthermore, since $\hat{q}$ is prime and $p+1 > 2$ is even (and therefore composite), it must be the case that $\hat{q} < p+1$, whence $\hat{q} - 1 < p$. However, since $q - 1 \leq \hat{q} - 1 < p$ for all $q \in Q$, we know that p cannot divide $\prod_{q\in Q}(q-1)$. Since p is prime, this implies that p divides $\prod_{r\in R} r^{e(r)}$. Then, since all $r \in R$ are prime, this would imply that $p = r$ for some $r \in R$. Since $Q \neq \emptyset$ and $p \in R$, we have by using Property 1.2 that

$$I(p) \leq I\left(p^{e(p)}\right) = \frac{\sum_{i=0}^{e(p)} p^i}{p^{e(p)}} < \prod_{r\in R} \frac{\sum_{i=0}^{e(r)} r^i}{r^{e(r)}} \prod_{q\in Q} \frac{q}{q-1},$$

which contradicts Equation (2.2). □

Because there are infinitely many odd primes, Theorem 2.6 tells us that there are infinitely many super solitary numbers. We now ask a logical next question: What is the natural density of super solitary numbers? If the conjecture of Anderson and Hickerson [1] is correct, then the natural density of super solitary numbers is zero since the set of super solitary numbers is a subset of the set of solitary numbers. We do not give a natural density of super solitary numbers here, but we do give some lemmata and theorems that may become useful tools in answering the question.

Lemma 2.7. *Let a and b be positive integers such that* $\gcd(a,b) = 1$. *Suppose*

$$\frac{a}{b} = \prod_{r\in R} \frac{\sum_{i=0}^{e(r)} r^i}{r^{e(r)}} \prod_{q\in Q} \frac{q}{q-1}, \tag{2.3}$$

where R and Q are finite sets of primes, $R \cap Q = \emptyset$, and $e(r) \in \mathbb{Z}^+$ for each $r \in R$. If $\hat{p} \in \mathbb{Z}^+$ satisfies

$$\frac{a}{b} < \frac{\hat{p}}{\hat{p}-1},$$

then $p \notin Q$ for any prime $p \leq \hat{p}$.

Proof. Suppose $\hat{p} \in \mathbb{Z}^+$ satisfies $\frac{a}{b} < \frac{\hat{p}}{\hat{p}-1}$. Let p be a prime less than or equal to $\hat{p}$. Then

$$\frac{p-1}{p} = 1 - \frac{1}{p} \leq 1 - \frac{1}{\hat{p}} = \frac{\hat{p}-1}{\hat{p}}.$$

Toward contradiction, assume that $p \in Q$. Using Equation (2.3), we see that

$$\prod_{r \in R} \frac{\sum_{i=0}^{e(r)} r^i}{r^{e(r)}} \prod_{q \in Q \setminus \{p\}} \frac{q}{q-1} = \frac{a}{b} \cdot \frac{p-1}{p} \leq \frac{a}{b} \cdot \frac{\hat{p}-1}{\hat{p}} < 1.$$

However, $\prod_{r \in R} \frac{\sum_{i=0}^{e(r)} r^i}{r^{e(r)}} \prod_{q \in Q \setminus \{p\}} \frac{q}{q-1} \geq 1$, so we have a contradiction, implying that $p \notin Q$. □

Theorem 2.8. *Let m be a positive integer. Suppose $I(m) = \frac{a}{b}$ in lowest terms, i.e., $a, b \in \mathbb{Z}^+$ and $\gcd(a,b) = 1$. Let $\hat{p}$ be the largest prime divisor of a. Suppose $I(m) < \frac{\hat{p}}{\hat{p}-1}$. Then m has a TFOFP if and only if m has a friend.*

Proof. If m has a friend n, then the sequence $n, n, n, \ldots$ is a TFOFP of m.

Now suppose m has a TFOFP. By Theorem 2.2, m has a TFOFP $\{s_k\}$ of the form

$$s_k = \prod_{r \in R} r^{e(r)} \prod_{q \in Q} q^k,$$

where R and Q are finite sets of primes, $R \cap Q = \emptyset$, and $e(r) \in \mathbb{Z}^+$ for each $r \in R$. Since $\{s_k\}$ is a TFOFP of m,

$$\frac{a}{b} = I(m) = \lim_{k \to \infty} I(s_k) = \prod_{r \in R} \frac{\sum_{i=0}^{e(r)} r^i}{r^{e(r)}} \prod_{q \in Q} \frac{q}{q-1}. \tag{2.4}$$

by Lemma 2.3.

Toward contradiction, assume $Q \neq \emptyset$. By Lemma 2.5, $\hat{q} = \max(Q)$ must divide a. Since $\hat{q}$ is prime, this implies $\hat{q} \leq \hat{p}$. By Lemma 2.7, $\hat{q} \notin Q$, contradicting $\hat{q} = \max(Q) \in Q$. Thus, $Q = \emptyset$. Therefore, each $s_k = \prod_{r \in R} r^{e(r)}$.

Because $\{s_k\}$ is a TFOFP of m, $m \neq \prod_{r\in R} r^{e(r)}$ and

$$I(m) = \lim_{k\to\infty} I(s_k) = \lim_{k\to\infty} I\left(\prod_{r\in R} r^{e(r)}\right) = I\left(\prod_{r\in R} r^{e(r)}\right).$$

Thus, $\prod_{r\in R} r^{e(r)}$ is a friend of m. □

Using Theorem 2.8, we can show that 15 has a TFOFP if and only if 15 has a friend since $I(15) = \frac{8}{5} < \frac{2}{1}$. Similarly, since $I(33) = \frac{16}{11} < \frac{2}{1}$, by Theorem 2.8, we know that 33 has a TFOFP if and only if 33 has a friend.

Note that 21 does not have a friend by Lemma 1.1 because $\gcd(21, \sigma(21)) = 1$. Since $I(21) = \frac{32}{21} < \frac{2}{1}$, 21 does not have a TFOFP by Theorem 2.8. Similarly, 35 does not have a friend by Lemma 1.1 since $\gcd(35, \sigma(35)) = 1$. Because $I(35) = \frac{48}{35} < \frac{3}{2}$, 35 does not have a TFOFP by Theorem 2.8.

Theorem 2.8 also provides another way to prove Theorem 2.6.

Alternative proof of Theorem 2.6. Let $\hat{p}$ be the largest prime divisor of $p+1$. Since $p+1$ is composite, $\hat{p} < p+1$, so

$$I(p) = \frac{p+1}{p} < \frac{\hat{p}}{\hat{p}-1}.$$

Thus, by Theorem 2.8, p has a TFOFP if and only if p has a friend. By Corollary 1.2, p does not have a friend, so p does not have a TFOFP. □

Theorem 2.9. 32 *is super solitary.*

Proof. Toward contradiction, assume that 32 has a TFOFP. Then by Theorem 2.2, 32 has a TFOFP $\{s_k\}$ of the form

$$s_k = \prod_{r\in R} r^{e(r)} \prod_{q\in Q} q^k,$$

where R and Q are finite sets of primes, $R \cap Q = \emptyset$, and $e(r) \in \mathbb{Z}^+$ for each $r \in R$. Since 32 is 2^5, it does not have a friend by Corollary 1.2. Therefore, $Q \neq \emptyset$, because otherwise if $Q = \emptyset$, then 32 would have a friend. By

Lemma 2.3 and Property 1.4,

$$\frac{63}{32} = I(m) = \lim_{k\to\infty} I(s_k) = I\left(\prod_{r\in R} r^{e(r)}\right)\prod_{q\in Q}\frac{q}{q-1}$$

Since $\frac{63}{32} < \frac{2}{1}$, $2 \notin Q$ by Lemma 2.7. Because $Q \neq \emptyset$, there exists $\hat{q} = \max(Q)$. By Lemma 2.5, $\hat{q}$ divides 63, so $\hat{q} = 3$ or $\hat{q} = 7$.

Suppose $\hat{q} = 3$. Then $Q = \{3\}$ since $3 = \max(Q)$ and $2 \notin Q$. Therefore,

$$\frac{63}{32} = I\left(\prod_{r\in R} r^{e(r)}\right)\frac{3}{2},$$

which is equivalent to

$$I\left(\prod_{r\in R} r^{e(r)}\right) = \frac{21}{16}.$$

Note that $16 < 21 < \sigma(16) = 31$ and $\gcd(16, 21) = 1$, so by Lemma 1.3, $\frac{21}{16}$ is not the abundancy index of any integer, contradicting the statement $I\left(\prod_{r\in R} r^{e(r)}\right) = \frac{21}{16}$. Thus, $\hat{q} \neq 3$.

Suppose $\hat{q} = 7$. We have

$$\frac{63}{32} = I\left(\prod_{r\in R} r^{e(r)}\right)\prod_{q\in Q\setminus\{7\}}\frac{q}{q-1}\cdot\frac{7}{6},$$

which is equivalent to

$$\frac{27}{16} = I\left(\prod_{r\in R} r^{e(r)}\right)\prod_{q\in Q\setminus\{7\}}\frac{q}{q-1}.$$

Notice that $16 < 27 < \sigma(16) = 31$ and $\gcd(16, 27) = 1$, so by Lemma 1.3, $\frac{27}{16}$ is not the abundancy index of any integer. This implies that $Q \setminus \{7\} \neq \emptyset$; otherwise, if $Q \setminus \{7\} = \emptyset$, then $I\left(\prod_{r\in R} r^{e(r)}\right) = \frac{27}{16}$, which produces a contradiction.

By Lemma 2.5, $\hat{q}_1 = \max(Q \backslash \{7\})$ divides 27. Thus, $\hat{q}_1 = 3$ and $Q = \{3, 7\}$ since $2 \notin Q$. Therefore,

$$\frac{63}{32} = I\left(\prod_{r \in R} r^{e(r)}\right) \frac{3}{2} \cdot \frac{7}{6},$$

which is equivalent to

$$I\left(\prod_{r \in R} r^{e(r)}\right) = \frac{9}{8}.$$

Observe that $8 < 9 < \sigma(8) = 15$ and $\gcd(8, 9) = 1$, so by Lemma 1.3, $\frac{9}{8}$ is not the abundancy index of any integer, contradicting the statement $I\left(\prod_{r \in R} r^{e(r)}\right) = \frac{9}{8}$. Therefore, $\hat{q} \neq 7$, which produces a contradiction that implies 32 does not have a TFOFP. □

We have mainly concentrated on super solitary numbers. We now give some results proving that certain numbers have TFOFPs.

Theorem 2.10. *Let p be prime and e be a positive integer. If $\sigma(p^e)$ is prime, then p^e has a TFOFP; namely, the sequence $\{s_k\}$, where*

$$s_k = p^{e-1} \cdot (\sigma(p^e))^k,$$

is a TFOFP of p^e.

Proof. Since p and $\sigma(p^e)$ are prime, by Lemma 2.3 and Property 1.4,

$$\begin{aligned}
\lim_{k \to \infty} I(s_k) &= \frac{\sum_{i=0}^{e-1} p^i}{p^{e-1}} \cdot \frac{\sigma(p^e)}{\sigma(p^e) - 1} = \frac{\sum_{i=0}^{e-1} p^i}{p^{e-1}} \cdot \frac{\sigma(p^e)}{\sum_{i=0}^{e} p^i - 1} \\
&= \frac{\sum_{i=0}^{e-1} p^i}{p^{e-1}} \cdot \frac{\sigma(p^e)}{\sum_{i=1}^{e} p^i} = \frac{\sum_{i=0}^{e-1} p^i}{p^{e-1}} \cdot \frac{\sigma(p^e)}{p \sum_{i=0}^{e-1} p^i} \\
&= \frac{\sigma(p^e)}{p^e} = I\left(p^e\right).
\end{aligned}$$

Since $p^{e-1} \neq p^e$, by Lemma 2.4, $\{s_k\}$ is a TFOFP of p^e. □

Corollary 2.11. *If $2^r - 1$ is a Mersenne prime, then 2^{r-1} has a TFOFP; namely, the sequence $\{s_k\}$, where*

$$s_k = 2^{r-2} \cdot (\sigma(2^{r-1}))^k = 2^{r-2} \cdot (2^r - 1)^k,$$

is a TFOFP of 2^{r-1}.

Proof. Apply Theorem 2.10 with $p = 2$ and $e = r - 1$, noting that $\sigma(2^{r-1}) = 2^r - 1$. □

The converses of Theorem 2.10 and Corollary 2.11 do not hold. For example, $\sigma(2^3) = 15 = 3 \cdot 5$, and 2^3 has the TFOFP $\{2 \cdot 5^k\}_{k=1}^{\infty}$. Similarly, $\sigma(3^8) = 9841 = 13 \cdot 757$, and 3^8 has the TFOFP $\{243 \cdot 757^k\}_{k=1}^{\infty}$.

Having discussed potential TFOFPs of prime powers, we now wish to explore possible TFOFPs of numbers which are not prime powers. The following theorem allows us to do so.

Theorem 2.12. *Suppose the positive integer m_1 has the TFOFP $x = \{x_k\}$ such that*

$$x_k = \prod_{r \in R} r^{e(r)} \prod_{q \in Q} q^k,$$

where R and Q are finite sets of primes, $R \cap Q = \emptyset$, and $e(r) \in \mathbb{Z}^+$ for each $r \in R$. Let m_2 be a positive integer such that

$$\gcd\left(m_1 \prod_{r \in R} r \prod_{q \in Q} q, m_2\right) = 1. \tag{2.5}$$

Then $m = m_1 m_2$ has the TFOFP $s = \{s_k\}$, where

$$s_k = x_k m_2 = m_2 \prod_{r \in R} r^{e(r)} \prod_{q \in Q} q^k.$$

Proof. Let x and s be the sequences stated in the theorem. Note that Equation (2.5) implies that $\gcd(m_1, m_2) = 1$ and $\gcd(x_k, m_2) = 1$ for each k. By weak multiplicativity of I (Property 1.3),

$$\begin{aligned} I(m) = I(m_1 m_2) = I(m_1)I(m_2) &= \lim_{k\to\infty} I(x_k)I(m_2) = \lim_{k\to\infty} I(x_k m_2) \\ &= \lim_{k\to\infty} I(s_k). \end{aligned}$$

Because $x_k \neq m_1$ for all k, $s_k = x_k m_2 \neq m_1 m_2 = m$ for all k. Only finitely many primes divide m_2. Since x is a TFOFP of m_1, $|P_x|$ is finite. Note that $P_s = P_x \bigcup \{\text{primes that divide } m_2\}$, so $|P_s|$ is finite and s is a TFOFP of m. □

Example 2.5. In Example 2.3, we showed that 2 has the TFOFP $\{3^k\}_{k=1}^{\infty}$. Let n be any positive integer such that $\gcd(2 \cdot 3, n) = 1$. Then by Theorem 2.12, the integer $2n$ has the TFOFP $\{3^k n\}_{k=1}^{\infty}$. See Example 2.4 for when $n = 5$.

Notice that the natural density of the set of positive integers coprime to $2 \cdot 3 = 6$ is

$$1 - \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{6}\right) = \frac{1}{3}.$$

This means the natural density of $\{2n \mid \gcd(n, 6) = 1\}$ is

$$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

Because each $m \in \{2n \mid \gcd(n, 6) = 1\}$ has a TFOFP, we conclude that the natural density of numbers with TFOFPs is at least $1/6$.

As shown in Example 2.5, the natural density of numbers with TFOFPs is positive. This result can also be concluded from the facts that friendly numbers have TFOFPs and the set of friendly numbers has a positive natural density. These observations lead us to ask the following question: Is natural density of numbers with TFOFPs equal to 1? If Anderson's and Hickerson's conjecture [1] is true, then natural density of numbers with TFOFPs is equal to 1. On the other hand, if natural density of numbers with TFOFPs is not equal to 1, then Anderson's and Hickerson's conjecture is false.

Table 1 on page 225 states whether an integer m, $1 \leq m \leq 24$, has a known TFOFP, has no TFOFP, or possibly an unknown TFOFP. If m has a TFOFP, then s_k is given so that $\{s_k\}$ is a TFOFP of m.

Having proven that p^j (where p is an odd prime) does not have a TFOFP when $j = 1$, we naturally move on to the following question: When does p^2 have a TFOFP? With Theorem 2.10, we have already shown that if $\sigma(p^2)$ is prime then p^2 has a TFOFP. However, we do not know if p^2 has a TFOFP if $\sigma(p^2)$ is composite. Toward an answer, we provide the following lemmata and theorem.

Lemma 2.13. *Let y be a positive integer such that $y+1$ is composite. Let q be a prime that divides $y+1$. Then*

$$\frac{y+1}{y} \cdot \frac{q-1}{q} \leq \frac{y-1}{y} < 1.$$

Proof. Observe that $q \leq \frac{y+1}{2}$, because $q \mid (y+1)$, q is prime, and $y+1$ is composite. This implies that $\frac{1}{q} \geq \frac{2}{y+1}$ and

$$\frac{q-1}{q} = 1 - \frac{1}{q} \leq 1 - \frac{2}{y+1} = \frac{y-1}{y+1}.$$

Therefore,

$$\frac{y+1}{y} \cdot \frac{q-1}{q} \leq \frac{y+1}{y} \cdot \frac{y-1}{y+1} = \frac{y-1}{y} < 1.$$

□

Lemma 2.14. *Let p be a prime and $j \geq 2$ be an integer. Suppose $\sigma(p^j)$ is composite and p^j has a TFOFP $\{s_k\}$ of the form*

$$s_k = \prod_{r \in R} r^{e(r)} \prod_{q \in Q} q^k,$$

where R and Q are finite sets of primes, $R \cap Q = \emptyset$, and $e(r) \in \mathbb{Z}^+$ for each $r \in R$. If $p \in R$, then $e(p) \leq j-2$.

Proof. Suppose $p \in R$. Since $\{s_k\}$ is a TFOFP of p^j, by Lemma 2.3,

$$\frac{\sigma(p^j)}{p^j} = I(p^j) = \lim_{k \to \infty} I(s_k) = \prod_{r \in R} \frac{\sigma\left(r^{e(r)}\right)}{r^{e(r)}} \prod_{q \in Q} \frac{q}{q-1}. \tag{2.6}$$

By Corollary 1.2, p^j is solitary, so $Q \neq \emptyset$.

Suppose $e(p) \geq j$. Then, since $Q \neq \emptyset$,

$$I(p^j) = \frac{\sigma(p^j)}{p^j} < \frac{\sigma(p^j)}{p^j} \prod_{q \in Q} \frac{q}{q-1} \leq \prod_{r \in R} \frac{\sigma\left(r^{e(r)}\right)}{r^{e(r)}} \prod_{q \in Q} \frac{q}{q-1} = I(p^j),$$

which is a contradiction, so $e(p) \leq j - 1$.

Now suppose $e(p) = j - 1$. Then by Equation (2.6),

$$\frac{\sigma(p^j)}{p^j} = \frac{\sigma(p^{j-1})}{p^{j-1}} \prod_{r \in R \setminus \{p\}} \frac{\sigma\left(r^{e(r)}\right)}{r^{e(r)}} \prod_{q \in Q} \frac{q}{q-1},$$

which implies

$$\frac{p\sigma(p^{j-1}) + 1}{p\sigma(p^{j-1})} = \frac{\sigma(p^j)}{p\sigma(p^{j-1})} = \prod_{r \in R \setminus \{p\}} \frac{\sigma\left(r^{e(r)}\right)}{r^{e(r)}} \prod_{q \in Q} \frac{q}{q-1}. \tag{2.7}$$

Let $\hat{q} = \max(Q)$. By Lemma 2.5, $\hat{q}$ divides $\sigma(p^j) = p\sigma(p^{j-1}) + 1$. From Equation (2.7), we see that

$$\frac{p\sigma(p^{j-1}) + 1}{p\sigma(p^{j-1})} = \frac{\hat{q}}{\hat{q}-1} \prod_{r \in R \setminus \{p\}} \frac{\sigma\left(r^{e(r)}\right)}{r^{e(r)}} \prod_{q \in Q \setminus \{\hat{q}\}} \frac{q}{q-1},$$

which implies

$$\frac{p\sigma(p^{j-1}) + 1}{p\sigma(p^{j-1})} \cdot \frac{\hat{q}-1}{\hat{q}} = \prod_{r \in R \setminus \{p\}} \frac{\sigma\left(r^{e(r)}\right)}{r^{e(r)}} \prod_{q \in Q \setminus \{\hat{q}\}} \frac{q}{q-1}.$$

By Lemma 2.13, $\frac{p\sigma(p^{j-1})+1}{p\sigma(p^{j-1})} \cdot \frac{\hat{q}-1}{\hat{q}} < 1$. However, $\prod_{r \in R \setminus \{p\}} \frac{\sigma\left(r^{e(r)}\right)}{r^{e(r)}} \prod_{q \in Q \setminus \{\hat{q}\}} \frac{q}{q-1} \geq 1$, so we have a contradiction. Therefore, $e(p) \neq j - 1$. □

Theorem 2.15. *Let p be an odd prime. Suppose $\sigma(p^2) = p^2 + p + 1$ is composite and p^2 has a TFOFP $\{s_k\}$ such that*

$$s_k = \prod_{r \in R} r^{e(r)} \prod_{q \in Q} q^k,$$

where R and Q are finite sets of primes, $R \cap Q = \emptyset$, and $e(r) \in \mathbb{Z}^+$ for each $r \in R$. Then there exist $q_1, q_2 \in Q$ such that $p < q_1 < q_2 \leq \frac{p^2+p+1}{3}$ and $q_1 \equiv q_2 \equiv 1 \pmod{p}$.

Proof. Since $\{s_k\}$ is a TFOFP of p^2, by Lemma 2.3,

$$\frac{p^2+p+1}{p^2} = \frac{\sigma(p^2)}{p^2} = I(p^2) = \lim_{k\to\infty} I(s_k) = \prod_{r\in R} \frac{\sigma\left(r^{e(r)}\right)}{r^{e(r)}} \prod_{q\in Q} \frac{q}{q-1},$$

which implies

$$(p^2+p+1)\prod_{r\in R} r^{e(r)} \prod_{q\in Q}(q-1) = p^2 \prod_{r\in R} \sigma\left(r^{e(r)}\right) \prod_{q\in Q} q.$$

By Lemma 2.14, $p \notin R$. (If $p \in R$, then $e(p) = 0$ by Lemma 2.14.) Therefore, $\gcd\left(p^2, \prod_{r\in R} r^{e(r)}\right) = 1$. Since $\gcd(p^2, p^2+p+1) = 1$, we see that p^2 divides $\prod_{q\in Q}(q-1)$.

Note $\frac{p^2+p+1}{p^2} < \frac{p}{p-1}$ by Property 1.6. Thus, by Lemma 2.7, $q > p$ for any $q \in Q$.

By Lemma 2.5, $\hat{q} = \max(Q)$ divides p^2+p+1, which is odd. Because p^2+p+1 is odd and composite and $\hat{q}$ is prime, $\hat{q} \leq \frac{p^2+p+1}{3}$. Therefore, $q \leq \hat{q} \leq \frac{p^2+p+1}{3}$ for any $q \in Q$.

Since $p \geq 3$ is an odd prime,

$$p^2+p+1 \leq p^2 + \frac{p^2}{3} + \frac{p^2}{9} = \frac{13}{9}p^2.$$

Therefore, for any $q \in Q$,

$$q \leq \frac{p^2+p+1}{3} \leq \frac{13}{27}p^2 < p^2.$$

Since $q-1 \leq q < p^2$ for any $q \in Q$, $p^2 \nmid (q-1)$ for any $q \in Q$. Therefore, since $p^2 \mid \prod_{q\in Q}(q-1)$, there exist $q_1, q_2 \in Q$ such that $p \mid (q_1-1)$, $p \mid (q_2-1)$, and $q_1 < q_2$. $\square$

m	Known TFOFP(s)	No TFOFP	TFOFP Unknown
1		✓	
2	$s_k = 3^k$		
3		✓	
4	$s_k = 2 \cdot 7^k$ or $s_k = 3^k \cdot 7^k$		
5		✓	
6	$s_k = 2^k$		
7		✓	
8	$s_k = 2 \cdot 5^k$ or $s_k = 3^k \cdot 5^k$		
9	$s_k = 3 \cdot 13^k$		
10	$s_k = 3^k \cdot 5$		
11		✓	
12	$s_k = 2 \cdot 3 \cdot 7^k$ or $s_k = 2^k \cdot 7^k$		
13		✓	
14	$s_k = 3^k \cdot 7$		
15			✓[3]
16	$s_k = 2^3 \cdot 31^k$ or $s_k = 3^k \cdot 5^k \cdot 31^k$		
17		✓	
18	$s_k = 2 \cdot 3 \cdot 13^k$ or $s_k = 2^k \cdot 13^k$		
19		✓	
20	$s_k = 2 \cdot 5 \cdot 7^k$ or $s_k = 3^k \cdot 5 \cdot 7^k$		
21		✓	
22	$s_k = 3^k \cdot 11$		
23		✓	
24	$s_k = 2 \cdot 3 \cdot 5^k$ or $s_k = 2^k \cdot 5^k$		

Table 1: Stating if $1 \leq m \leq 24$ has a known TFOFP, has no TFOFP, or possibly an unknown TFOFP. If m has a TFOFP, then s_k is given so that $\{s_k\}$ is a TFOFP of m.

[3]15 has a TFOFP if and only if 15 has a friend.

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