International Journal of Mathematics and
Computer Science, **11**(2016), no. 1, 1–21

On the solution of bilateral quaternion polynomial equations

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(Received October 19, 2015, Accepted November 3, 2015)

Abstract

We show that bilateral quaternion polynomial equations of the form

$$\sum_{n=0}^{k} (a_n q^n b_n + c_n q^n d_n) = 0$$

where k is an arbitrary nonnegative integer can be reduced to a system of four real polynomial equations in four unknowns and we consider the solution of these systems with the use of Mathematica version 9. The number of solutions is studied by finding a Gröbner basis for the ideal generated by the family of these four polynomials and considering the cardinality of its variety. The numerical solution of the generalized quaternion algebraic Riccati equation

$$qaq - qb - cq - m = 0$$

is considered separately.

Key words and phrases: Quaternions, bilateral quaternion equations, Newton-Raphson method.
AMS (MOS) Subject Classifications: 11R52, 12E15, 12Y05, 15A66.
ISSN 1814-0432, 2016, http://ijmcs.future-in-tech.net

1 Introduction

In analogy with complex numbers, (real) *quaternions* are defined as numbers q of the form $q = x_1\mathbf{1} + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, where $x_1, x_2, x_3, x_4 \in \mathbb{R}$ and $\mathbf{i^2} = \mathbf{j^2} = \mathbf{k^2} = \mathbf{ijk} = -\mathbf{1}$. In the quaternion number system $\mathbb{H}$, the conjugate, the modulus, and the multiplicative inverse of the quaternion q are defined, respectively, by

$$\bar{q} = x_1\mathbf{1} - x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k}\ ,\ |q| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}\ ,\ q^{-1} = \frac{\bar{q}}{|q|^2}\ ,\ |q| \neq 0$$

Quaternion multiplication is in general non-commutative. It is well known that each quaternion q can be represented as the (2×2) complex matrix

$$q = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \tag{1.1}$$

where $i^2 = -1$. In this note using results obtained in [2] and the general outline suggested there for the solution of polynomial quaternion equations, we study the numerical solution of bilateral quaternion equations of the form

$$\sum_{n=0}^{k} (a_n q^n b_n + c_n q^n d_n) = 0 \tag{1.2}$$

where k is an arbitrary nonnegative integer and for each $n = 0, 1, ..., k$, a_n, b_n, c_n, d_n are quaternions, using Mathematica version 9. Notice that our form of equations includes the classical and higher order quaternion Sylvester equation $aq + bq = c$ and $aq^n + q^n b = c$, respectively, studied in [2, 6, 8, 5]. We look separately at the quaternion equation

$$qaq - qb - cq - m = 0 \tag{1.3}$$

which for $c = \bar{b}$ becomes the algebraic quaternion Riccati equation [15, 8]. The solution, of quaternion polynomial equations with real quaternion coefficients has been studied extensively, starting with the classic result of [3] (see also [13]) where it was shown that quaternion polynomials of degree n of the type

$$f(x) = a_0 x a_1 x \cdots x a_n + \phi(x) \tag{1.4}$$

where $a_i \neq 0$ for $i = 0, 1, ..., n$ and $\phi(x)$ is a sum of a finite number of similar monomials $b_0 x b_1 x \cdots x b_k$ where $k < n$, have at least one root. However, as pointed out in [3], the proof does not apply to bilateral quaternion polynomials of degree k with more than one term of degree k, as is the case in the general form of (1.2). The Sylvester equation $aq + bq = c$ for suitable a, b, c would be an example of a quaternion equation with two first order terms that has no solution [5]. Lateral and bilateral quaternion equations have been studied by many authors [6, 8, 10, 15, 4] and several methods have been proposed for their numerical solution. Typically these methods involve finding the eigenvalues of a certain companion matrix, canonically associated with the equation, or solving some other n-th degree polynomial equation. Therefore any attempt to solve a quaternion equation of degree ≥ 3 must rely on numerical methods. In this note we will treat each equation of the form (1.2) separately and study the existence of one, finitely many, or infinitely many solutions by reducing it to a family $\mathcal{F}$ of four real polynomials in four real unknowns. Surprisingly, regarding the expected number of solutions of such systems, very little is known, for example Khovanskii's Theorem [17] stating that for a system of n polynomials in n variables involving m distinct monomials in total, the number of isolated roots in the positive orthant $(\mathbb{R}_+)^n$ is at most $2^{\binom{m}{2}}(n+1)^m$ which is usually a very unrealistic upper bound since the number of roots is actually much smaller. In the case when the coefficients of the system are rational, using the concept of a Gröbner basis [16] to compute the cardinality of the variety $\mathcal{V}(\mathcal{F})$ of $\mathcal{F}$, some conclusions can be made about the number of **complex** roots of the system. In the particular case when there are no complex roots then (1.2) has no solution. Finding the Gröbner basis as well as all symbolic calculations in this note were carried out with the use of Mathematica version 9. It should be pointed out that the Mathematica command *Solve* for solving systems of equations uses the idea of a Gröbner basis.

2 Some facts about Gröbner bases

Following [16], if $\mathbb{K} \subseteq \mathbb{C}$ is a field, we denote by $\mathbb{K}[x_1, ..., x_n]$ the ring of polynomials in the n variables $x_1, ..., x_n$ with coefficients in $\mathbb{K}$. Let $\mathcal{F}$ be a family of polynomials in $x_1, ..., x_n$. We denote by $\langle \mathcal{F} \rangle$ the *ideal generated by* $\mathcal{F}$, i.e.,

$$\langle \mathcal{F} \rangle = \{\sum_{i=1}^{r} p_i f_i \,:\, f_i \in \mathcal{F}, p_i \in \mathbb{K}[x_1, ..., x_n], i = 1, ..., r \in \mathbb{N}\}$$

We equip $\mathbb{K}[x_1, ..., x_n]$ with a *term order* on the monomials $x^a := x_1^{a_1} \cdots x_n^{a_n}$, $a_i \in \mathbb{N}$, for example the *degree lexicographic order* with $x_1 < x_2 < x_3 < ...$ in which $x^a < x^b$ if and only if $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or if $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the first coordinates a_i and b_i in a and b from the left which are different, satisfy $a_i > b_i$, see also [1]. For example, $x_1 x_2 < x_2^2$ because $x_1 x_2 = x_1^1 x_2^1$, $x_2^2 = x_1^0 x_2^2$, $1+1 = 0+2$, and the first coordinates of $a = (1,1)$ and $b = (0,2)$ from the left which are different, satisfy $a_1 = 1 > 0 = b_1$. Given a term order, the unique *initial term* of each polynomial f is denoted by $in_<(f)$ and is the largest monomial x^a which appears in f with a nonzero coefficient. If I is an ideal in $\mathbb{K}[x_1, ..., x_n]$ then its *initial ideal* in $\mathbb{K}[x_1, ..., x_n]$ is denoted by $in_<(I)$ and is defined as the ideal generated by the initial terms of the polynomials in I. A monomial $x^a \notin in_<(I)$ is called *standard.* A finite subset $\mathcal{G}$ of I whose initial terms generate $in_<(I)$ is called a Gröbner basis for I . If $\mathcal{G}$ is a Gröbner basis for $\mathcal{F}$ then $\langle \mathcal{F} \rangle = \langle \mathcal{G} \rangle$. The *variety* of $\mathcal{F}$ is the set $V(\mathcal{F})$ of all common complex roots of the polynomials in $\mathcal{F}$. We have $V(\mathcal{F}) = V(\langle \mathcal{F} \rangle) = V(\langle \mathcal{G} \rangle) = V(\mathcal{G})$. Moreover, by *Hilbert's Nullstellensatz*, $V(\mathcal{F}) = \emptyset$ if and only if $\mathcal{G} = \{1\}$. The cardinality of $V(\mathcal{G})$, when roots are counted with their multiplicities, equals the number of standard monomials in $\mathcal{G}$. The cardinality of $V(\mathcal{G})$ is finite if and only if the number of standard monomials in $\mathcal{G}$ is finite.

3 Real system form of a general bilateral quaternion polynomial equation

Let $n \in 0, 1, 2, ...$ and $q = x_1\mathbf{1} + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} = x_1\mathbf{1} + \vec{q} \in \mathbb{H}$ where $\vec{q} := x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$. As noticed in [2] if q has the matrix form (1.1) then, since the set of matrices of the form (1.1) is closed under matrix multiplication and addition,

$$q^n = \begin{pmatrix} x_1(n) + ix_2(n) & x_3(n) + ix_4(n) \\ -x_3(n) + ix_4(n) & x_1(n) - ix_2(n) \end{pmatrix}$$

where, assuming $\vec{q} \neq \mathbf{0}$,

$$x_1(n) = \text{Re } (x_1 + i|\vec{q}|)^n \ , \ x_2(n) = \frac{x_2}{|\vec{q}|} \text{Im } ((x_1 + i|\vec{q}|)^n)$$

$$x_3(n) = \frac{x_3}{|\vec{q}|} \text{Im } (x_1 + i|\vec{q}|)^n) \ , \ x_4(n) = \frac{x_4}{|\vec{q}|} \text{Im } ((x_1 + i|\vec{q}|)^n)$$

while if $\vec{q} = \mathbf{0}$, i.e. if we are interested in real solutions q, then

$$q^n = x_1^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e.,

$$x_1(n) = x_1^n, \;\; x_2(n) = x_3(n) = x_4(n) = 0$$

Letting

$$a_n = \begin{pmatrix} a_1(n) + ia_2(n) & a_3(n) + ia_4(n) \\ -a_3(n) + ia_4(n) & a_1(n) - ia_2(n) \end{pmatrix}$$

$$b_n = \begin{pmatrix} b_1(n) + ib_2(n) & b_3(n) + ib_4(n) \\ -b_3(n) + ib_4(n) & b_1(n) - ib_2(n) \end{pmatrix}$$

$$c_n = \begin{pmatrix} c_1(n) + ic_2(n) & c_3(n) + ic_4(n) \\ -c_3(n) + ic_4(n) & c_1(n) - ic_2(n) \end{pmatrix}$$

$$d_n = \begin{pmatrix} d_1(n) + id_2(n) & d_3(n) + id_4(n) \\ -d_3(n) + id_4(n) & d_1(n) - id_2(n) \end{pmatrix}$$

after carrying out all matrix operations and equating the real and imaginary parts of the entries of the (first row is enough) resulting (2×2) matrix to zero, every bilateral quaternion polynomial equation of the form (1.2) can be reduced to a system of only four real polynomial equations

$$f_i(x_1, x_2, x_3, x_4) = 0 \;, \;\; i = 1, 2, 3, 4$$

where

$$f_1(x_1, x_2, x_3, x_4) = \sum_{n=0}^{k} (a_1(n)b_1(n)x_1(n) - a_2(n)b_2(n)x_1(n) - a_3(n)b_3(n)x_1(n) - a_4(n)b_4(n)x_1(n)$$

$$-a_2(n)b_1(n)x_2(n) - a_1(n)b_2(n)x_2(n) - a_4(n)b_3(n)x_2(n) + a_3(n)b_4(n)x_2(n) - a_3(n)b_1(n)x_3(n)$$

$$+a_4(n)b_2(n)x_3(n) - a_1(n)b_3(n)x_3(n) - a_2(n)b_4(n)x_3(n) - a_4(n)b_1(n)x_4(n) - a_3(n)b_2(n)x_4(n)$$

$$+a_2(n)b_3(n)x_4(n) - a_1(n)b_4(n)x_4(n) + c_1(n)x_1(n)d_1(n) - c_2(n)x_2(n)d_1(n) - c_3(n)x_3(n)d_1(n)$$

$$-c_4(n)x_4(n)d_1(n) - c_2(n)x_1(n)d_2(n) - c_1(n)x_2(n)d_2(n) + c_4(n)x_3(n)d_2(n) - c_3(n)x_4(n)d_2(n)$$

$$-c_3(n)x_1(n)d_3(n) - c_4(n)x_2(n)d_3(n) - c_1(n)x_3(n)d_3(n) + c_2(n)x_4(n)d_3(n) - c_4(n)x_1(n)d_4(n)$$

$$+c_3(n)x_2(n)d_4(n) - c_2(n)x_3(n)d_4(n) - c_1(n)x_4(n)d_4(n))$$

$$f_2(x_1, x_2, x_3, x_4)$$

$$= \sum_{n=0}^{k}(a_2(n)b_1(n)x_1(n)+a_1(n)b_2(n)x_1(n)-a_4(n)b_3(n)x_1(n)+a_3(n)b_4(n)x_1(n)$$
$$+a_1(n)b_1(n)x_2(n)-a_2(n)b_2(n)x_2(n)+a_3(n)b_3(n)x_2(n)+a_4(n)b_4(n)x_2(n)-a_4(n)b_1(n)x_3(n)$$
$$-a_3(n)b_2(n)x_3(n)-a_2(n)b_3(n)x_3(n)+a_1(n)b_4(n)x_3(n)+a_3(n)b_1(n)x_4(n)-a_4(n)b_2(n)x_4(n)$$
$$-a_1(n)b_3(n)x_4(n)-a_2(n)b_4(n)x_4(n)+c_2(n)x_1(n)d_1(n)+c_1(n)x_2(n)d_1(n)-c_4(n)x_3(n)d_1(n)$$
$$+c_3(n)x_4(n)d_1(n)+c_1(n)x_1(n)d_2(n)-c_2(n)x_2(n)d_2(n)-c_3(n)x_3(n)d_2(n)-c_4(n)x_4(n)d_2(n)$$
$$-c_4(n)x_1(n)d_3(n)+c_3(n)x_2(n)d_3(n)-c_2(n)x_3(n)d_3(n)-c_1(n)x_4(n)d_3(n)+c_3(n)x_1(n)d_4(n)$$
$$+c_4(n)x_2(n)d_4(n)+c_1(n)x_3(n)d_4(n)-c_2(n)x_4(n)d_4(n))$$

$$f_3(x_1,x_2,x_3,x_4)$$
$$= \sum_{n=0}^{k}(a_3(n)b_1(n)x_1(n)+a_4(n)b_2(n)x_1(n)+a_1(n)b_3(n)x_1(n)-a_2(n)b_4(n)x_1(n)$$
$$+a_4(n)b_1(n)x_2(n)-a_3(n)b_2(n)x_2(n)-a_2(n)b_3(n)x_2(n)-a_1(n)b_4(n)x_2(n)+a_1(n)b_1(n)x_3(n)$$
$$+a_2(n)b_2(n)x_3(n)-a_3(n)b_3(n)x_3(n)+a_4(n)b_4(n)x_3(n)-a_2(n)b_1(n)x_4(n)+a_1(n)b_2(n)x_4(n)$$
$$-a_4(n)b_3(n)x_4(n)-a_3(n)b_4(n)x_4(n)+c_3(n)x_1(n)d_1(n)+c_4(n)x_2(n)d_1(n)+c_1(n)x_3(n)d_1(n)$$
$$-c_2(n)x_4(n)d_1(n)+c_4(n)x_1(n)d_2(n)-c_3(n)x_2(n)d_2(n)+c_2(n)x_3(n)d_2(n)+c_1(n)x_4(n)d_2(n)$$
$$+c_1(n)x_1(n)d_3(n)-c_2(n)x_2(n)d_3(n)-c_3(n)x_3(n)d_3(n)-c_4(n)x_4(n)d_3(n)-c_2(n)x_1(n)d_4(n)$$
$$-c_1(n)x_2(n)d_4(n)+c_4(n)x_3(n)d_4(n)-c_3(n)x_4(n)d_4(n))$$

$$f_4(x_1,x_2,x_3,x_4)$$
$$= \sum_{n=0}^{k}(a_4(n)b_1(n)x_1(n)-a_3(n)b_2(n)x_1(n)+a_2(n)b_3(n)x_1(n)+a_1(n)b_4(n)x_1(n)$$
$$-a_3(n)b_1(n)x_2(n)-a_4(n)b_2(n)x_2(n)+a_1(n)b_3(n)x_2(n)-a_2(n)b_4(n)x_2(n)+a_2(n)b_1(n)x_3(n)$$
$$-a_1(n)b_2(n)x_3(n)-a_4(n)b_3(n)x_3(n)-a_3(n)b_4(n)x_3(n)+a_1(n)b_1(n)x_4(n)+a_2(n)b_2(n)x_4(n)$$
$$+a_3(n)b_3(n)x_4(n)-a_4(n)b_4(n)x_4(n)+c_4(n)x_1(n)d_1(n)-c_3(n)x_2(n)d_1(n)+c_2(n)x_3(n)d_1(n)$$
$$+c_1(n)x_4(n)d_1(n)-c_3(n)x_1(n)d_2(n)-c_4(n)x_2(n)d_2(n)-c_1(n)x_3(n)d_2(n)+c_2(n)x_4(n)d_2(n)$$
$$+c_2(n)x_1(n)d_3(n)+c_1(n)x_2(n)d_3(n)-c_4(n)x_3(n)d_3(n)+c_3(n)x_4(n)d_3(n)+c_1(n)x_1(n)d_4(n)$$
$$-c_2(n)x_2(n)d_4(n)-c_3(n)x_3(n)d_4(n)-c_4(n)x_4(n)d_4(n))$$

4 Complex system form of a general bilateral quaternion polynomial equation

As in Section 3, for $n \in 0, 1, 2, ...$ and $q = x_1\mathbf{1} + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} = x_1\mathbf{1} + \vec{q} \in \mathbb{H}$ having the matrix form (1.1) we obtain

$$q^n = \begin{pmatrix} z(n) & w(n) \\ -\bar{w}(n) & \bar{z}(n) \end{pmatrix}$$

where, assuming $\lambda := i\sqrt{|w|^2 + (\mathrm{Im\,z})^2} \neq 0$, i.e. if $\vec{q} \neq \mathbf{0}$,

$$z(n) = \frac{1}{2\lambda}\left((i\,\mathrm{Im\,z} + \lambda)(\mathrm{Re\,z} + \lambda)^{\mathrm{n}} - (\mathrm{i\,Im\,z} - \lambda)(\mathrm{Re\,z} - \lambda)^{\mathrm{n}}\right)$$

$$w(n) = \frac{w}{2\lambda}\left((\mathrm{Re\,z} + \lambda)^{\mathrm{n}} - (\mathrm{Re\,z} - \lambda)^{\mathrm{n}}\right)$$

while if $\lambda = 0$, i.e. if $\vec{q} = \mathbf{0}$, then

$$q^n = \left(\frac{\mathrm{Re\,z}}{2}\right)^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Letting

$$a_n = \begin{pmatrix} a_1(n) & a_2(n) \\ -\overline{a_2(n)} & \overline{a_1(n)} \end{pmatrix} \quad b_n = \begin{pmatrix} b_1(n) & b_2(n) \\ -\overline{b_2(n)} & \overline{b_1(n)} \end{pmatrix}$$

$$c_n = \begin{pmatrix} c_1(n) & c_2(n) \\ -\overline{c_2(n)} & \overline{c_1(n)} \end{pmatrix} \quad d_n = \begin{pmatrix} d_1(n) & d_2(n) \\ -\overline{d_2(n)} & \overline{d_1(n)} \end{pmatrix}$$

every bilateral quaternion polynomial equation of the form (1.2) can be reduced to an undetermined system of only two complex (non polynomial) equations

$$f_i(z, w) = 0\ ,\ i = 1, 2$$

where

$$f_1(z, w) = \sum_{n=0}^{k}(-a_2(n)b_1(n)\overline{w(n)} - a_2(n)\overline{b_2(n)z(n)} - c_2(n)\overline{d_2(n)z(n)}$$

$$-c_2(n)\overline{w(n)}d_1(n) - a_1(n)\overline{b_2(n)}w(n) - c_1(n)\overline{d_2(n)}w(n) + a_1(n)b_1(n)z(n) + c_1(n)d_1(n)z(n))$$

$$f_2(z,w)=\sum_{n=0}^{k}(-a_2(n)b_2(n)\overline{w(n)}+a_2(n)\overline{b_1(n)z(n)}+c_2(n)\overline{d_1(n)z(n)}$$

$$-c_2(n)\overline{w(n)}d_2(n)+a_1(n)\overline{b_1(n)}w(n)+c_1(n)\overline{d_1(n)}w(n)+a_1(n)b_2(n)z(n)+c_1(n)d_2(n)z(n))$$

To extend the above system to a system of four polynomial equations with complex coefficients, we let

$$q=\begin{pmatrix} z & w \\ -\alpha & \beta \end{pmatrix} \ \alpha,\beta,z,w\in\mathbb{C}$$

in which case

$$q^n=\begin{pmatrix} z(n) & w(n) \\ \alpha(n) & \beta(n) \end{pmatrix}$$

where for

$$\mu=\sqrt{-4\alpha w+(\beta-z)^2}\neq 0$$

$$z(n)=\frac{1}{2^{1+n}\mu}((\beta+\mu-z)(\beta-\mu+z)^n+(-\beta+\mu+z)(\beta+\mu+z)^n)$$

$$w(n)=\frac{w}{\mu 2^n}((\beta+\mu+z)^n-(\beta-\mu+z)^n$$

$$\alpha(n)=\frac{\alpha}{\mu 2^n}((\beta-\mu+z)^n-(\beta+\mu+z)^n$$

$$\beta(n)=\frac{1}{2^{1+n}\mu}((-\beta+\mu+z)(\beta-\mu+z)^n+(\beta+\mu-z)(\beta+\mu+z)^n)$$

and (1.2) is reduced to the system

$$f_i(z,w,\alpha,\beta)=0\ ,\ i=1,2,3,4$$

where

$$f_1(z,w,\alpha,\beta)=\sum_{n=0}^{k}(-a_1(n)\overline{b_2(n)}w(n)-c_1(n)\overline{d_2(n)}w(n)+a_1(n)b_1(n)z(n)$$

$$+c_1(n)d_1(n)z(n)+a_2(n)b_1(n)\alpha(n)+c_2(n)d_1(n)\alpha(n)-a_2(n)\overline{b_2(n)}\beta(n)-c_2(n)\overline{d_2(n)}\beta(n))$$

$$f_2(z,w,\alpha,\beta)=\sum_{n=0}^{k}(a_1(n)\overline{b_1(n)}w(n)+c_1(n)\overline{d_1(n)}w(n)+a_1(n)b_2(n)z(n)$$

$$+c_1(n)d_2(n)z(n)+a_2(n)b_2(n)\alpha(n)+c_2(n)d_2(n)\alpha(n)+a_2(n)\overline{b_1(n)}\beta(n)+c_2(n)\overline{d_1(n)}\beta(n))$$

$$f_3(z,w,\alpha,\beta)=\sum_{n=0}^{k}(\overline{a_2(n)b_2(n)}w(n)+\overline{c_2(n)d_2(n)}w(n)-b_1(n)\overline{a_2(n)}z(n)$$

$$-\overline{c_2(n)}d_1(n)z(n)+b_1(n)\overline{a_1(n)}\alpha(n)+\overline{c_1(n)}d_1(n)\alpha(n)-\overline{a_1(n)b_2(n)}\beta(n)-\overline{c_1(n)d_2(n)}\beta(n))$$

$$f_4(z,w,\alpha,\beta)=\sum_{n=0}^{k}(-\overline{a_2(n)b_1(n)}w(n)-\overline{c_2(n)d_1(n)}w(n)-b_2(n)\overline{a_2(n)}z(n)$$

$$-\overline{c_2(n)}d_2(n)z(n)+b_2(n)\overline{a_1(n)}\alpha(n)+\overline{c_1(n)}d_2(n)\alpha(n)+\overline{a_1(n)b_1(n)}\beta(n)+\overline{c_1(n)d_1(n)}\beta(n))$$

and after solving it as system of complex polynomial equations, we see if there are solutions (z,w,α,β) with $\alpha=\bar{w}$ and $\beta=\bar{z}$.

5 The algebraic Riccati equation

Let $q=x_1\mathbf{1}+x_2\mathbf{i}+x_3\mathbf{j}+x_4\mathbf{k}=x_1\mathbf{1}+\vec{q}\in\mathbb{H}$, i.e.,

$$q=\begin{pmatrix} x_1+ix_2 & x_3+ix_4) \\ -x_3+ix_4 & x_1-ix_2 \end{pmatrix}$$

Letting

$$a=\begin{pmatrix} a_1+ia_2 & a_3+ia_4 \\ -a_3+ia_4 & a_1-ia_2 \end{pmatrix}\ ,\ b=\begin{pmatrix} b_1+ib_2 & b_3+ib_4 \\ -b_3+ib_4 & b_1-ib_2 \end{pmatrix}$$

$$c=\begin{pmatrix} c_1+ic_2 & c_3+ic_4 \\ -c_3+ic_4 & c_1-ic_2 \end{pmatrix}\ ,\ m=\begin{pmatrix} m_1+im_2 & m_3+im_4 \\ -m_3+im_4 & m_1-im_2 \end{pmatrix}$$

the quaternion equation (1.3) is reduced to the system of real polynomial equations

$$f_i(x_1,x_2,x_3,x_4)=0\ ,\ i=1,2,3,4$$

where

$$f_1(x_1,x_2,x_3,x_4)=-m_1-b_1x_1-c_1x_1+a_1x_1^2+b_2x_2+c_2x_2-2a_2x_1x_2$$

$$-a_1x_2^2+b_3x_3+c_3x_3-2a_3x_1x_3-a_1x_3^2+b_4x_4+c_4x_4-2a_4x_1x_4-a_1x_4^2$$

$$f_2(x_1,x_2,x_3,x_4) = -m_2 - b_2x_1 - c_2x_1 + a_2x_1^2 - b_1x_2 - c_1x_2 + 2a_1x_1x_2 - a_2x_2^2$$
$$-b_4x_3 + c_4x_3 - 2a_3x_2x_3 + a_2x_3^2 + b_3x_4 - c_3x_4 - 2a_4x_2x_4 + a_2x_4^2$$

$$f_3(x_1,x_2,x_3,x_4) = -m_3 - b_3x_1 - c_3x_1 + a_3x_1^2 + b_4x_2 - c_4x_2 + a_3x_2^2$$
$$-b_1x_3 - c_1x_3 + 2a_1x_1x_3 - 2a_2x_2x_3 - a_3x_3^2 - b_2x_4 + c_2x_4 - 2a_4x_3x_4 + a_3x_4^2$$

$$f_4(x_1,x_2,x_3,x_4) = -m_4 - b_4x_1 - c_4x_1 + a_4x_1^2 - b_3x_2 + c_3x_2 + a_4x_2^2$$
$$+b_2x_3 - c_2x_3 + a_4x_3^2 - b_1x_4 - c_1x_4 + 2a_1x_1x_4 - 2a_2x_2x_4 - 2a_3x_3x_4 - a_4x_4^2$$

In complex form, if

$$q = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \ , \ m = \begin{pmatrix} M_1 & M_2 \\ -\bar{M}_2 & \bar{M}_1 \end{pmatrix}$$
$$a = \begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix} \ , \ b = \begin{pmatrix} B_1 & B_2 \\ -\bar{B}_2 & \bar{B}_1 \end{pmatrix} \ c = \begin{pmatrix} C_1 & C_2 \\ -\bar{C}_2 & \bar{C}_1 \end{pmatrix}$$

(1.3) is reduced to the system of complex equations

$$f_i(z,w) = 0 \ , \ i = 1,2,3,4$$

where

$$f_1(z,w) = -M_1 - B_1z - C_1z + A_1z^2 - wz\bar{A}_2 + w\bar{B}_2 + C_2\bar{w} - A_2z\bar{w} - w\bar{A}_1\bar{w}$$
$$f_2(z,w) = -M_2 - C_1w - B_2z + A_1wz - w^2\bar{A}_2 - w\bar{B}_1 - C_2\bar{z} + A_2z\bar{z} + w\bar{A}_1\bar{z}$$
$$f_3(z,w) = z\bar{C}_2 + \bar{M}_2 + B_1\bar{w} - A_1z\bar{w} + \bar{C}_1\bar{w} + A_2\bar{w}^2 - z\bar{z}\bar{A}_2 + \bar{B}_2\bar{z} - \bar{A}_1\bar{w}\bar{z}$$
$$f_4(z,w) = w\bar{C}_2 - \bar{M}_1 + B_2\bar{w} - A_1w\bar{w} - \bar{A}_2w\bar{z} - \bar{B}_1\bar{z} - \bar{C}_1\bar{z} - A_2\bar{w}\bar{z} + \bar{A}_1\bar{z}^2$$

which letting $\alpha = \bar{w}$ and $\beta = \bar{z}$ extends to the system of complex polynomial equations

$$f_i(z,w,\alpha,\beta) = 0 \ , \ i = 1,2,3,4$$

where

$$f_1(z,w,\alpha,\beta) = -M_1 - B_1z - C_1z + A_1z^2 - wz\bar{A}_2 + w\bar{B}_2 + C_2\alpha - A_2z\alpha - w\bar{A}_1\alpha$$
$$f_2(z,w,\alpha,\beta) = -M_2 - C_1w - B_2z + A_1wz - w^2\bar{A}_2 - w\bar{B}_1 - C_2\beta + A_2z\beta + w\bar{A}_1\beta$$
$$f_3(z,w,\alpha,\beta) = z\bar{C}_2 + \bar{M}_2 + B_1\alpha - A_1z\alpha + \bar{C}_1\alpha + A_2\alpha^2 - z\beta\bar{A}_2 + \bar{B}_2\beta - \bar{A}_1\alpha\beta$$
$$f_4(z,w) = w\bar{C}_2 - \bar{M}_1 + B_2\alpha - A_1w\alpha - \bar{A}_2w\beta - \bar{B}_1\beta - \bar{C}_1\beta - A_2\alpha\beta + \bar{A}_1\beta^2$$

and we are interested in solutions (z,w,α,β) with $\bar{w} = \alpha$ and $\beta = \bar{z}$.

6 Examples

Example 1.

In the case of the quaternion equation

$$aq^3 + q^3 b = c\ ,\ \ q = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4) \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}$$

$$a = 1 + \mathbf{i} + 3\mathbf{j} - 4\mathbf{k} = \begin{pmatrix} 1+i & 3-4i \\ -3-4i & 1-i \end{pmatrix}$$

$$b = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} = \begin{pmatrix} 2i & -2+2i \\ 2+2i & -2i \end{pmatrix}$$

$$c = -1 + 1\mathbf{i} + 6\mathbf{j} + 0\mathbf{k} = \begin{pmatrix} -1+i & 6 \\ -6 & -1-i \end{pmatrix}$$

we obtain the system

$$f_1(x_1,x_2,x_3,x_4) = 1 + x_1^3 - 9x_1^2x_2 - 3x_1x_2^2 + 3x_2^3 - 3x_1^2x_3$$
$$+x_2^2x_3 - 3x_1x_3^2 + 3x_2x_3^2 + x_3^3 + 6x_1^2x_4$$
$$-2x_2^2x_4 - 2x_3^2x_4 - 3x_1x_4^2 + 3x_2x_4^2 + x_3x_4^2 - 2x_4^3 = 0$$

$$f_2(x_1,x_2,x_3,x_4) = -1 + 3x_1^3 + 3x_1^2x_2 - 9x_1x_2^2 - x_2^3 + 18x_1^2x_3$$
$$-6x_2^2x_3 - 9x_1x_3^2 - x_2x_3^2 - 6x_3^3 + 15x_1^2x_4$$
$$-5x_2^2x_4 - 5x_3^2x_4 - 9x_1x_4^2 - x_2x_4^2 - 6x_3x_4^2 - 5x_4^3 = 0$$

$$f_3(x_1,x_2,x_3,x_4) = 6 - x_1^3 + 18x_1^2x_2 + 3x_1x_2^2 - 6$$
$$x_2^3 - 3x_1^2x_3 + x_2^2x_3 + 3x_1x_3^2 - 6x_2x_3^2 + x_3^3$$
$$-3x_1^2x_4 + x_2^2x_4 + x_3^2x_4 + 3x_1x_4^2 - 6x_2x_4^2 + x_3x_4^2 + x_4^3 = 0$$

$$f_4(x_1,x_2,x_3,x_4) = 2x_1^3 + 15x_1^2x_2 - 6x_1x_2^2$$
$$-5x_2^3 + 3x_1^2x_3 - x_2^2x_3 - 6x_1x_3^2 - 5x_2x_3^2 - x_3^3$$
$$-3x_1^2x_4 + x_2^2x_4 + x_3^2x_4 - 6x_1x_4^2 - 5x_2x_4^2 - x_3x_4^2 + x_4^3 = 0$$

The Mathematica version 9 command FindRoot, for an initial guess $(x_1, x_2, x_3, x_4) = (1, 0, 0, 0)$, produces the solution

$$(x_1, x_2, x_3, x_4) = (1.23628, -0.214625, 0.298941, -0.385813)$$

$$q = 1.23628 - 0.214625\mathbf{i} + 0.298941\mathbf{j} - 0.385813\mathbf{k}$$

For an initial guess $(x_1, x_2, x_3, x_4) = (-1, 0, 0, 0)$, it produces the solution

$$(x_1, x_2, x_3, x_4) = (-1.07989, -0.323664, 0.450817, -0.581824)$$

$$q = -1.07989 - 0.323664\mathbf{i} + 0.450817\mathbf{j} - 0.581824\mathbf{k}$$

For an initial guess $(x_1, x_2, x_3, x_4) = (0, 0, 1, 0)$, it produces the solution

$$(x_1, x_2, x_3, x_4) = (-0.156393, 0.538288, -0.749759, 0.967637)$$

$$q = -0.156393 + 0.538288\mathbf{i} - 0.749759\mathbf{j} + 0.967637\mathbf{k}$$

while for or the initial choice $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ the method run into a singular Jacobian. A slight variation $(x_1, x_2, x_3, x_4) = (0.01, 0, 0, 0)$ produced the solution

$$(x_1, x_2, x_3, x_4) = (1.23628, -0.214625, 0.298941, -0.385813)$$

$$q = 1.23628 - 0.214625\mathbf{i} + 0.298941\mathbf{j} - 0.385813\mathbf{k}$$

The command

$$GroebnerBasis[\{f_1, f_2, f_3, f_4\}, \{x_1, x_2, x_3, x_4\}]$$

produces the Groebner basis

$$g_1(x_1, x_2, x_3, x_4) = -40812436757196811351 - 87282629347203853672 2x_4^3$$

$$-2595083882442126374208x_4^6 + 3982437379941048630784x_4^9$$

$$g_2(x_1, x_2, x_3, x_4) = 151x_3 + 117x_4$$

$$g_3(x_1, x_2, x_3, x_4) = 151x_2 - 84x_4$$

$$g_4(x_1, x_2, x_3, x_4) = 306079851124102221x_1 + 1494541173065598080x_4$$

$$+831757654628886658 4x_4^4 - 10940762032805078656x_4^7$$

By Gauss's Fundamental Theorem, there are 9 **complex** roots x_4 of the first member of the above Groebner basis. For each one of them we find a unique answer for x_1, x_2, x_3. Thus a total of 9 complex solutions (x_1, x_2, x_3, x_4). Using the NSolve command we find that the solutions x_4 of $g_1(x_1, x_2, x_3, x_4)$ are

$$x_4 = -0.581824,\ x_4 = -0.483819 - 0.837998i,\ x_4 = -0.483819 + 0.837998i$$

$$x_4 = -0.385813,\ x_4 = 0.192907 - 0.334124i,\ x_4 = 0.192907 + 0.334124i$$

$$x_4 = 0.290912 - 0.503874i,\ x_4 = 0.290912 + 0.503874i,\ x_4 = 0.967637$$

There are three real x_4 roots of the first member $g_1(x_1, x_2, x_3, x_4)$ of the Groebner basis. For each one of them we find a unique answer for x_1, x_2, x_3. Thus a total of 3 quaternion roots.

Example 2.

For the quaternion equation

$$aq^3 + q^3 b = c\ ,\ \ q = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}$$

$$a = 1 + \mathbf{i} + 0\mathbf{j} + \mathbf{k} = \begin{pmatrix} 1+i & i \\ i & 1-i \end{pmatrix}$$

$$b = -1 + \mathbf{i} + 0\mathbf{j} + \mathbf{k} = \begin{pmatrix} -1+i & i \\ i & -1-i \end{pmatrix}$$

$$c = 1 + 0\mathbf{i} + 0\mathbf{j} + \mathbf{k} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

we obtain the system

$$f_1(x_1, x_2, x_3, x_4) = -1 - 6x_1^2 x_2 + 2x_2^3 + 2x_2 x_3^2 - 6x_1^2 x_4$$

$$+2x_2^2 x_4 + 2x_3^2 x_4 + 2x_2 x_4^2 + 2x_4^3 = 0$$

$$f_2(x_1, x_2, x_3, x_4) = 2x_1^3 - 6x_1 x_2^2 - 6x_1 x_3^2 - 6x_1 x_4^2 = 0$$

$$f_3(x_1, x_2, x_3, x_4) = 0$$

$$f_4(x_1, x_2, x_3, x_4) = -1 + 2x_1^3 - 6x_1 x_2^2 - 6x_1 x_3^2 - 6x_1 x_4^2 = 0$$

Since

$$\det F'(X) = \det \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \le i,j \le 4} = 0$$

the Newton-Raphson method does not work. However, since

$$GroebnerBasis[\{f_1, f_2, f_3, f_4\}, \{x_1, x_2, x_3, x_4\}] = \{1\}$$

we conclude that there are no complex roots (x_1, x_2, x_3, x_4), thus no real roots (x_1, x_2, x_3, x_4), thus no quaternion roots $q = x_1 + x_2 i + x_3 j + x_4 k$.

Example 3.

For a, b, c, q as in the previous example and

$$m = 1 + 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} = \begin{pmatrix} 1+2i & 3+4i \\ -3+4i & 1-2i \end{pmatrix}$$

the quaternion equation $qaq - qb - cq - m = 0 = 0$ is equivalent to the system

$$f_1(x_1, x_2, x_3, x_4) = -1 + x_1^2 + x_2 - 2x_1x_2 - x_2^2 - x_3^2 + 2x_4 - 2x_1x_4 - x_4^2 = 0$$

$$f_2(x_1, x_2, x_3, x_4) = -2 - x_1 + x_1^2 + 2x_1x_2 - x_2^2 + x_3^2 - 2x_2x_4 + x_4^2 = 0$$

$$f_3(x_1, x_2, x_3, x_4) = -3 + 2x_1x_3 - 2x_2x_3 - x_4 - 2x_3x_4 = 0$$

$$f_4(x_1, x_2, x_3, x_4) = -4 - 2x_1 + x_1^2 + x_2^2 + x_3 + x_3^2 + 2x_1x_4 - 2x_2x_4 - x_4^2 = 0$$

For an initial guess $(x_1, x_2, x_3, x_4) = (1, 0, 0, 0)$, Newton-Raphson produces the solution

$$(x_1, x_2, x_3, x_4) = (1.66603, -0.31918, 1.20394, 0.52236)$$

$$q = 1.66603 - 0.31918\mathbf{i} + 1.20394\mathbf{j} + 0.52236\mathbf{k}$$

while for an initial guess $(x_1, x_2, x_3, x_4) = (-1, 0, 0, 0)$, it produces the solution

$$(x_1, x_2, x_3, x_4) = (-1.08445, 0.51304, -0.979374, -0.13464)$$

$$q = -1.08445 + 0.51304\mathbf{i} - 0.979374\mathbf{j} - 0.13464\mathbf{k}$$

To see that these are the only solutions we find the Groebner basis

$$g_1(x_1, x_2, x_3, x_4) = -681 - 3698x_4 + 9707x_4^2 - 2440x_4^3 + 3688x_4^4 + 832x_4^5 + 240x_4^6$$

$$g_2(x_1, x_2, x_3, x_4) = 846546102 + 1556340276x_3 - 4930830129x_4 + 525829117x_4^2$$

$$-1798724714x_4^3 - 371568676x_4^4 - 104185320x_4$$

$$g_3(x_1,x_2,x_3,x_4) = 533702337 + 1556340276x_2 + 1880515239x_4 - 491831966x_4^2$$
$$+1121988448x_4^3 + 268281416x_4^4 + 63221520x_4^5$$

$$g_4(x_1,x_2,x_3,x_4) = 802878642 + 1556340276x_1 - 6352246905x_4 + 1289632621x_4^2$$
$$-2642462246x_4^3 - 620897956x_4^4 - 171336120x_4^5$$

The roots of $g_1(x_1,x_2,x_3,x_4)$ are

$$x_4 = -2.30085 - 3.58513i,\ x_4 = -2.30085 + 3.58513i,\ x_4 = -0.13464$$

$$x_4 = 0.373657 - 1.44347i,\ x_4 = 0.373657 + 1.44347i,\ x_4 = 0.52236$$

and arguing as in Example 1 we conclude that there are only two quaternion roots.

Example 4.

If in Example 3 we replace c by b we obtain the Algebraic Riccati Equation $qaq - qb - bq - m = 0 = 0$ which is equivalent to the system

$$f_1(x_1,x_2,x_3,x_4) = -1 + 2x_1 + x_1^2 + 2x_2 - 2x_1x_2 - x_2^2 - x_3^2 + 2x_4 - 2x_1x_4 - x_4^2 = 0$$

$$f_2(x_1,x_2,x_3,x_4) = -2 - 2x_1 + x_1^2 + 2x_2 + 2x_1x_2 - x_2^2 + x_3^2 - 2x_2x_4 + x_4^2 = 0$$

$$f_3(x_1,x_2,x_3,x_4) = -3 + 2x_3 + 2x_1x_3 - 2x_2x_3 - 2x_3x_4 = 0$$

$$f_4(x_1,x_2,x_3,x_4) = -4 - 2x_1 + x_1^2 + x_2^2 + x_3^2 + 2x_4 + 2x_1x_4 - 2x_2x_4 - x_4^2 = 0$$

For an initial guess $(x_1,x_2,x_3,x_4) = (1,0,0,0)$, Newton-Raphson produces the solution

$$(x_1,x_2,x_3,x_4) = (1.29194, 0.132536, 1.27344, 0.981494)$$

$$q = 1.29194 + 0.132536\mathbf{i} + 1.27344\mathbf{j} + 0.981494\mathbf{k}$$

an initial guess $(x_1,x_2,x_3,x_4) = (-1,0,0,0)$ runs into a singular Jacobian and an initial guess $(x_1,x_2,x_3,x_4) = (-3,1,1,0)$ produces the solution

$$(x_1,x_2,x_3,x_4) = (-0.625277, 1.2008, -1.27344, 0.35184)$$

$$q = -0.625277 + 1.2008\mathbf{i} - 1.27344\mathbf{j} + 0.35184\mathbf{k}$$

A Groebner basis is

$$g_1(x_1,x_2,x_3,x_4) = 244 - 1112x_4 + 1490x_4^2 - 984x_4^3 + 369x_4^4$$

$$g_2(x_1,x_2,x_3,x_4) = 1560 + 484x_3 - 3324x_4 + 2214x_4^2 - 1107x_4^3$$
$$g_3(x_1,x_2,x_3,x_4) = -520 + 242x_2 + 866x_4 - 738x_4^2 + 369x_4^3$$
$$g_4(x_1,x_2,x_3,x_4) = 1076 + 484x_1 - 2840x_4 + 2214x_4^2 - 1107x_4^3$$

The roots of $g_1(x_1,x_2,x_3,x_4)$ are

$x_4 = 0.35184$, $x_4 = 0.666667-1.2126i$, $x_4 = 0.666667+1.2126i$, $x_4 = 0.981494$

and, as in the previous examples, we conclude that there are only two quaternion roots.

Example 5.

As shown by Hamilton in 1899, the quaternion equation $q^2 + 1 = 0$ has infinitely many roots given by

$$q = x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}\ ,\ x_2^2 + x_3^2 + x_4^2 = 1$$

The equivalent polynomial system viewing, for example, $q^2 + 1 = 0$ as an Algebraic Riccati equation with $a = 1$, $b = c = 1$ and $m = -1$, is

$$f_1(x_1,x_2,x_3,x_4) = 1 + x_1^2 - x_2^2 - x_3^2 - x_4^2 = 0$$

$$f_2(x_1,x_2,x_3,x_4) = 2x_1x_2 = 0$$
$$f_3(x_1,x_2,x_3,x_4) = 2x_1x_3 = 0$$
$$f_4(x_1,x_2,x_3,x_4) = 2x_1x_4 = 0$$

The traditional algebraic argument is that $f_2 = f_3 = f_4 = 0$ implies $x_1 = 0$ or $x_2 = x_3 = x_4 = 0$. If $x_1 = 0$ then $f_1 = 0$ implies $x_2^2 + x_3^2 + x_4^2 = 1$. If $x_1 \neq 0$ then $f_2 = f_3 = f_4 = 0$ implies $x_2 = x_3 = x_4 = 0$ and $f_1 = 0$ implies $1 + x_1^2 = 0$ which cannot be solved in $\mathbb{R}$. Here

$$\det F'(X) = \det\left(\frac{\partial f_i}{\partial x_j}\right)_{1\le i,j\le 4} = 16x_1^2(x_1^2 + x_2^2 + x_3^2 + x_4^2) = 0 \Leftrightarrow x_1 = 0$$

For most initial choices the FindRoot command runs into a singular Jacobian. For an initial guess $(x_1,x_2,x_3,x_4) = (2,0,1,0)$ it produces the solution

$$(x_1,x_2,x_3,x_4) = (1.20371\cdot 10^{-33}, 0, 1, 0)$$

$$q = 1.20371\cdot 10^{-33} + 0\mathbf{i} + 1\mathbf{j} + 0\mathbf{k}$$

etc.
Using the Solve command, Mathematica returns the solutions to the system $f_1 = f_2 = f_3 = f_4 = 0$:

$$x_1 = 0, x_4 = -\sqrt{1 - x_2^2 - x_3^2} \;\; ; \;\; x_1 = 0, x_4 = \sqrt{1 - x_2^2 - x_3^2}$$

$$x_1 = -i, x_2 = x_3 = x_4 = 0 \;\; ; \;\; x_1 = i, x_2 = x_3 = x_4 = 0$$

The last two quadruples are, of course, immediately rejected as real quaternion solutions.
A Groebner basis consists of

$$g_1(x_1, x_2, x_3, x_4) = -x_4 + x_2^2 x_4 + x_3^2 x_4 + x_4^3$$

$$g_2(x_1, x_2, x_3, x_4) = -x_3 + x_2^2 x_3 + x_3^3 + x_3 x_4^2$$

$$g_3(x_1, x_2, x_3, x_4) = -x_2 + x_2^3 + x_2 x_3^2 + x_2 x_4^2$$

$$g_4(x_1, x_2, x_3, x_4) = x_1 x_2 \,,\; g_5(x_1, x_2, x_3, x_4) = x_1 x_3 \,,\; g_6(x_1, x_2, x_3, x_4) = x_1 x_4$$

$$g_7(x_1, x_2, x_3, x_4) = 1 + x_1^2 - x_2^2 - x_3^2 - x_4^2$$

Since the set $\{g_i : i = 1, ..., 7\}$, does not contain for each x_j, where $j = 1, 2, 3, 4$ (for example for x_1) a polynomial with a leading monomial that is a power of x_j (without any other variable appearing in the leading term) the system $f_1 = f_2 = f_3 = f_4 = 0$ has infinitely many **complex** solutions. The Solve command returns for the system $g_1 = ... = g_7 = 0$ the same solutions as for the system $f_1 = f_2 = f_3 = f_4 = 0$ above.

Example 6.

For a very general example, in the sense of Section 3 let $k = 4$ and for $n = 1, 2, 3, 4$ let

$$a_n = \begin{pmatrix} n + i(n+1) & (n-1) + in \\ -n + 1 + in & n - i(n+1) \end{pmatrix}, \; b_n = \begin{pmatrix} 2n + i(3n-1) & n + 2 - in^4 \\ -n - 2 - in^4 & 2n - i(3n-1) \end{pmatrix}$$

$$c_n = \begin{pmatrix} n^2 + i(1-n) & n^2 - 1 - in \\ -n^2 + 1 - in & n^2 - i(1-n) \end{pmatrix}, \; d_n = \begin{pmatrix} 4n - 3 + i(n^2+1) & 1 - n^3 + in^2 \\ -1 + n^3 + in^2 & 4n - 3 - i(n^2+1) \end{pmatrix}$$

while for $n = 0$ we take $a_0 = b_0 = c_0 = d_0 = 0$. Then equation (1.2) is equivalent to the real system $f_i(x_1, x_2, x_3, x_4) = 0$, $i = 1, 2, 3, 4$, where

$$f_1(x_1, x_2, x_3, x_4) = x_1 + 75x_1^2 + 555x_1^3 + 2251x_1^4 - 11x_2 - 126x_1 x_2 - 909x_1^2 x_2$$

$$-4484x_1^3x_2-75x_2^2-1665x_1x_2^2-13506x_1^2x_2^2+303x_2^3+4484x_1x_2^3+2251x_2^4-x_3+106x_1x_3$$
$$+1413x_1^2x_3+8276x_1^3x_3-471x_2^2x_3-8276x_1x_2^2x_3-75x_3^2-1665x_1x_3^2$$
$$-13506x_1^2x_3^2+303x_2x_3^2+4484x_1x_2x_3^2+4502x_2^2x_3^2-471x_3^3-8276x_1x_3^3+2251x_3^4$$
$$+5x_4+34x_1x_4+441x_1^2x_4+2876x_1^3x_4-147x_2^2x_4-2876x_1x_2^2x_4-147x_3^2x_4-2876x_1x_3^2x_4$$
$$-75x_4^2-1665x_1x_4^2-13506x_1^2x_4^2+303x_2x_4^2+4484x_1x_2x_4^2$$
$$+4502x_2^2x_4^2-471x_3x_4^2-8276x_1x_3x_4^2+4502x_3^2x_4^2-147x_4^3-2876x_1x_4^3+2251x_4^4$$

$$f_2(x_1,x_2,x_3,x_4)=5x_1+11x_1^2-63x_1^3-487x_1^4-3x_2-78x_1x_2-1143x_1^2x_2-7116x_1^3x_2$$
$$-11x_2^2+189x_1x_2^2+2922x_1^2x_2^2+381x_2^3+7116x_1x_2^3-487x_2^4-7x_3-106x_1x_3-963x_1^2x_3-5020x_1^3x_3$$
$$+321x_2^2x_3+5020x_1x_2^2x_3-11x_3^2+189x_1x_3^2$$
$$+2922x_1^2x_3^2+381x_2x_3^2+7116x_1x_2x_3^2-974x_2^2x_3^2+321x_3^3+5020x_1x_3^3$$
$$-487x_3^4-x_4+182x_1x_4+1953x_1^2x_4+10220x_1^3x_4$$
$$-651x_2^2x_4-10220x_1x_2^2x_4-651x_3^2x_4-10220x_1x_3^2x_4-11x_4^2+189x_1x_4^2$$
$$+2922x_1^2x_4^2+381x_2x_4^2+7116x_1x_2x_4^2-974x_2^2x_4^2$$
$$+321x_3x_4^2+5020x_1x_3x_4^2-974x_3^2x_4^2-651x_4^3-10220x_1x_4^3-487x_4^4$$

$$f_3(x_1,x_2,x_3,x_4)=5x_1+51x_1^2+201x_1^3+539x_1^4-5x_2-50x_1x_2$$
$$-45x_1^2x_2+964x_1^3x_2-51x_2^2-603x_1x_2^2-3234x_1^2x_2^2+15x_2^3$$
$$-964x_1x_2^3+539x_2^4+5x_3+30x_1x_3+117x_1^2x_3+332x_1^3x_3-39x_2^2x_3$$
$$-332x_1x_2^2x_3-51x_3^2-603x_1x_3^2-3234x_1^2x_3^2+15x_2x_3^2-964x_1x_2x_3^2+1078x_2^2x_3^2$$
$$-39x_3^3-332x_1x_3^3+539x_3^4-3x_4+10x_1x_4+315x_1^2x_4+2268x_1^3x_4$$
$$-105x_2^2x_4-2268x_1x_2^2x_4-105x_3^2x_4-2268x_1x_3^2x_4-51x_4^2-603x_1x_4^2-3234x_1^2x_4^2$$
$$+15x_2x_4^2-964x_1x_2x_4^2+1078x_2^2x_4^2-39x_3x_4^2-332x_1x_3x_4^2+1078x_3^2x_4^2-105x_4^3-2268x_1x_4^3+539x_4^4$$

$$f_4(x_1,x_2,x_3,x_4)=7x_1-19x_1^2-195x_1^3-857x_1^4+5x_2+26x_1x_2+135x_1^2x_2$$
$$+596x_1^3x_2+19x_2^2+585x_1x_2^2+5142x_1^2x_2^2-45x_2^3-596x_1x_2^3-857x_2^4$$

$$-3x_3 - 82x_1x_3 - 333x_1^2x_3 - 252x_1^3x_3 + 111x_2^2x_3 + 252x_1x_2^2x_3 + 19x_3^2$$

$$+585x_1x_3^2 + 5142x_1^2x_3^2 - 45x_2x_3^2 - 596x_1x_2x_3^2 - 1714x_2^2x_3^2 + 111x_3^3 + 252x_1x_3^3$$

$$-857x_3^4 + 9x_4 + 122x_1x_4 + 549x_1^2x_4 + 1620x_1^3x_4 - 183x_2^2x_4 - 1620x_1x_2^2x_4$$

$$-183x_3^2x_4 - 1620x_1x_3^2x_4 + 19x_4^2 + 585x_1x_4^2 + 5142x_1^2x_4^2 - 45x_2x_4^2 - 596x_1x_2x_4^2$$

$$-1714x_2^2x_4^2 + 111x_3x_4^2 + 252x_1x_3x_4^2 - 1714x_3^2x_4^2 - 183x_4^3 - 1620x_1x_4^3 - 857x_4^4$$

The $GroebnerBasis[\{f_1, f_2, f_3, f_4\}, \{x_1, x_2, x_3, x_4\}$ command produces a polynomial $g_1(x_1, x_2, x_3, x_4)$ that is of degree 58 in x_4 only, a polynomial $g_2(x_1, x_2, x_3, x_4)$ of degree 57 in x_4 and of degree 1 in x_3, a polynomial $g_3(x_1, x_2, x_3, x_4)$ of degree 57 in x_4 and of degree 1 in x_2, and a polynomial $g_4(x_1, x_2, x_3, x_4)$ of degree 57 in x_4 and of degree 1 in x_1. The coefficients of these polynomials are too huge to list here. Using the NSolve command we find that only 6 of the 58 roots x_4 of $g_1(x_1, x_2, x_3, x_4)$ are real. They are, using a four decimals round-up,

$$x_4 = -0.1291, x_4 = -0.0570, x_4 = -0.0385, x_4 = -0.0044, x_4 = 0, x_4 = 0.1516$$

For each value of x_4 above we use the NSolve command to solve the system $g_2 = g_3 = g_4 = 0$ for x_1, x_2, x_3 and we find the six quaternion solutions of

$$\sum_{n=1}^{4} (a_n q^n b_n + c_n q^n d_n) = 0$$

as follows:

$$q = 0.0411 - 0.1484\mathbf{i} - 0.0867\mathbf{j} - 0.1291\mathbf{k}\,,\ q = -0.2362 - 0.0425\mathbf{i} - 0.1104\mathbf{j} - 0.0570\mathbf{k}$$

$$q = 0.1897 - 0.1209\mathbf{i} + 0.1163\mathbf{j} - 0.0385\mathbf{k}\,,\ q = 0.0265 - 0.1108\mathbf{i} - 0.0216\mathbf{j} - 0.0438\mathbf{k}$$

$$q = 0 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}\,,\ q = -0.1600 + 0.0410\mathbf{i} + 0.0800\mathbf{j} + 0.1516\mathbf{k}$$

These roots can also be found using Newton-Raphson and the

$$FindRoot[\{f_1, f_2, f_3, f_4\}, \{\{x_1, ...\}, \{x_2, ...\}, \{x_3, ...\}, \{x_4, ...\}\}]$$

command for the initial choices of (x_1, x_2, x_3, x_4): $(0, 0, 1, 0)$, $(-2, -1, -1, -1)$, $(15, 0, 0, 0)$, $(-0.01, -0.1, 0, 0)$, $(0, 0, 0, 0)$, and $(-1, 0, -1, -1)$ respectively.

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