Reverse Zagreb indices of cartesian product of graphs

Süleyman Ediz, Murat Cancan

Education Faculty
Yüzüncü Yıl University
Van 65080, Turkey

email: suleymanediz@yyu.edu.tr, mcancan@gmail.com

(Received October 7, 2015, Accepted January 7, 2016)

Abstract

In this paper, some exact expressions for the reverse Zagreb indices of Cartesian product of two simple connected graphs are determined. We apply our results to compute the reverse Zagreb indices of arbitrary \(C_4\) tube and \(C_4\) torus.

1 Introduction

Let \(G\) be a simple connected graph with \(|V(G)| = n\) vertices and \(|E(G)| = m\) edges. The number of edges incident to the vertex \(v\) is \(d_v\). A vertex of degree one is said to be a pendent vertex. If \(m = n + c - 1\), then \(G\) is called a \(c\)-cyclic graph. In particular, if \(c = 0\), then \(G\) is called a tree. The path \(P_n\) is the \(n\)-vertex tree in which exactly two vertices have degree one. The cycle graph \(C_n\) is the 1-cycle graph of order \(n\), in which all vertices have degree two. We write \(\Delta\) and \(\delta\) for the largest and the smallest of all degrees of vertices of \(G\) respectively. The Cartesian product \(G \times P\) of graphs \(G\) and \(P\) has the vertex set \(V(G \times P) = V(G) \times V(P)\). And \((a, b)(c, d)\) is an edge of \(G \times P\) if \(a = c\) and \(bd \in E(P)\), or \(ac \in E(G)\) and \(b = d\). Let \(R\) and \(S\) denote a \(C_4\) tube and torus respectively. Then, \(R = P_n \times C_m, S = C_k \times C_m, k, m \geq 3\) and \(n \geq 2\).

Key words and phrases: Reverse vertex degree, Reverse Zagreb indices, Cartesian product of graphs.

AMS (MOS) Subject Classifications: 05C05, 05C76.

ISSN 1814-0432, 2016, http://ijmcs.future-in-tech.net
Zagreb indices were introduced more than forty years ago by Gutman and Trinajestic [1]. They are defined as

\[ M_1 = M_1(G) = \sum_{v \in V(G)} d_v^2 \quad (1.1) \]

and

\[ M_2 = M_2(G) = \sum_{uv \in E(G)} d_u d_v \quad (1.2) \]

respectively. In formula (1.2), \( uv \) denotes an edge connecting the vertices \( u \) and \( v \). For details of the mathematical theory and chemical applications of Zagreb indices, see the surveys [2, 3, 4, 5]. Reverse vertex degree and reverse Zagreb indices were introduced in [6]. Maximum and minimum graphs with respect to the first reverse Zagreb alpha index and minimum graphs with respect to the first reverse Zagreb beta index and the second reverse Zagreb index determined in [6]. To find maximum graphs with respect to the first reverse Zagreb beta index and the second reverse Zagreb index remain open problems.

**Definition 1.1.** ([6]) Let \( G \) be a simple connected graph and \( v \) be a vertex of \( G \). Then, the reverse vertex degree \( c_v \) of the vertex \( v \) is defined as \( c_v = \Delta - d_v + 1 \).

**Definition 1.2.** ([6]) Let \( G \) be a simple connected graph. Then the total reverse vertex degree \( TR(G) \) of the graph \( G \) is the sum of all the reverse vertex degrees of the vertices of \( G \).

**Lemma 1.3.** ([6]) Let \( G \) be a simple connected graph with \( n \) vertex and \( m \) edges. Then

\[ TR(G) = \sum_{v \in V(G)} c_v = n (\Delta + 1) - 2m \]

**Definition 1.4.** ([6]) Let \( G \) be a simple connected graph. Then the first reverse Zagreb alpha index of \( G \) defined as

\[ CM^{\alpha}_1(G) = \sum_{v \in V(G)} c_v^2 \quad (1.3) \]

**Definition 1.5.** ([6]) Let \( G \) be a simple connected graph. Then the first reverse Zagreb beta index of \( G \) defined as

\[ CM^{\beta}_1(G) = \sum_{uv \in E(G)} (c_u + c_v) \quad (1.4) \]
Definition 1.6. ([6]) Let $G$ be a simple connected graph. Then, the second reverse Zagreb index of $G$ defined as

$$CM_2(G) = \sum_{uv \in E(G)} c_u c_v$$

The chemical applications of these novel indices have been investigated in [7]. Also maximum chemical graphs with respect to the first reverse Zagreb index were determined in [7].

The first and second Zagreb indices of the cartesian product of graphs were computed in [8]. The other topological indices of the product of graphs are in [9, 10, 11]. In this paper, some exact expressions for the reverse Zagreb indices of cartesian product of two simple connected graphs are determined. We apply our results to compute the reverse Zagreb indices of arbitrary $C_4$ tube and $C_4$ torus.

2 Reverse Zagreb indices of Cartesian product of graphs

Before we proceed, we give the following crucial Lemma 2.1 whose proof was given in [12].

Lemma 2.1. ([12]) Let $G$ and $H$ be two simple connected graphs and $(a, b) \in E(G \times H)$. Then

a) $|V(G \times H)| = |V(G)||V(H)|$,

b) $|E(G \times H)| = |V(G)||E(H)| + |E(G)||V(H)|$,

c) $d_{G\times H}(a, b) = d_a + d_b$.

From the Lemma 2.1, we give the following corollary

Corollary 2.2. Let $G$ and $H$ be two connected simple graphs. Then $\Delta_{G\times H} = \Delta_G + \Delta_H$.

Proof. Let $d_a = \Delta_G$ and $d_b = \Delta_H$. Then $(a, b) \in V(G \times H)$. Clearly from Lemma 2.1c, $d_{G\times H}(a, b) = d_a + d_b$.\qed

Proposition 2.3. Let $(a, b) \in V(G \times H)$. Then $c_{(a,b)} = c_a + c_b - 1$.

Proof. From Definition 1.1, $c_{(a,b)} = \Delta_{G\times H} - d_{(a,b)} + 1$. From Lemma 2.1 and Corollary 2.2, $c_{(a,b)} = \Delta_G + \Delta_H - d_a - d_b + 1 = \Delta_G - d_a + 1 + \Delta_H - d_b + 1 - 1 = c_a + c_b - 1$.\qed
Proposition 2.4. ([6]) Let $P_n$ be a path and $C_n$ be a cycle with $n \geq 3$. Then
a) $CM^\alpha_1(P_n) = n + 6$ and $CM^\alpha_1(C_n) = n$,
b) $CM^\alpha_2(P_n) = 2n$ and $CM^\alpha_2(C_n) = 2n$,
c) $CM^2_2(P_n) = n + 1$ and $CM^2_2(C_n) = n$.

Theorem 2.5. Let $G$ and $H$ be two connected simple graphs. Then

$$CM^\alpha_1(G \times H) = |V(H)|CM^\alpha_1(G) + |V(G)|CM^\alpha_1(H) + 2TR(G)TR(H) - 2V(H)TR(G) - 2V(G)TR(H) + |V(G)||V(H)|.$$ 

Proof. By Proposition 2.3, $c_{(a,b)} = c_a + c_b - 1$. So

$$CM^\alpha_1(G \times H) = \sum_{a \in V(G)} \sum_{b \in V(H)} [(c_a + c_b) - 1]^2$$

$$= \sum_{a \in V(G)} \sum_{b \in V(H)} [(c_a + c_b)^2 - 2(c_a + c_b) + 1]$$

$$= \sum_{a \in V(G)} \sum_{b \in V(H)} [c_a^2 + 2c_ac_b + c_b^2]$$

$$-2 \sum_{a \in V(G)} \sum_{b \in V(H)} [c_a + c_b] + \sum_{a \in V(G)} \sum_{b \in V(H)} 1$$

$$= |V(H)|CM^\alpha_1(G) + |V(G)|CM^\alpha_1(H) + 2TR(G)TR(H) - 2V(H)TR(G) - 2V(G)TR(H) + |V(G)||V(H)|.$$ 

Lemma 2.6. Let $P_n$ be a path and $C_n$ be a cycle with $n \geq 3$. Then $TR(P_n) = n + 2$ and $TR(C_n) = n$.

Proof. The proof is a direct consequence of Lemma 1.3.

Corollary 2.7. Let $R = P_n \times C_m$ be a $C_4$-tube. Then

$$CM^\alpha_1(R) = m(n + 6) = CM^\alpha_1(C_m)CM^\alpha_1(P_n)$$

Proof. From Theorem 2.5, we can directly write

$$CM^\alpha_1(R) = CM^\alpha_1(P_n \times C_m) = |V(C_m)|CM^\alpha_1(P_n) + |V(P_n)|CM^\alpha_1(C_m) + 2TR(P_n)TR(C_m) - 2|V(C_m)||TR(P_n)| - 2|V(P_n)||TR(C_m)| + |V(C_m)||V(P_n)|$$

By using Proposition 2.4 and Lemma 2.6, we get

$$= m(n + 6) + nm + 2(n + 2)m - 2m(n + 2) - 2nm + mn = m(n + 6).$$
Corollary 2.8. Let $S = C_n \times C_m$ be a $C_4$-torus. Then

$$CM_1^\alpha (S) = mn = CM_1^\alpha (C_m) CM_1^\alpha (C_n)$$

Proof. From the Theorem 2.5, we have

$$CM_1^\alpha (S) = CM_1^\alpha (C_n \times C_m) = |V (C_m)| CM_1^\alpha (C_n) + |V (C_n)| CM_1^\alpha (C_m)$$

$$+2TR(C_n)TR(C_m) - 2|V (C_m)| TR(C_n) - 2|V (C_n)| TR(C_m) + |V (C_m)| |V (C_n)|$$

By using Proposition 2.4 and Lemma 2.6, we get

$$= mn + nm + 2mn - 2mn - 2mn + mn = mn.$$

It is very surprising to see that $CM_1^\alpha (R) = CM_1^\alpha (C_m) CM_1^\alpha (P_n)$ and $CM_1^\alpha (S) = CM_1^\alpha (C_m) CM_1^\alpha (C_n)$.

Theorem 2.9. Let $G$ and $H$ be two connected simple graphs. Then

$$CM_1^\alpha (G \times H) = 2|E (H)| TR(G) + |V (G)| CM_1^\alpha (H) - 2|E (H)| |V (G)|$$

$$+ 2|E (G)| TR(H) + |V (H)| CM_1^\alpha (G) - 2|E (G)| |V (H)|$$

Proof. By Proposition 2.3, $c_{(a,b)} = c_a + c_b - 1$. So

$$CM_1^\alpha (G \times H) = \sum_{(a,b)(c,d)\in E(G\times H)} (c_{(a,b)} + c_{(c,d)})$$

$$= \sum_{u \in V(G)bd \in E(H)} \sum_{v \in V(H)} [(c_u + c_b - 1) + (c_u + c_d - 1)]$$

$$+ \sum_{u \in V(H)ac \in E(G)} [(c_a + c_v - 1) + (c_c + c_v - 1)]$$

$$= \sum_{u \in V(G)bd \in E(H)} \sum_{v \in V(H)} [2c_u + (c_b + c_d) - 2]$$

$$+ \sum_{u \in V(H)ac \in E(G)} [2c_v + (c_a + c_c) - 2]$$

$$= 2|E (H)| TR(G) + |V (G)| CM_1^\alpha (H) - 2|E (H)| |V (G)|$$

$$+ 2|E (G)| TR(H) + |V (H)| CM_1^\alpha (G) - 2|E (G)| |V (H)|$$
As a direct consequence of Theorem 2.9, we have the following corollaries whose proofs will not be provided.

**Corollary 2.10.** Let $R = P_n \times C_m$ be a $C_4$-tube. Then
$$CM_2(R) = 4mn.$$ 

**Corollary 2.11.** Let $S = C_n \times C_m$ be a $C_4$-torus. Then
$$CM_2(S) = 4mn.$$ 

**Theorem 2.12.** Let $G$ and $H$ be two connected simple graphs. Then

$$CM_2(G \times H) = 3|E(H)|CM_1^a(G) + 3|E(G)|CM_1^a(H) + |V(G)|CM_2(H)$$
$$+ |V(H)|CM_2(G) - 2TR(G)|E(H)| - 2TR(H)|E(G)|$$
$$- |V(G)|CM_1^b(H) - |V(H)|CM_1^b(G) + |V(G)||E(H)| + |V(H)||E(G)|$$

**Proof.** By Proposition 2.3, $c_{(a,b)} = c_a + c_b - 1$. So

$$CM_2(G \times H) = \sum_{(a,b)(c,d) \in E(G \times H)} c_{(a,b)}c_{(c,d)}$$

$$= \sum_{u \in V(G)} \sum_{b \in E(H)} (c_u + c_b - 1) (c_u + c_d - 1)$$
$$+ \sum_{v \in V(H)} \sum_{a \in E(G)} (c_a + c_v - 1) (c_c + c_v - 1)$$

$$= \sum_{u \in V(G)} \sum_{b \in E(H)} (c_u^2 + c_u (c_d + c_b) + c_b c_d - 2c_u - c_b - c_d + 1)$$
$$+ \sum_{v \in V(H)} \sum_{a \in E(G)} (c_a^2 + c_v (c_a + c_c) + c_a c_c - 2c_v - c_a - c_c + 1)$$

$$= |E(H)|CM_1^a(G) + |E(G)|CM_1^a(H) + 2|E(G)|CM_1^a(H)$$
$$+ 2|E(H)|CM_1^a(G) + |V(G)|CM_2(H) + |V(H)||CM_2(G)$$
$$- 2TR(G)|E(H)| - 2TR(H)|E(G)| - |V(G)||CM_1^b(H) - |V(H)||CM_1^b(G)$$
$$+ |V(G)||E(H)| + |V(H)||E(G)|$$

$$= 3|E(H)|CM_1^a(G) + 3|E(G)|CM_1^a(H) + |V(G)||CM_2(H) + |V(H)||CM_2(G)$$
$$- 2TR(G)|E(H)| - 2TR(H)|E(G)| - |V(G)||CM_1^b(H) - |V(H)||CM_1^b(G)$$
$$+ |V(G)||E(H)| + |V(H)||E(G)|$$

$\square$
As a direct consequence of Theorem 2.12, we also give the following corollaries without proofs.

**Corollary 2.13.** Let $R = P_n \times C_m$ be a $C_4$-tube. Then

$$CM_2(R) = m(n + 16).$$

**Corollary 2.14.** Let $S = C_n \times C_m$ be a $C_4$-torus. Then

$$CM_2(S) = 2mn.$$

**References**


