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Reverse Zagreb indices of cartesian product of graphs

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Abstract

In this paper, some exact expressions for the reverse Zagreb indices of Cartesian product of two simple connected graphs are determined. We apply our results to compute the reverse Zagreb indices of arbitrary C_4 tube and C_4 torus.

1 Introduction

Let G be a simple connected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The number of edges incident to the vertex v is d_v. A vertex of degree one is said to be a pendent vertex. If $m = n+c-1$, then G is called a c-cyclic graph. In particular, if $c = 0$, then G is called a tree. The path P_n is the n-vertex tree in which exactly two vertices have degree one. The cycle graph C_n is the 1-cycle graph of order n, in which all vertices have degree two. We write Δ and δ for the largest and the smallest of all degrees of vertices of G respectively. The Cartesian product $G \times P$ of graphs G and P has the vertex set $V(G \times P) = V(G) \times V(P)$. And $(a, b)(c, d)$ is an edge of $G \times P$ if $a = c$ and $bd \in E(P)$, or $ac \in E(G)$ and $b = d$. Let R and S denote a C_4 tube and torus respectively. Then, $R = P_n \times C_m$, $S = C_k \times C_m$, $k, m \geq 3$ and $n \geq 2$.

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Zagreb indices were introduced more than forty years ago by Gutman and Trinajestic [1]. They are defined as

$$M_1 = M_1(G) = \sum_{v \in V(G)} d_v^2 \tag{1.1}$$

and

$$M_2 = M_2(G) = \sum_{uv \in E(G)} d_u\, d_v \tag{1.2}$$

respectively. In formula (1.2), uv denotes an edge connecting the vertices u and v. For details of the mathematical theory and chemical applications of Zagreb indices, see the surveys [2, 3, 4, 5]. Reverse vertex degree and reverse Zagreb indices were introduced in [6]. Maximum and minimum graphs with respect to the first reverse Zagreb alpha index and minimum graphs with respect to the first reverse Zagreb beta index and the second reverse Zagreb index determined in [6]. To find maximum graphs with respect to the first reverse Zagreb beta index and the second reverse Zagreb index remain open problems.

Definition 1.1. *([6]) Let G be a simple connected graph and v be a vertex of G. Then, the reverse vertex degree c_v of the vertex v is defined as $c_v = \Delta - d_v + 1$.*

Definition 1.2. *([6]) Let G be a simple connected graph. Then the total reverse vertex degree $TR(G)$ of the graph G is the sum of all the reverse vertex degrees of the vertices of G.*

Lemma 1.3. *([6]) Let G be a simple connected graph with n vertex and m edges. Then*

$$TR(G) = \sum_{v \in V(G)} c_v = n\,(\Delta + 1) - 2m$$

Definition 1.4. *([6]) Let G be a simple connected graph. Then the first reverse Zagreb alpha index of G defined as*

$$CM_1^{\alpha}(G) = \sum_{v \in V(G)} c_v^2 \tag{1.3}$$

Definition 1.5. *([6]) Let G be a simple connected graph. Then the first reverse Zagreb beta index of G defined as*

$$CM_1^{\beta}(G) = \sum_{uv \in E(G)} (c_u + c_v) \tag{1.4}$$

Definition 1.6. *([6]) Let G be a simple connected graph. Then, the second reverse Zagreb index of G defined as*

$$CM_2(G) = \sum_{uv \in E(G)} c_u c_v \tag{1.5}$$

The chemical applications of these novel indices have been investigated in [7]. Also maximum chemical graphs with respect to the first reverse Zagreb index were determined in [7].
The first and second Zagreb indices of the cartesian product of graphs were computed in [8]. The other topological indices of the product of graphs are in [9, 10, 11]. In this paper, some exact expressions for the reverse Zagreb indices of cartesian product of two simple connected graphs are determined. We apply our results to compute the reverse Zagreb indices of arbitrary C_4 tube and C_4 torus.

2 Reverse Zagreb indices of Cartesian product of graphs

Before we proceed, we give the following crucial Lemma 2.1 whose proof was given in [12].

Lemma 2.1. *([12]) Let G and H be two simple connected graphs and $(a, b) \in E(G \times H)$. Then*
a)$|V(G \times H| = |V(G)||V(H)|$,
b)$|E(G \times H| = |V(G)||E(H)| + |E(G)||V(H)|$,
c)$d_{G \times H}(a, b) = d_a + d_b$.

From the Lemma 2.1, we give the following corollary

Corollary 2.2. *Let G and H be two connected simple graphs. Then $\Delta_{G \times H} = \Delta_G + \Delta_H$.*

Proof. Let $d_a = \Delta_G$ and $d_b = \Delta_H$. Then $(a, b) \in V(G \times H)$. Clearly from Lemma 2.1c, $d_{G \times H}(a, b) = d_a + d_b$. □

Proposition 2.3. *Let $(a, b) \in V(G \times H)$. Then $c_{(a,b)} = c_a + c_b - 1$.*

Proof. From Definition 1.1, $c_{(a,b)} = \Delta_{G \times H} - d_{(a,b)} + 1$. From Lemma 2.1 and Corollary 2.2, $c_{(a,b)} = \Delta_G + \Delta_H - d_a - d_b + 1 = \Delta_G - d_a + 1 + \Delta_H - d_b + 1 - 1 = c_a + c_b - 1$. □

Proposition 2.4. *([6]) Let P_n be a path and C_n be a cycle with $n \geq 3$. Then*
a)$CM_1^{\alpha}(P_n) = n + 6$ and $CM_1^{\alpha}(C_n) = n$,
b)$CM_1^{\beta}(P_n) = 2n$ and $CM_1^{\beta}(C_n) = 2n$,
c)$CM_2(P_n) = n + 1$ and $CM_2(C_n) = n$.

Theorem 2.5. *Let G and H be two connected simple graphs. Then*

$$CM_1^{\alpha}(G \times H) = |V(H)| \, CM_1^{\alpha}(G) + |V(G)| \, CM_1^{\alpha}(H)$$

$$+2TR(G)TR(H) - 2V(H) \, TR(G) - 2V(G) \, TR(H) + |V(G)| \, |V(H)| .$$

Proof. By Proposition 2.3, $c_{(a,b)} = c_a + c_b - 1$. So

$$CM_1^{\alpha}(G \times H) = \sum_{a \in V(G)} \sum_{b \in V(H)} [(c_a + c_b) - 1]^2$$

$$= \sum_{a \in V(G)} \sum_{b \in V(H)} [(c_a + c_b)^2 - 2(c_a + c_b) + 1]$$

$$= \sum_{a \in V(G)} \sum_{b \in V(H)} [c_a^2 + 2c_a c_b + c_b^2]$$

$$-2 \sum_{a \in V(G)} \sum_{b \in V(H)} [c_a + c_b] + \sum_{a \in V(G)} \sum_{b \in V(H)} 1$$

$$= |V(H)| \, CM_1^{\alpha}(G) + |V(G)| \, CM_1^{\alpha}(H)$$

$$+2TR(G)TR(H) - 2V(H) \, TR(G) - 2V(G) \, TR(H) + |V(G)| \, |V(H)|$$

. □

Lemma 2.6. *Let P_n be a path and C_n be a cycle with $n \geq 3$. Then*
$TR(P_n) = n + 2$ and $TR(C_n) = n$.

Proof. The proof is a direct consequence of Lemma 1.3. □

Corollary 2.7. *Let $R = P_n \times C_m$ be a C_4-tube. Then*

$$CM_1^{\alpha}(R) = m(n + 6) = CM_1^{\alpha}(C_m) \, CM_1^{\alpha}(P_n)$$

Proof. From Theorem 2.5, we can directly write

$$CM_1^{\alpha}(R) = CM_1^{\alpha}(P_n \times C_m) = |V(C_m)| \, CM_1^{\alpha}(P_n) + |V(P_n)| \, CM_1^{\alpha}(C_m)$$

$$+2TR(P_n)TR(C_m) - 2|V(C_m)| \, TR(P_n) - 2|V(P_n)| \, TR(C_m) + |V(C_m)| \, |V(P_n)|$$

By using Proposition 2.4 and Lemma 2.6, we get

$$= m(n + 6) + nm + 2(n + 2)m - 2m(n + 2) - 2nm + mn = m(n + 6) .$$

□

Corollary 2.8. *Let $S = C_n \times C_m$ be a C_4-torus. Then*

$$CM_1^{\alpha}(S) = mn = CM_1^{\alpha}(C_m)\, CM_1^{\alpha}(C_n)$$

Proof. From the Theorem 2.5, we have

$$CM_1^{\alpha}(S) = CM_1^{\alpha}(C_n \times C_m) = |V(C_m)|\, CM_1^{\alpha}(C_n) + |V(C_n)|\, CM_1^{\alpha}(C_m)$$

$$+2TR(C_n)TR(C_m) - 2|V(C_m)|\, TR(C_n) - 2|V(C_n)|\, TR(C_m) + |V(C_m)|\, |V(C_n)|$$

By using Proposition 2.4 and Lemma 2.6, we get

$$= mn + nm + 2mn - 2mn - 2mn + mn = mn.$$

□

It is very surprising to see that $CM_1^{\alpha}(R) = CM_1^{\alpha}(C_m)\, CM_1^{\alpha}(P_n)$ and $CM_1^{\alpha}(S) = CM_1^{\alpha}(C_m)\, CM_1^{\alpha}(C_n)$.

Theorem 2.9. *Let G and H be two connected simple graphs. Then*

$$CM_1^{\beta}(G \times H) = 2|E(H)|\, TR(G) + |V(G)|\, CM_1^{\beta}(H) - 2|E(H)|\, |V(G)|$$

$$+2|E(G)|\, TR(H) + |V(H)|\, CM_1^{\beta}(G) - 2|E(G)|\, |V(H)|$$

Proof. By Proposition 2.3, $c_{(a,b)} = c_a + c_b - 1$. So

$$CM_1^{\beta}(G \times H) = \sum_{(a,b)(c,d) \in E(G \times H)} \left(c_{(a,b)} + c_{(c,d)}\right)$$

$$= \sum_{u \in V(G)} \sum_{bd \in E(H)} [(c_u + c_b - 1) + (c_u + c_d - 1)]$$

$$+ \sum_{v \in V(H)} \sum_{ac \in E(G)} [(c_a + c_v - 1) + (c_c + c_v - 1)]$$

$$= \sum_{u \in V(G)} \sum_{bd \in E(H)} [2c_u + (c_b + c_d) - 2]$$

$$+ \sum_{v \in V(H)} \sum_{ac \in E(G)} [2c_v + (c_a + c_c) - 2]$$

$$= 2|E(H)|\, TR(G) + |V(G)|\, CM_1^{\beta}(H) - 2|E(H)|\, |V(G)|$$

$$+2|E(G)|\, TR(H) + |V(H)|\, CM_1^{\beta}(G) - 2|E(G)|\, |V(H)|$$

□

As a direct consequence of Theorem 2.9, we have the following corollaries whose proofs will not be provided.

Corollary 2.10. *Let $R = P_n \times C_m$ be a C_4-tube. Then*

$$CM_2(R) = 4mn.$$

Corollary 2.11. *Let $S = C_n \times C_m$ be a C_4-torus. Then*

$$CM_2(S) = 4mn.$$

Theorem 2.12. *Let G and H be two connected simple graphs. Then*

$$\begin{aligned} CM_2(G \times H) &= 3|E(H)|\, CM_1^{\alpha}(G) + 3|E(G)|\, CM_1^{\alpha}(H) + |V(G)|\, CM_2(H) \\ &+ |V(H)|\, CM_2(G) - 2TR(G)|E(H)| - 2TR(H)|E(G)| \\ &- |V(G)|\, CM_1^{\beta}(H) - |V(H)|\, CM_1^{\beta}(G) + |V(G)||E(H)| + |V(H)||E(G)| \end{aligned}$$

Proof. By Proposition 2.3, $c_{(a,b)} = c_a + c_b - 1$. So

$$\begin{aligned} CM_2(G \times H) &= \sum_{(a,b)(c,d) \in E(G \times H)} c_{(a,b)} c_{(c,d)} \\ &= \sum_{u \in V(G)} \sum_{bd \in E(H)} (c_u + c_b - 1)(c_u + c_d - 1) \\ &+ \sum_{v \in V(H)} \sum_{ac \in E(G)} (c_a + c_v - 1)(c_c + c_v - 1) \\ &= \sum_{u \in V(G)} \sum_{bd \in E(H)} \left(c_u^2 + c_u(c_d + c_b) + c_b c_d - 2c_u - c_b - c_d + 1\right) \\ &+ \sum_{v \in V(H)} \sum_{ac \in E(G)} \left(c_v^2 + c_v(c_a + c_c) + c_a c_c - 2c_v - c_a - c_c + 1\right) \\ &= |E(H)|\, CM_1^{\alpha}(G) + |E(G)|\, CM_1^{\alpha}(H) + 2|E(G)|\, CM_1^{\alpha}(H) \\ &+ 2|E(H)|\, CM_1^{\alpha}(G) + |V(G)|\, CM_2(H) + |V(H)|\, CM_2(G) \\ &- 2TR(G)|E(H)| - 2TR(H)|E(G)| - |V(G)|\, CM_1^{\beta}(H) - |V(H)|\, CM_1^{\beta}(G) \\ &+ |V(G)||E(H)| + |V(H)||E(G)| \\ &= 3|E(H)|\, CM_1^{\alpha}(G) + 3|E(G)|\, CM_1^{\alpha}(H) + |V(G)|\, CM_2(H) + |V(H)|\, CM_2(G) \\ &- 2TR(G)|E(H)| - 2TR(H)|E(G)| - |V(G)|\, CM_1^{\beta}(H) - |V(H)|\, CM_1^{\beta}(G) \\ &+ |V(G)||E(H)| + |V(H)||E(G)| \end{aligned}$$

□

As a direct consequence of Theorem 2.12, we also give the following corollaries without proofs.

Corollary 2.13. *Let $R = P_n \times C_m$ be a C_4-tube. Then*

$$CM_2(R) = m(n + 16).$$

Corollary 2.14. *Let $S = C_n \times C_m$ be a C_4-torus. Then*

$$CM_2(S) = 2mn.$$

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