International Journal of Mathematics and
Computer Science, **11**(2016), no. 1, 23–32

On Tribonacci functions and Tribonacci numbers

Maryam Naderi Parizi[1], Madjid Eshaghi Gordji[2]

[1]Department of Mathematics
Payame Noor University
Iran

[2]Department of Mathematics
Semnan University
Semnan, Iran

email: mrym_ndr@yahoo.com, madjid.eshaghi@gmail.com

(Received August 29, 2015, Accepted September 27, 2015)

Abstract

In this paper, we consider Tribonacci functions on the real numbers $\mathbb{R}$; i.e, functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$f(x+3) = f(x+2) + f(x+1) + f(x).$$

We develop the notion of Tribonacci functions using the concept of f-even and f-odd functions. Moreover, we show that if f is a Tribonachi function, then $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \beta$ such that β is one of the roots of equation $x^3 - x^2 - x - 1 = 0$ for which β is greater than one.

1 Introduction

Fibonacci numbers have been studied in many different forms for centuries. As a result, the literature on the subject is incredibly vast. One of the amazing qualities of these numbers is the variety of mathematical models where

Key words and phrases: Tribonacci function, Tribonacci number, f-even, f-odd.
AMS (MOS) Subject Classifications: 11B39, 39A10.
ISSN 1814-0432, 2016, http://ijmcs.future-in-tech.net

they play some sort of role and where their properties are of importance in elucidating the ability of the model under discussion to explain whatever implications are inherent in it. The fact that the ratio of successive Fibonacci numbers approaches the Golden ratio rather quickly as they go to infinity probably has a good deal to do with the observation made in the previous sentence. Surveys and connections of the type just mentioned are provided in [1] and [2]. Hyers-Ulam stability of Fibonacci functional equation was studied in [6]. Surprisingly, novel perspectives are still available and will presumably continue to be so for the future as long as mathematical investigations continue to be made. Han, Kim, and Neggers [3,4] studied a Fibonacci norm of positive integers and Fibonacci sequences groupoids in arbitrary groupoids. They considered Fibonacci functions using the concept of f-even and f-odd functions. Moreover, in [7], Han, Kim, and Neggers showed that if f is a Fibonacci function, then $\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$. In this paper we consider Tribonacci functions on the real numbers $\mathbb{R}$; i.e., functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ for all $x \in \mathbb{R}$, $f(x+3) = f(x+2) + f(x+1) + f(x)$. We develop these for Tribonacci functions. Moreover, we show that if f is a Tribonacci function, then $\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = \beta$ such that β is one of roots of equation $x^3 - x^2 - x - 1 = 0$ and β is greater than one.

2 Tribonacci functions

A function f defined on the real numbers is called a Tribonacci function if it satisfies the formula

$$f(x) = f(x-1) + f(x-2) + f(x-3) \tag{2.1}$$

for all $x \in \mathbb{R}$,where $\mathbb{R}$ is the set of real numbers.

Example 2.1. *Let $f(x) = a^x$ be a Tribonacci function on $\mathbb{R}$ where $a > 0$. Then*

$$a^{x+3} = f(x+3) = f(x+2) + f(x+1) + f(x) = a^x(a^2 + a + 1).$$

since $a > 0$, we have $a^3 = a^2 + a + 1$ and $a = \beta$ such that β is root of the equation $x^3 - x^2 - x - 1 = 0$ and $\beta > 1$. Hence $f(x) = \beta^x$ is a Tribonacci function of this type on $\mathbb{R}$.

Example 2.2. *Let* $\{u_n\}_{-\infty}^{\infty}$, $\{v_n\}_{-\infty}^{\infty}$ *and* $\{w_n\}_{-\infty}^{\infty}$ *be full Tribonacci sequences. We define a function* $f(x)$ *by* $f(x) = u_{[x]} + v_{[x]}t + w_{[x]}t^2$ *where* $t = x - [x] \in (0, 1)$. *Then*

$$f(x+3) = u_{[x]+3} + v_{[x]+3}t + w_{[x]+3}t^2$$
$$= (u_{[x]+2} + u_{[x]+1} + u_{[x]}) + (v_{[x]+2} + v_{[x]+1} + v_{[x]})t + (w_{[x]+2} + w_{[x]+1} + w_{[x]})t^2$$
$$= (u_{[x]+2} + v_{[x]+2}t + w_{[x]+2}t^2) + (u_{[x]+1} + v_{[x]+1}t + w_{[x]+1}t^2) + (u_{[x]} + v_{[x]}t + w_{[x]}t^2)$$
$$= f(x+2) + f(x+1) + f(x)$$

for any $x \in \mathbb{R}$. *This proves that* f *is a Tribonacci function. Note that if a Tribonacci function is differentiable on* $\mathbb{R}$, *then its derivative is also a Tribonacci function.*

Proposition 2.3. *Let* $\mathbb{R}$ *be a Tribonacci function. Define* $g(x) := f(x + t + t^2)$, *where* $t \in \mathbb{R}$ *for any* $x \in \mathbb{R}$. *Then* g *is also a Tribonacci function.*

Proof.

$$\begin{aligned} g(x+3) &= f(x+3+t+t^2) \\ &= f(x+t+t^2+2) + f(x+t+t^2+1) + f(x+t+t^2) \\ &= g(x+2) + g(x+1) + g(x) \end{aligned}$$

□

Example 2.4. *Since* $f(x) = \beta^x$ *is Tribonacci,* $g(x) = (\beta)^{x+t+t^2} = (\beta)^{t+t^2} f(x)$ *is also a Tribonacci function, where* $t \in \mathbb{R}$.

Theorem 2.5. *Let* $f(x)$ *be a Tribonacci function and let* $\{F_n\}$, $\{F'_n\}$ *and* $\{F''_n\}$ *be sequences of Tribonacci numbers with* $F_0 = 0$, $F_1 = 1$, $F_2 = 2$,$F_3 = 3$

and $F'_0 = 0,\ F'_1 = 0, F'_2 = 1, F'_3 = 1$ *and* $F''_0 = 1,\ F''_1 = -1, F''_2 = 1, F''_3 = 1$. *Then*

$$f(x+n) = F'_n f(x+2) + F_{n-2} f(x+1) + F''_n f(x)$$

for any $x \in \mathbb{R}$ *and* $n \geq 3$ *an integer such that* $F_n = F'_n + F'_{n+1}$ *and* $F''_n = F'_{n-1}$.

Proof. If $n = 3$, then

$$f(x+3) = f(x+2) + f(x+1) + f(x) = F'_3 f(x+2) + F_1 f(x+1) + F''_3 f(x)$$

and if $n = 4$, then we have

$$f(x+4) = f(x+3) + f(x+2) + f(x+1) = F'_3 f(x+2) + F_1 f(x+1) + F''_3 f(x) + f(x+1) +$$

$$0f(x+1) + 0f(x) + 0f(x+2) + f(x+1) + 0f(x)$$

$$= (F'_3 + F'_2 + F'_1) f(x+2) + (F_1 + F'_0 + F_{-1}) f(x+1) + (F''_3 + F''_2 + F''_1) f(x)$$

$$= F'_4 f(x+2) + F_2 f(x+1) + F''_4 f(x).$$

If $n = 5$, then we have

$$f(x+5) = f(x+4) + f(x+3) + f(x+2) = F'_4 f(x+2) + F_2 f(x+1) + F''_4 f(x) + F'_3 f(x+2) +$$

$$F_1 f(x+1) + F''_3 f(x) + f(x+2) = (F'_4 + F'_3 + F'_2) f(x+2) + (F_2 + F_1 + F_0) f(x+1) +$$

$$(F''_4 + F''_3 + F''_2) f(x) = F'_5 f(x+2) + F_3 f(x+1) + F''_5 f(x)$$

Assume that it holds for the cases of n, $n+1$ and $n+2$. Then

$$f(x+n+3) = f(x+n+2) + f(x+n+1) + f(x+n) = (F'_{n+2} f(x+2) + F_n f(x+1) + F''_{n+2}) f(x)$$

$$+ (F'_{n+1} f(x+2) + F_{n-1} f(x) + F''_{n+1} f(x)) + (F'_n f(x+2) + F_{n-2} f(x+1) + F''_n f(x)) = (F'_{n+2} +$$

$$F'_{n+1} + F'_n) f(x+2) + (F_n + F_{n-1} + F_{n-2}) f(x+1) + (F''_{n+2} +$$

$$F''_{n+1} + F''_n) f(x) = F'_{n+3} f(x+2) + F_{n+1} f(x+1) + F''_{n+3} f(x)$$

proving the theorem. □

Corollary 2.6. *If $\{F'_n\}$ is the sequence of Tribonacci numbers with $F'_1 = 0$ and $F'_2 = F'_3 = 1$ and β is root of the equation $x^3 - x^2 - x - 1 = 0$ and $\beta > 1$, then*

$$\beta^n = F'_n\beta^2 + (F'_{n-2} + F'_{n-1})\beta + F'_{n-1} \tag{2.2}$$

Proof. As we have seen in example (2.1), $f(x) = \beta^x$ is a Tribonacci function. By applying theorem (2.5), we have

$$\begin{aligned}\beta^{x+n} &= f(x+n) = F'_n f(x+2) + (F'_{n-2} + F'_{n-1}) f(x+1) + F'_{n-1} f(x)\\ &= F'_n\beta^x + 2 + (F'_{n-2} + F'_{n-1})\beta^x + 1 + F'_{n-1}\beta^x,\end{aligned}$$

proving

$$\beta^n = F'_n\beta^2 + (F'_{n-2} + F'_{n-1})\beta + F'_{n-1}$$

□

Theorem 2.7. *Let $\{u_n\}$ be the full Tribonacci sequence. Then*

$$U_{[x+n]} = F'_n U_{[x]+2} + (F'_{n-2} + F'_{n-1}) U_{[x]+1} + F'_{n-1} U_{[x]}, \tag{2.3}$$

$$U_{[x+n]-1} = F'_n U_{[x]+1} + (F'_{n-2} + F'_{n-1}) U_{[x]} + F'_{n-1} U_{[x]-1}, \tag{2.4}$$

$$U_{[x+n]-2} = F'_n U_{[x]} + (F'_{n-2} + F'_{n-1}) U_{[x]-1} + F'_{n-1} U_{[x]-2} \tag{2.5}$$

Proof. The map $f(x) := U_{[x]} + U_{[x]-1}t + U_{[x]-2}t^2$ discussed in example (2.2) is a Tribonacci function. If we apply theorem (2.5), then we obtain

$$
\begin{aligned}
& U_{[x+n]} + U_{[x+n]-1}t + U_{[x+n]-2}t^2 = f(x+n) \\
& = F'_n f(x+2) + (F'_{n-2} + F'_{n-1}) f(x+1) + F'_{n-1} f(x) \\
& = F'_n [U_{[x+2]} + U_{[x+2]-1}t + U_{[x+2]-2}t^2] \\
& + (F'_{n-2} + F'_{n-1})[U_{[x+1]} + U_{[x+1]-1}t + U_{[x+1]-2}t^2] \\
& + F'_{n-1}[U_{[x]} + U_{[x]-1}t + U_{[x]-2}t^2] \\
& = F'_n U_{[x]+2} + (F'_{n-2} + F'_{n-1}) U_{[x]+1} + F'_{n-1} U_{[x]} \\
& + (F'_n U_{[x]+1} + (F'_{n-2} + F'_{n-1}) U_{[x]} + F'_{n-1} U_{[x]-1})t \\
& + (F'_n U_{[x]} + (F'_{n-2} + F'_{n-1}) U_{[x]-1} + F'_{n-1} U_{[x]-2})t^2
\end{aligned}
$$

proving the theorem. □

Corollary 2.8. *If* $n \geq 3$, *then*

$$F_{[x+n]} = F'_n F_{[x]+2} + (F'_{n-2} + F'_{n-1}) F_{[x]+1} + F'_{n-1} F_{[x]} \tag{2.6}$$

and

$$F_{[x+n]-1} = F'_n F_{[x]+1} + (F'_{n-2} + F'_{n-1}) F_{[x]} + F'_{n-1} F_{[x]-1} \tag{2.7}$$

and

$$F_{[x+n]-2} = F'_n F_{[x]} + (F'_{n-2} + F'_{n-1}) F_{[x]-1} + F'_{n-1} F_{[x]-2} \tag{2.8}$$

Corollary 2.9.

$$F_{[n+3]} = F'_n F_5 + (F'_{n-2} + F'_{n-1}) F_2 + F'_{n-1} F_3$$

Proof. Let $x := 3$ in (2.6) or $x := 4$ in (2.7) or $x := 5$ in (2.7). □

3 f-even and f-odd functions

In this section, we develop the notion of Tribonacci functions using the concept of f-even and f-odd functions.

Definition 3.1. *Let $a(x)$ be a real-valued function of real variable such that if $a(x)h(x) = 0$ and $h(x)$ is continuous then $h(x) \equiv 0$. The map $a(x)$ is said to be an f-even function (resp., f-odd function) if $a(x+1) = a(x)$(resp., $a(x+1) = -a(x)$) for any $x \in \mathbb{R}$.*

Example 3.2. *If $a(x) = x - [x]$, then $a(x)h(x) \equiv 0$ implies $h(x) \equiv 0$ if $x \notin \mathbb{Z}$. By continuity of $h(x)$, it follows that*

$$h(n) = \lim_{x \to n} h(x) = 0$$

for any integer n, and hence $h(x) \equiv 0$. Since $a(x+1) = (x+1) - [x+1] = (x+1) - ([x]+1) = x - [x] = a(x)$, we see that $a(x)$ is an f-even function.

Theorem 3.3. *Let $f(x) = a(x)g(x)$ be a function, where $a(x)$ is an f-even function and $g(x)$ is a continuous function. Then $f(x)$ is a Tribonacci function if and only if $g(x)$ is a Tribonacci function.*

Proof. Suppose that $f(x)$ is a Tribonacci function. Then

$$a(x)g(x+3) = a(x+3)g(x+3) = f(x+3) =$$

$$f(x+2) + f(x+1) + f(x) = a(x)(g(x+2) + g(x+1) + g(x)).$$

Hence $a(x)[g(x+3) - g(x+2) - g(x+1) - g(x)] \equiv 0$ and $g(x+3) - g(x+2) - g(x+1) - g(x) \equiv 0$; i.e., $g(x+3) = g(x+2) + g(x+1) + g(x)$ and $g(x)$ is a Tribonacci function. On the other hand, if $g(x)$ is any Tribonacci function then $g(x+3) = g(x+2) + g(x+1) + g(x)$ implies that

$$\begin{aligned} f(x+3) = a(x)g(x+3) &= a(x)(g(x+2) + g(x+1) + g(x) \\ &= a(x+2)g(x+2) + a(x+1)g(x+1) + a(x)g(x) \\ &= f(x+2) + f(x+1) + f(x) \end{aligned}$$

thus f is also a Tribonacci function. □

4 Quotients of Tribonacci functions

In this section, we discuss the limit of the quotient of a Tribonacci function.

Theorem 4.1. *If $f(x)$ is a Tribonacci function, then the limit of quotient $\frac{f(x+1)}{f(x)}$ exists.*

Proof. if we consider a quotient $\frac{f(x+1)}{f(x)}$ of a Tribonacci function $f(x)$

We let $\alpha := f(x)$, $\beta := f(x+1)$ and $\gamma := f(x+2)$. By theorem (2.5)

$$f(x+n) = F'_n f(x+2) + F_{n-2} f(x+1) + F''_n f(x) =$$

$$F'_n f(x+2) + (F'_{n-2} - F'_{n-1}) f(x+1) + F'_{n-1} f(x)$$

for any $n \geq 3$. Given $x' \in \mathbb{R}$, there exist $x \in \mathbb{R}$ and $n \in Z$ such that $x' = x + n$. Hence

$$\lim_{x' \to \infty} \frac{f(x'+1)}{f(x')} = \lim_{n \to \infty} \frac{f(x+n+1)}{f(x+n)}$$

$$= \lim_{n \to \infty} \frac{F'_{n+1}\gamma + (F'_{n-1} - F'_n)\beta + F'_n\alpha}{F'_n\gamma + (F'_{n-2} - F'_{n-1})\beta + F'_{n-1}\alpha}$$

$$= \lim_{n \to \infty} \frac{F'_n}{F'_{n-1}} \frac{\frac{F'_{n+1}}{F'_n}\gamma + (\frac{F'_{n-1}}{F'_n} - 1)\beta + \alpha}{\frac{F'_{n-1}}{F'_n}\gamma + (\frac{F'_{n-2}}{F'_{n-1}} - 1)\beta + \alpha}$$

$$= \varphi[\frac{\varphi\gamma + (\frac{1}{\varphi} - 1)\beta + \alpha}{\varphi\gamma + (\frac{1}{\varphi} - 1)\beta + \alpha}] = \varphi$$

where $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$ such that φ is a root of the equation $x^3 - x^2 - x - 1 = 0$ for which $1 < \varphi < 2$. Thus $\lim_{x\to\infty} \frac{f(x+1)}{f(x)}$. Consider $f(x) > 0$, $f(x+1) > 0$, $f(x+2) > 0$. We may change $\frac{f(x+1)}{f(x)}$ by $\frac{f(\delta+2n+1)}{f(\delta+2n)}$ since any real number $x(> 0)$ can be written $x = \delta + 2n$ for some $\delta \in \mathbb{R}$ and $n \in \mathbb{N}$. Consider the sequence $\{\frac{f(\delta+2n+1)}{f(\delta+2n)}\}_{n=1}^{\infty}$.

$$\begin{aligned} \frac{f(\delta+2n+1)}{f(\delta+2n)} &= \frac{f(\delta+2n) + f(\delta+2n-1) + f(\delta+2n-2)}{f(\delta+2n)} \\ &= 1 + \frac{f(\delta+2n-1) + f(\delta+2n-2)}{f(\delta+2n)} < 2, \end{aligned}$$

since

$$\frac{f(\delta+2n-1) + f(\delta+2n-2) + f(\delta+2n-3)}{f(\delta+2n)} < 1.$$

We claim that $\{\frac{f(\delta+2n+1)}{f(\delta+2n)}\}_{n=1}^{\infty}$ is monotonically increasing. Since

$$\begin{aligned} \frac{f(\delta+2(n+1)+1)}{f(\delta+2(n+1))} - \frac{f(\delta+2n+1)}{f(\delta+2n)} &= \frac{f(\delta+2n+3)}{f(\delta+2n+2)} - \frac{f(\delta+2n+1)}{f(\delta+2n)} \\ &= \frac{f(\delta+2n+3)f(\delta+2n) - f(\delta+2n+1)f(\delta+2n+2)}{f(\delta+2n+2)f(\delta+2n)}, \end{aligned}$$

we show that the numerator of the equation is nonnegative.

$$\begin{aligned} & f(\delta+2n+3)f(\delta+2n) - f(\delta+2n+2)f(\delta+2n+1) \\ &= (f(\delta+2n+2) + f(\delta+2n+1) + f(\delta+2n))f(\delta+2n) - f(\delta+2n+2)f(\delta+2n+1) \\ &= f(\delta+2n+2)[f(\delta+2n) - f(\delta+2n+1)] + f(\delta+2n+1)f(\delta+2n) + [f(\delta+2n)]^2 \\ &> f(\delta+2n)[f(\delta+2n) - f(\delta+2n+1)] + f(\delta+2n+1)f(\delta+2n) + [f(\delta+2n)]^2 \\ &> 3[f(\delta+2n)]^2 \geq 0, \end{aligned}$$

which shows that the sequence is monotonically increasing. By the monotone convergence theorem, $\lim_{n\to\infty} \frac{f(\delta+2n+1)}{f(\delta+2n)} = \lim_{x\to\infty} \frac{f(x+1)}{f(x)}$ exists. $\square$

References

[1] K. Atanasov et al., New Visual perspectives on Fibonacci numbers, *World Sci. Pub. Co.,* New Jersey, 2002.

[2] R. A. Dunlap, The Golden Ratio and Fibonacci numbers, *World Sci. Pub. Co.,* New Jersey, 1997.

[3] J. S. Han, H. S. Kim, J. Neggers, The Fibonacci norm of a positive integer n-observations and conjectures, *Int. J. Number Theory,* **6,** 2010, 371–385.

[4] J. S. Han, H. S. Kim, J. Neggers, Fibonacci sequenses in groupoids, *Advances in Difference Equations,* **19,** 2012, doi:10.1186-11847-2012-19.

[5] J. S. Han, H. S. Kim, J. Neggers, On Fibonacci functions with Fibonacci numbers, *Advances in Difference Equations,* 2012.

[6] S. M. Jung, Hyers-Ulam stability of Fibonacci functional equation, *Bull. Iranian Math. Soc.,* **35,** 2009, 217–227.

[7] H. S. Kim, J. Neggers, Fibonacci Means and Golden Section Mean, *Computers and Mathematics with Applications,* **56,** 2008, 228–232.