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On Octonion polynomial equations

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Abstract

We show how a general octonion polynomial equation, in the octonion unknown(s) x, of the form

$$\sum_{k=1}^{n}\left(\left(a_k x^k\right) b_k + c_k\left(x^k d_k\right) + \left(A_k x^k\right) B_k + C_k\left(x^k D_k\right)\right) = E$$

where $n \in \{1, 2, ...\}$ and E and all coefficients are octonions, can be reduced to a system of eight polynomial equations with real coefficients in eight real unknowns. We consider the solution of these systems with the use of Mathematica version 9 and use the concept of a Gröbner basis to count the number of solutions of such equations.

1 Introduction

Using the *Cayley-Dickson construction* [10, 1, 5] we can define the set $\mathbb{O}$ of *octonions* or *Cayley numbers* as

$$\mathbb{O} = \{(p, q) \,/\, p, q \in \mathbb{H}\}$$

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where $\mathbb{H}$ is the set of *quaternions*, i.e.,

$$\mathbb{H} = \{q = a + bi + cj + dk \,/\, a, b, c, d \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}$$

with conjugate $q^* = a - bi - cj - dk$ and norm $\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}$. Octonion addition and scalar multiplication by $\lambda \in \mathbb{R}$ are defined pairwise, i.e.,

$$(p_1, q_1) + (p_2, q_2) = (p_1 + p_2, q_1 + q_2)\,,\ \lambda(p, q) = (\lambda p, \lambda q)$$

and the, in general non-associative, product of two octonions (p_1, q_1) and (p_2, q_2) can be defined [10] as

$$(p_1, q_1)(p_2, q_2) = (p_1 p_2 - q_2^* q_1, q_2 p_1 + q_1 p_2^*)$$

With these operations $\mathbb{R}$ becomes a nonassociative , alternative *normed division algebra* over the reals [1, 9]. The conjugate $(p, q)^*$ and norm $\|(p, q)\|$ of an octonion (p, q) are defined by

$$(p, q)^* := (p^*, -q)$$

$$\|(p, q)\|^2 := (p, q)^*(p, q) = (\|p\|^2 + \|q\|^2, 0)$$

with multiplication identity element $(1, 0)$ and multiplicative inverse

$$(p, q)^{-1} := \frac{1}{\|p\|^2 + \|q\|^2}(p, q)^*$$

for each nonzero octonion (p, q). Using the quaternion complex matrix representation

$$q = a + bi + cj + dk = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix} \tag{1.1}$$

in what follows we will think of octonions as pairs of such matrices. The study of octonion equations has been of recent interest [2, 6, 7, 8]. In this note we consider polynomial octonion equations of the form

$$\sum_{k=1}^{n} \left(\left(a_k x^k \right) b_k + c_k \left(x^k d_k \right) + \left(A_k x^k \right) B_k + C_k \left(x^k D_k \right) \right) = E \tag{1.2}$$

where $n \in \{1,2,...\}$ and x, E and all coefficients are octonions. If $x=(p,q)$ is a solution of equation (1.2) where

$$p=\begin{pmatrix} x_1+ix_2 & x_3+ix_4 \\ -x_3+ix_4 & x_1-ix_2 \end{pmatrix},\ q=\begin{pmatrix} x_5+ix_6 & x_7+ix_8 \\ -x_7+ix_8 & x_5-ix_6 \end{pmatrix}$$

then, as in [3] and [4], equation (1.2) can be reduced to a, generally non-linear, system of eight real polynomial equations

$$f_i=f_i(x_1,x_2,...,x_8)=0\ ,\ i=1,2,...,8 \tag{1.3}$$

Because the algebra of octonions is alternative, the n-th power x^n of an octonion $x=(p,q)$ can be defined via repeated multiplication without worrying about associativity.

2 The equations $(ax)b=c$ and $a(xb)=c$

Letting

$$a=\left(\begin{pmatrix} a_1+ia_2 & a_3+ia_4 \\ -a_3+ia_4 & a_1-ia_2 \end{pmatrix},\begin{pmatrix} a_5+ia_6 & a_7+ia_8 \\ -a_7+ia_8 & a_5-ia_6 \end{pmatrix}\right)$$

$$b=\left(\begin{pmatrix} b_1+ib_2 & b_3+ib_4 \\ -b_3+ib_4 & b_1-ib_2 \end{pmatrix},\begin{pmatrix} b_5+ib_6 & b_7+ib_8 \\ -b_7+ib_8 & b_5-ib_6 \end{pmatrix}\right)$$

$$c=\left(\begin{pmatrix} c_1+ic_2 & c_3+ic_4 \\ -c_3+ic_4 & c_1-ic_2 \end{pmatrix},\begin{pmatrix} c_5+ic_6 & c_7+ic_8 \\ -c_7+ic_8 & c_5-ic_6 \end{pmatrix}\right)$$

and

$$x=\left(\begin{pmatrix} x_1+ix_2 & x_3+ix_4 \\ -x_3+ix_4 & x_1-ix_2 \end{pmatrix},\begin{pmatrix} x_5+ix_6 & x_7+ix_8 \\ -x_7+ix_8 & x_5-ix_6 \end{pmatrix}\right)$$

the octonion equation $(ax)b=c$ is reduced to the linear system

$$f_1(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)=-c_1+(a_1b_1-a_2b_2-a_3b_3-a_4b_4-a_5b_5-a_6b_6-a_7b_7-a_8b_8)x_1$$

$$+(-a_2b_1-a_1b_2-a_4b_3+a_3b_4-a_6b_5+a_5b_6+a_8b_7-a_7b_8)x_2$$

$$+(-a_3b_1 + a_4b_2 - a_1b_3 - a_2b_4 - a_7b_5 - a_8b_6 + a_5b_7 + a_6b_8)x_3$$
$$+(-a_4b_1 - a_3b_2 + a_2b_3 - a_1b_4 - a_8b_5 + a_7b_6 - a_6b_7 + a_5b_8)x_4$$
$$+(-a_5b_1 + a_6b_2 + a_7b_3 + a_8b_4 - a_1b_5 - a_2b_6 - a_3b_7 - a_4b_8)x_5$$
$$+(-a_6b_1 - a_5b_2 + a_8b_3 - a_7b_4 + a_2b_5 - a_1b_6 + a_4b_7 - a_3b_8)x_6$$
$$+(-a_7b_1 - a_8b_2 - a_5b_3 + a_6b_4 + a_3b_5 - a_4b_6 - a_1b_7 + a_2b_8)x_7$$
$$+(-a_8b_1 + a_7b_2 - a_6b_3 - a_5b_4 + a_4b_5 + a_3b_6 - a_2b_7 - a_1b_8)x_8 = 0$$
$$f_2(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$$
$$= c_2 + (-a_2b_1 - a_1b_2 + a_4b_3 - a_3b_4 + a_6b_5 - a_5b_6 - a_8b_7 + a_7b_8)x_1$$
$$+(-a_1b_1 + a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 - a_8b_8)x_2$$
$$+(a_4b_1 + a_3b_2 + a_2b_3 - a_1b_4 + a_8b_5 - a_7b_6 + a_6b_7 - a_5b_8)x_3$$
$$+(-a_3b_1 + a_4b_2 + a_1b_3 + a_2b_4 - a_7b_5 - a_8b_6 + a_5b_7 + a_6b_8)x_4$$
$$+(a_6b_1 + a_5b_2 - a_8b_3 + a_7b_4 + a_2b_5 - a_1b_6 - a_4b_7 + a_3b_8)x_5$$
$$+(-a_5b_1 + a_6b_2 + a_7b_3 + a_8b_4 + a_1b_5 + a_2b_6 - a_3b_7 - a_4b_8)x_6$$
$$+(-a_8b_1 + a_7b_2 - a_6b_3 - a_5b_4 + a_4b_5 + a_3b_6 + a_2b_7 + a_1b_8)x_7$$
$$+(a_7b_1 + a_8b_2 + a_5b_3 - a_6b_4 - a_3b_5 + a_4b_6 - a_1b_7 + a_2b_8)x_8 = 0$$
$$f_3(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = -c_3 + (a_3b_1 + a_4b_2 + a_1b_3 - a_2b_4 - a_7b_5 - a_8b_6 + a_5b_7 + a_6b_8)x_1$$
$$+(a_4b_1 - a_3b_2 - a_2b_3 - a_1b_4 + a_8b_5 - a_7b_6 + a_6b_7 - a_5b_8)x_2 +$$
$$(a_1b_1 + a_2b_2 - a_3b_3 + a_4b_4 + a_5b_5 + a_6b_6 + a_7b_7 + a_8b_8)x_3$$
$$+(-a_2b_1 + a_1b_2 - a_4b_3 - a_3b_4 - a_6b_5 + a_5b_6 + a_8b_7 - a_7b_8)x_4$$
$$+(-a_7b_1 - a_8b_2 - a_5b_3 + a_6b_4 - a_3b_5 - a_4b_6 + a_1b_7 + a_2b_8)x_5$$
$$+(-a_8b_1 + a_7b_2 - a_6b_3 - a_5b_4 + a_4b_5 - a_3b_6 - a_2b_7 + a_1b_8)x_6$$
$$+(a_5b_1 - a_6b_2 - a_7b_3 - a_8b_4 - a_1b_5 + a_2b_6 - a_3b_7 + a_4b_8)x_7$$
$$+(a_6b_1 + a_5b_2 - a_8b_3 + a_7b_4 - a_2b_5 - a_1b_6 - a_4b_7 - a_3b_8)x_8 = 0$$
$$f_4(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = c_4 + (-a_4b_1 + a_3b_2 - a_2b_3 - a_1b_4 + a_8b_5 - a_7b_6 + a_6b_7 - a_5b_8)x_1$$

$$+(a_3b_1 + a_4b_2 - a_1b_3 + a_2b_4 + a_7b_5 + a_8b_6 - a_5b_7 - a_6b_8)x_2$$
$$+(-a_2b_1 + a_1b_2 + a_4b_3 + a_3b_4 - a_6b_5 + a_5b_6 + a_8b_7 - a_7b_8)x_3$$
$$+(-a_1b_1 - a_2b_2 - a_3b_3 + a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 - a_8b_8)x_4$$
$$+(a_8b_1 - a_7b_2 + a_6b_3 + a_5b_4 + a_4b_5 - a_3b_6 + a_2b_7 - a_1b_8)x_5$$
$$+(-a_7b_1 - a_8b_2 - a_5b_3 + a_6b_4 + a_3b_5 + a_4b_6 + a_1b_7 + a_2b_8)x_6$$
$$+(a_6b_1 + a_5b_2 - a_8b_3 + a_7b_4 - a_2b_5 - a_1b_6 + a_4b_7 + a_3b_8)x_7$$
$$+(-a_5b_1 + a_6b_2 + a_7b_3 + a_8b_4 + a_1b_5 - a_2b_6 - a_3b_7 + a_4b_8)x_8 = 0$$
$$f_5(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = -c_5+(a_5b_1+a_6b_2+a_7b_3+a_8b_4+a_1b_5-a_2b_6-a_3b_7-a_4b_8)x_1$$
$$+(a_6b_1 - a_5b_2 - a_8b_3 + a_7b_4 - a_2b_5 - a_1b_6 - a_4b_7 + a_3b_8)x_2$$
$$+(a_7b_1 + a_8b_2 - a_5b_3 - a_6b_4 - a_3b_5 + a_4b_6 - a_1b_7 - a_2b_8)x_3$$
$$+(a_8b_1 - a_7b_2 + a_6b_3 - a_5b_4 - a_4b_5 - a_3b_6 + a_2b_7 - a_1b_8)x_4$$
$$+(a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 - a_5b_5 + a_6b_6 + a_7b_7 + a_8b_8)x_5$$
$$+(-a_2b_1 + a_1b_2 - a_4b_3 + a_3b_4 - a_6b_5 - a_5b_6 + a_8b_7 - a_7b_8)x_6$$
$$+(-a_3b_1 + a_4b_2 + a_1b_3 - a_2b_4 - a_7b_5 - a_8b_6 - a_5b_7 + a_6b_8)x_7$$
$$+(-a_4b_1 - a_3b_2 + a_2b_3 + a_1b_4 - a_8b_5 + a_7b_6 - a_6b_7 - a_5b_8)x_8 = 0$$
$$f_6(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = c_6+(-a_6b_1+a_5b_2-a_8b_3+a_7b_4-a_2b_5-a_1b_6-a_4b_7+a_3b_8)x_1$$
$$+(a_5b_1 + a_6b_2 - a_7b_3 - a_8b_4 - a_1b_5 + a_2b_6 + a_3b_7 + a_4b_8)x_2$$
$$+(-a_8b_1 + a_7b_2 + a_6b_3 - a_5b_4 + a_4b_5 + a_3b_6 - a_2b_7 + a_1b_8)x_3$$
$$+(a_7b_1 + a_8b_2 + a_5b_3 + a_6b_4 - a_3b_5 + a_4b_6 - a_1b_7 - a_2b_8)x_4$$
$$+(-a_2b_1 + a_1b_2 - a_4b_3 + a_3b_4 + a_6b_5 + a_5b_6 + a_8b_7 - a_7b_8)x_5$$
$$+(-a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 + a_6b_6 - a_7b_7 - a_8b_8)x_6$$
$$+(-a_4b_1 - a_3b_2 + a_2b_3 + a_1b_4 - a_8b_5 + a_7b_6 + a_6b_7 + a_5b_8)x_7$$
$$+(a_3b_1 - a_4b_2 - a_1b_3 + a_2b_4 + a_7b_5 + a_8b_6 - a_5b_7 + a_6b_8)x_8 = 0$$
$$f_7(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = -c_7+(a_7b_1-a_8b_2-a_5b_3+a_6b_4+a_3b_5-a_4b_6+a_1b_7+a_2b_8)x_1$$

$$+(-a_8b_1 - a_7b_2 - a_6b_3 - a_5b_4 + a_4b_5 + a_3b_6 - a_2b_7 + a_1b_8)x_2$$
$$+(-a_5b_1 + a_6b_2 - a_7b_3 + a_8b_4 + a_1b_5 - a_2b_6 - a_3b_7 - a_4b_8)x_3$$
$$+(a_6b_1 + a_5b_2 - a_8b_3 - a_7b_4 - a_2b_5 - a_1b_6 - a_4b_7 + a_3b_8)x_4$$
$$+(a_3b_1 - a_4b_2 - a_1b_3 + a_2b_4 - a_7b_5 + a_8b_6 - a_5b_7 - a_6b_8)x_5$$
$$+(-a_4b_1 - a_3b_2 + a_2b_3 + a_1b_4 - a_8b_5 - a_7b_6 - a_6b_7 + a_5b_8)x_6$$
$$+(a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5 + a_6b_6 - a_7b_7 + a_8b_8)x_7$$
$$+(a_2b_1 - a_1b_2 + a_4b_3 - a_3b_4 + a_6b_5 - a_5b_6 - a_8b_7 - a_7b_8)x_8 = 0$$
$$f_8(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$$
$$= c_8 + (-a_8b_1 - a_7b_2 + a_6b_3 + a_5b_4 - a_4b_5 - a_3b_6 + a_2b_7 - a_1b_8)x_1$$
$$+(-a_7b_1 + a_8b_2 - a_5b_3 + a_6b_4 + a_3b_5 - a_4b_6 + a_1b_7 + a_2b_8)x_2$$
$$+(a_6b_1 + a_5b_2 + a_8b_3 + a_7b_4 - a_2b_5 - a_1b_6 - a_4b_7 + a_3b_8)x_3$$
$$+(a_5b_1 - a_6b_2 - a_7b_3 + a_8b_4 - a_1b_5 + a_2b_6 + a_3b_7 + a_4b_8)x_4$$
$$+(-a_4b_1 - a_3b_2 + a_2b_3 + a_1b_4 + a_8b_5 + a_7b_6 - a_6b_7 + a_5b_8)x_5$$
$$+(-a_3b_1 + a_4b_2 + a_1b_3 - a_2b_4 - a_7b_5 + a_8b_6 + a_5b_7 + a_6b_8)x_6$$
$$+(a_2b_1 - a_1b_2 + a_4b_3 - a_3b_4 + a_6b_5 - a_5b_6 + a_8b_7 + a_7b_8)x_7$$
$$+(-a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 + a_8b_8)x_8 = 0$$

Similarly, the octonion equation $a(xb) = c$ is reduced to the linear system

$$f_1(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = -c_1 + (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 - a_8b_8)x_1$$
$$+(-a_2b_1 - a_1b_2 - a_4b_3 + a_3b_4 - a_6b_5 + a_5b_6 + a_8b_7 - a_7b_8)x_2$$
$$+(-a_3b_1 + a_4b_2 - a_1b_3 - a_2b_4 - a_7b_5 - a_8b_6 + a_5b_7 + a_6b_8)x_3$$
$$+(-a_4b_1 - a_3b_2 + a_2b_3 - a_1b_4 - a_8b_5 + a_7b_6 - a_6b_7 + a_5b_8)x_4$$
$$+(-a_5b_1 + a_6b_2 + a_7b_3 + a_8b_4 - a_1b_5 - a_2b_6 - a_3b_7 - a_4b_8)x_5$$
$$+(-a_6b_1 - a_5b_2 + a_8b_3 - a_7b_4 + a_2b_5 - a_1b_6 + a_4b_7 - a_3b_8)x_6$$
$$+(-a_7b_1 - a_8b_2 - a_5b_3 + a_6b_4 + a_3b_5 - a_4b_6 - a_1b_7 + a_2b_8)x_7$$

$$+(-a_8b_1 + a_7b_2 - a_6b_3 - a_5b_4 + a_4b_5 + a_3b_6 - a_2b_7 - a_1b_8)x_8 = 0$$

$$f_2(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = c_2+(-a_2b_1-a_1b_2+a_4b_3-a_3b_4+a_6b_5-a_5b_6-a_8b_7+a_7b_8)x_1$$
$$+(-a_1b_1 + a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 - a_8b_8)x_2$$
$$+(a_4b_1 + a_3b_2 + a_2b_3 - a_1b_4 - a_8b_5 + a_7b_6 - a_6b_7 + a_5b_8)x_3$$
$$+(-a_3b_1 + a_4b_2 + a_1b_3 + a_2b_4 + a_7b_5 + a_8b_6 - a_5b_7 - a_6b_8)x_4$$
$$+(a_6b_1 + a_5b_2 + a_8b_3 - a_7b_4 + a_2b_5 - a_1b_6 + a_4b_7 - a_3b_8)x_5$$
$$+(-a_5b_1 + a_6b_2 - a_7b_3 - a_8b_4 + a_1b_5 + a_2b_6 + a_3b_7 + a_4b_8)x_6$$
$$+(-a_8b_1 + a_7b_2 + a_6b_3 + a_5b_4 - a_4b_5 - a_3b_6 + a_2b_7$$
$$+a_1b_8)x_7 + (a_7b_1 + a_8b_2 - a_5b_3 + a_6b_4 + a_3b_5 - a_4b_6 - a_1b_7 + a_2b_8)x_8 = 0$$

$$f_3(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = -c_3+(a_3b_1+a_4b_2+a_1b_3-a_2b_4-a_7b_5-a_8b_6+a_5b_7+a_6b_8)x_1$$
$$+(a_4b_1 - a_3b_2 - a_2b_3 - a_1b_4 - a_8b_5 + a_7b_6 - a_6b_7 + a_5b_8)x_2$$
$$+(a_1b_1 + a_2b_2 - a_3b_3 + a_4b_4 + a_5b_5 + a_6b_6 + a_7b_7 + a_8b_8)x_3$$
$$+(-a_2b_1 + a_1b_2 - a_4b_3 - a_3b_4 + a_6b_5 - a_5b_6 - a_8b_7 + a_7b_8)x_4$$
$$+(-a_7b_1 + a_8b_2 - a_5b_3 - a_6b_4 - a_3b_5 + a_4b_6 + a_1b_7 - a_2b_8)x_5$$
$$+(-a_8b_1 - a_7b_2 - a_6b_3 + a_5b_4 - a_4b_5 - a_3b_6 + a_2b_7 + a_1b_8)x_6$$
$$+(a_5b_1 + a_6b_2 - a_7b_3 + a_8b_4 - a_1b_5 - a_2b_6 - a_3b_7 - a_4b_8)x_7$$
$$+(a_6b_1 - a_5b_2 - a_8b_3 - a_7b_4 + a_2b_5 - a_1b_6 + a_4b_7 - a_3b_8)x_8 = 0$$

$$f_4(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = c_4+(-a_4b_1+a_3b_2-a_2b_3-a_1b_4+a_8b_5-a_7b_6+a_6b_7-a_5b_8)x_1$$
$$+(a_3b_1 + a_4b_2 - a_1b_3 + a_2b_4 - a_7b_5 - a_8b_6 + a_5b_7 + a_6b_8)x_2$$
$$+(-a_2b_1 + a_1b_2 + a_4b_3 + a_3b_4 + a_6b_5 - a_5b_6 - a_8b_7 + a_7b_8)x_3$$
$$+(-a_1b_1 - a_2b_2 - a_3b_3 + a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 - a_8b_8)x_4$$
$$+(a_8b_1 + a_7b_2 - a_6b_3 + a_5b_4 + a_4b_5 + a_3b_6 - a_2b_7 - a_1b_8)x_5$$
$$+(-a_7b_1 + a_8b_2 + a_5b_3 + a_6b_4 - a_3b_5 + a_4b_6 + a_1b_7 - a_2b_8)x_6$$
$$+(a_6b_1 - a_5b_2 + a_8b_3 + a_7b_4 + a_2b_5 - a_1b_6 + a_4b_7 - a_3b_8)x_7$$

$$+(-a_5b_1 - a_6b_2 - a_7b_3 + a_8b_4 + a_1b_5 + a_2b_6 + a_3b_7 + a_4b_8)x_8 = 0$$

$$f_5(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) = -c_5+(a_5b_1+a_6b_2+a_7b_3+a_8b_4+a_1b_5-a_2b_6-a_3b_7-a_4b_8)x_1$$

$$+(a_6b_1 - a_5b_2 + a_8b_3 - a_7b_4 - a_2b_5 - a_1b_6 + a_4b_7 - a_3b_8)x_2$$

$$+(a_7b_1 - a_8b_2 - a_5b_3 + a_6b_4 - a_3b_5 - a_4b_6 - a_1b_7 + a_2b_8)x_3$$

$$+(a_8b_1 + a_7b_2 - a_6b_3 - a_5b_4 - a_4b_5 + a_3b_6 - a_2b_7 - a_1b_8)x_4$$

$$+(a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 - a_5b_5 + a_6b_6 + a_7b_7 + a_8b_8)x_5$$

$$+(-a_2b_1 + a_1b_2 + a_4b_3 - a_3b_4 - a_6b_5 - a_5b_6 - a_8b_7 + a_7b_8)x_6$$

$$+(-a_3b_1 - a_4b_2 + a_1b_3 + a_2b_4 - a_7b_5 + a_8b_6 - a_5b_7 - a_6b_8)x_7$$

$$+(-a_4b_1 + a_3b_2 - a_2b_3 + a_1b_4 - a_8b_5 - a_7b_6 + a_6b_7 - a_5b_8)x_8 = 0$$

$$f_6(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) = c_6+(-a_6b_1+a_5b_2-a_8b_3+a_7b_4-a_2b_5-a_1b_6-a_4b_7+a_3b_8)x_1$$

$$+(a_5b_1 + a_6b_2 + a_7b_3 + a_8b_4 - a_1b_5 + a_2b_6 - a_3b_7 - a_4b_8)x_2$$

$$+(-a_8b_1 - a_7b_2 + a_6b_3 + a_5b_4 - a_4b_5 + a_3b_6 + a_2b_7 + a_1b_8)x_3$$

$$+(a_7b_1 - a_8b_2 - a_5b_3 + a_6b_4 + a_3b_5 + a_4b_6 - a_1b_7 + a_2b_8)x_4$$

$$+(-a_2b_1 + a_1b_2 + a_4b_3 - a_3b_4 + a_6b_5 + a_5b_6 - a_8b_7 + a_7b_8)x_5$$

$$+(-a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 + a_6b_6 - a_7b_7 - a_8b_8)x_6$$

$$+(-a_4b_1 + a_3b_2 - a_2b_3 + a_1b_4 + a_8b_5 + a_7b_6 + a_6b_7 - a_5b_8)x_7$$

$$+(a_3b_1 + a_4b_2 - a_1b_3 - a_2b_4 - a_7b_5 + a_8b_6 + a_5b_7 + a_6b_8)x_8 = 0$$

$$f_7(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) = -c_7+(a_7b_1-a_8b_2-a_5b_3+a_6b_4+a_3b_5-a_4b_6+a_1b_7+a_2b_8)x_1$$

$$+(-a_8b_1 - a_7b_2 + a_6b_3 + a_5b_4 - a_4b_5 - a_3b_6 - a_2b_7 + a_1b_8)x_2$$

$$+(-a_5b_1 - a_6b_2 - a_7b_3 - a_8b_4 + a_1b_5 + a_2b_6 - a_3b_7 + a_4b_8)x_3$$

$$+(a_6b_1 - a_5b_2 + a_8b_3 - a_7b_4 + a_2b_5 - a_1b_6 - a_4b_7 - a_3b_8)x_4$$

$$+(a_3b_1 + a_4b_2 - a_1b_3 - a_2b_4 - a_7b_5 - a_8b_6 - a_5b_7 + a_6b_8)x_5$$

$$+(-a_4b_1 + a_3b_2 - a_2b_3 + a_1b_4 + a_8b_5 - a_7b_6 - a_6b_7 - a_5b_8)x_6$$

$$+(a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5 + a_6b_6 - a_7b_7 + a_8b_8)x_7$$

$$+(a_2b_1 - a_1b_2 - a_4b_3 + a_3b_4 - a_6b_5 + a_5b_6 - a_8b_7 - a_7b_8)x_8 = 0$$

$$f_8(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = c_8 + (-a_8b_1 - a_7b_2 + a_6b_3 + a_5b_4 - a_4b_5 - a_3b_6 + a_2b_7 - a_1b_8)x_1$$

$$+(-a_7b_1 + a_8b_2 + a_5b_3 - a_6b_4 - a_3b_5 + a_4b_6 + a_1b_7 + a_2b_8)x_2$$

$$+(a_6b_1 - a_5b_2 + a_8b_3 - a_7b_4 + a_2b_5 - a_1b_6 + a_4b_7 + a_3b_8)x_3$$

$$+(a_5b_1 + a_6b_2 + a_7b_3 + a_8b_4 - a_1b_5 - a_2b_6 - a_3b_7 + a_4b_8)x_4$$

$$+(-a_4b_1 + a_3b_2 - a_2b_3 + a_1b_4 + a_8b_5 - a_7b_6 + a_6b_7 + a_5b_8)x_5$$

$$+(-a_3b_1 - a_4b_2 + a_1b_3 + a_2b_4 + a_7b_5 + a_8b_6 - a_5b_7 + a_6b_8)x_6$$

$$+(a_2b_1 - a_1b_2 - a_4b_3 + a_3b_4 - a_6b_5 + a_5b_6 + a_8b_7 + a_7b_8)x_7$$

$$+(-a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 + a_8b_8)x_8 = 0$$

In both cases the determinant of the coefficient matrix of the $x_i's$ is

$$D = (a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2+a_7^2+a_8^2)^4(b_1^2+b_2^2+b_3^2+b_4^2+b_5^2+b_6^2+b_7^2+b_8^2)^4 = (\|a\|\,\|b\|)^8$$

Thus, if both octonions a and b are nonzero then the octonion equations $(ax)b = c$ and $a(xb) = c$ have a unique solution. The form of the determinant D is justified by the fact that, since $\mathbb{O}$ is a composition algebra, equations $(ax)b = c$ and $a(xb) = c$ imply that $\|a\,\|x\|\,\|b\| = \|c\|$. If $c = \mathbf{0}$ then $x = \mathbf{0}$ is a solution. If $c \neq \mathbf{0}$ then a solution x exists provided that $\|a\|, \|b\| \neq 0$.

Example 1.

This example was considered in [2]. For

$$a_1 = b_1 = 1, a_2 = b_2 = 4, a_3 = b_3 = -1, a_4 = b_4 = 0$$

$$a_5 = b_5 = 3, a_6 = b_6 = 1, a_7 = b_7 = -7, a_8 = b_8 = 5$$

$$c_1 = -1, c_2 = 2, c_3 = 3, c_4 = 5, c_5 = -3, c_6 = 0, c_7 = -1, c_8 = 0$$

the system becomes

$$\begin{aligned}
-100x_1 - 8x_2 + 2x_3 - 6x_5 - 2x_6 + 14x_7 - 10x_8 &= -1\\
-8x_1 - 70x_2 - 8x_3 + 24x_5 + 8x_6 - 56x_7 + 40x_8 &= -2\\
-2x_1 + 8x_2 + 100x_3 + 6x_5 + 2x_6 - 14x_7 + 10x_8 &= 3\\
-102x_4 &= -5\\
6x_1 - 24x_2 + 6x_3 + 84x_5 - 6x_6 + 42x_7 - 30x_8 &= -3\\
-2x_1 + 8x_2 - 2x_3 + 6x_5 - 100x_6 - 14x_7 + 10x_8 &= 0\\
-14x_1 + 56x_2 - 14x_3 + 42x_5 + 14x_6 + 4x_7 + 70x_8 &= -1\\
-10x_1 + 40x_2 - 10x_3 + 30x_5 + 10x_6 - 70x_7 - 52x_8 &= 0
\end{aligned}$$

The determinant of the coefficient matrix is equal to $11716593810022656 \neq 0$ and the unique solution of $a(xa) = (ax)a = axa = c$ is

$$x_1 = \frac{53}{5202}, x_2 = \frac{47}{2601}, x_3 = \frac{155}{5202}, x_4 = \frac{5}{102}$$
$$x_5 = -\frac{53}{1734}, x_6 = -\frac{1}{2601}, x_7 = -\frac{37}{5202}, x_8 = -\frac{5}{2601}$$

Example 2.

For

$$a_1 = 1, a_2 = -1, a_3 = 3, a_4 = -1, a_5 = -2, a_6 = 3, a_7 = 2, a_8 = 1$$
$$b_1 = 4, b_2 = -3, b_3 = 2, b_4 = 1, b_5 = -1, b_6 = -3, b_7 = 1, b_8 = -2$$
$$c_1 = -6, c_2 = 3, c_3 = -2, c_4 = 5, c_5 = 4, c_6 = 2, c_7 = -1, c_8 = 0$$

the linear system for $a(xb) = c$ is

$$\begin{aligned}
3x_1 + 26x_2 - 13x_3 + 6x_4 - 3x_5 - 9x_6 - 3x_7 - 19x_8 &= -6\\
-12x_1 + x_2 - 20x_3 - 5x_4 + 27x_5 + x_6 - x_7 + 7x_8 &= -3\\
15x_1 + 2x_2 - 7x_3 - 14x_4 - 5x_5 - x_6 - 27x_7 + 11x_8 &= -2\\
9x_2 - 12x_3 - 7x_4 - 15x_5 - 7x_6 + 19x_7 + 21x_8 &= -5\\
-21x_1 + 13x_2 + 19x_3 - 13x_4 + x_5 - 12x_6 - 7x_7 + 4x_8 &= 4\\
-9x_1 - 13x_2 - 7x_3 + 19x_4 - 6x_5 - 23x_6 - 10x_7 + 5x_8 &= -2\\
15x_1 + 13x_2 + 13x_3 + 17x_4 + 15x_5 - 4x_6 + x_7 + 16x_8 &= -1\\
15x_1 - 9x_2 + 3x_3 - 15x_4 + 10x_5 - 23x_6 + 10x_7 - 9x_8 &= 0
\end{aligned}$$

and its unique solution is

$$x_1 = -\frac{31}{450}, x_2 = -\frac{143}{1350}, x_3 = \frac{289}{1350}, x_4 = -\frac{13}{270}$$

$$x_5 = \frac{23}{1350}, x_6 = \frac{1}{15}, x_7 = -\frac{29}{1350}, x_8 = -\frac{22}{675}$$

Similarly, the linear system for $(ax)b = c$ becomes

$$\begin{aligned}
3x_1 + 26x_2 - 13x_3 + 6x_4 - 3x_5 - 9x_6 - 3x_7 - 19x_8 &= -6 \\
-12x_1 + x_2 - 12x_3 - 11x_4 + 17x_5 + x_6 - 25x_7 + 5x_8 &= -3 \\
15x_1 + 10x_2 - 7x_3 + 14x_4 + 5x_5 - 5x_6 - x_7 + 27x_8 &= -2 \\
15x_2 + 16x_3 - 7x_4 + 25x_5 + 5x_6 + 13x_7 - x_8 &= -5 \\
-21x_1 + 3x_2 + 9x_3 + 27x_4 + x_5 + 8x_6 - 5x_7 &= 4 \\
-9x_1 - 13x_2 - 11x_3 + 7x_4 + 14x_5 - 23x_6 + 14x_7 - 3x_8 &= -2 \\
15x_1 - 11x_2 - 13x_3 + 11x_4 + 13x_5 + 20x_6 + x_7 - 12x_8 &= -1 \\
15x_1 - 7x_2 + 19x_3 + 7x_4 + 6x_5 - 15x_6 - 18x_7 - 9x_8 &= 0
\end{aligned}$$

and its unique solution is

$$x_1 = -\frac{31}{450}, x_2 = -\frac{41}{270}, x_3 = \frac{119}{1350}, x_4 = \frac{29}{450}$$

$$x_5 = -\frac{41}{270}, x_6 = \frac{47}{675}, x_7 = -\frac{19}{1350}, x_8 = \frac{34}{675}$$

For both systems, the determinant of the coefficient matrix is $3321506250000 \neq 0$.

3 The general case

Letting

$$x^k = \left(\begin{pmatrix} x_1(k) + ix_2(k) & x_3(k) + ix_4(k) \\ -x_3(k) + ix_4(k) & x_1(k) - ix_2(k) \end{pmatrix}, \begin{pmatrix} x_5(k) + ix_6(k) & x_7(k) + ix_8(k) \\ -x_7(k) + ix_8(k) & x_5(k) - ix_6(k) \end{pmatrix} \right)$$

$$E = \left(\begin{pmatrix} E_1 + iE_2 & E_3 + iE_4 \\ -E_3 + iE_4 & E_1 - iE_2 \end{pmatrix}, \begin{pmatrix} E_5 + iE_6 & E_7 + iE_8 \\ -E_7 + iE_8 & E_5 - iE_6 \end{pmatrix} \right)$$

and for $y \in \{a, b, c, d, A, B, C, D\}$,

$$y_k = \left(\begin{pmatrix} y_1(k) + iy_2(k) & y_3(k) + iy_4(k) \\ -y_3(k) + iy_4(k) & y_1(k) - iy_2(k) \end{pmatrix}, \begin{pmatrix} y_5(k) + iy_6(k) & y_7(k) + iy_8(k) \\ -y_7(k) + iy_8(k) & y_5(k) - iy_6(k) \end{pmatrix} \right)$$

the general octonion polynomial equation (1.2) is reduced to a system of the form (1.3). As in [3] and [4], after carrying out the octonion multiplications and additions we end up with two equal pairs of quaternions. Due to the special form of the matrices that represent quaternions, it suffices to equate the real and imaginary parts of the corresponding entries in the first rows. It is not hard to derive formulas for the functions f_i using Mathematica but the expressions are rather long. Instead, we will illustrate the situation with an example.

Example 3.

For $k = 1, 2$ letting

$$a_1(k) = k + 1, a_2(k) = 2k - 1, a_3(k) = k - 1, a_4(k) = -2k$$

$$a_5(k) = -k, a_6(k) = k, a_7(k) = 3k, a_8(k) = k - 1$$

$$b_1(k) = k - 1, b_2(k) = k - 2, b_3(k) = 2k, b_4(k) = 2k + 3$$

$$b_5(k) = k, b_6(k) = k + 1, b_7(k) = -k, b_8(k) = k + 1$$

$$c_1(k) = k, c_2(k) = 3k + 2, c_3(k) = k, c_4(k) = k + 1$$

$$c_5(k) = 3k, c_6(k) = -k + 1, c_7(k) = k + 1, c_8(k) = -2k + 1$$

$$d_1(k) = 4k - 1, d_2(k) = k + 2, d_3(k) = -k, d_4(k) = 2k + 1$$

$$d_5(k) = -2k, d_6(k) = -k, d_7(k) = 3k, d_8(k) = k$$

$$A_1(k) = k, A_2(k) = -2k - 1, A_3(k) = 2k + 1, A_4(k) = 2k$$

$$A_5(k) = k, A_6(k) = -k, A_7(k) = -2k, A_8(k) = k + 1$$

$$B_1(k) = k + 2, B_2(k) = -k + 2, B_3(k) = k, B_4(k) = k + 3$$

$$B_5(k) = -k, B_6(k) = -k+1, B_7(k) = k-3, B_8(k) = k$$
$$C_1(k) = 3k, C_2(k) = -k+2, C_3(k) = -k, C_4(k) = k-1$$
$$C_5(k) = k, C_6(k) = k-1, C_7(k) = k-2, C_8(k) = 2k+1$$
$$D_1(k) = k-1, D_2(k) = -k+2, D_3(k) = k, D_4(k) = k+1$$
$$D_5(k) = k, D_6(k) = -k+3, D_7(k) = -k, D_8(k) = 2k-3$$
$$E_1 = -1, E_2 = 2, E_3 = 3, E_4 = 1, E_5 = -2, E_6 = -1, E_7 = 2, E_8 = -3$$

$$x^2 = \left(\begin{pmatrix} x_1(2) + ix_2(2) & x_3(2) + ix_4(2) \\ -x_3(2) + ix_4(2) & x_1(2) - ix_2(2) \end{pmatrix}, \begin{pmatrix} x_5(2) + ix_6(2) & x_7(2) + ix_8(2) \\ -x_7(2) + ix_8(2) & x_5(2) - ix_6(2) \end{pmatrix} \right)$$

where

$$x_1(2) = x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2 - x_8^2,\ x_2(2) = 2x_1x_2$$

$x_3(2) = 2x_1x_3,\ x_4(2) = 2x_1x_4,\ x_5(2) = 2x_1x_5,\ x_6(2) = 2x_1x_6,\ x_7(2) = 2x_1x_7,\ x_8(2) = 2x_1x_8$ equation (1.2) becomes

$$\sum_{k=1}^{2} \left(\left(a_k x^k\right) b_k + c_k \left(x^k d_k\right) + \left(A_k x^k\right) B_k + C_k \left(x^k D_k\right) \right) = E \qquad (3.4)$$

and yields the system $\{f_i = f_i(x_1, ..., x_8) = 0, i = 1, ..., 8\}$ where

$f_1 = 1-12x_1-18x_1^2-16x_2-160x_1x_2+18x_2^2-18x_3-130x_1x_3+18x_3^2-46x_4-190x_1x_4+18x_4^2-x_5$
$-76x_1x_5+18x_5^2-29x_6-86x_1x_6+18x_6^2+9x_7+78x_1x_7+18x_7^2-47x_8-258x_1x_8+18x_8^2$

$f_2 = -2+6x_1+14x_1^2+32x_1x_2-14x_2^2+24x_3+126x_1x_3-14x_3^2-4x_4+18x_1x_4-14x_4^2-9x_5$
$-46x_1x_5-14x_5^2-13x_6-56x_1x_6-14x_6^2-33x_7-246x_1x_7-14x_7^2-7x_8-10x_1x_8-14x_8^2$

$f_3 = -3+20x_1+49x_1^2-24x_2-102x_1x_2-49x_2^2+22x_3+136x_1x_3-49x_3^2-12x_4-72x_1x_4-49x_4^2+x_5$
$+2x_1x_5-49x_5^2+13x_6+194x_1x_6-49x_6^2+9x_7+36x_1x_7-49x_7^2-33x_8-230x_1x_8-49x_8^2$

$f_4 = -1+24x_1+91x_1^2-36x_2-210x_1x_2-91x_2^2-6x_3-24x_1x_3-91x_3^2+16x_4+120x_1x_4-91x_4^2$
$-x_5+122x_1x_5-91x_5^2+17x_6+134x_1x_6-91x_6^2+15x_7+38x_1x_7-91x_7^2-11x_8-91x_8^2$

$f_5 = 2+27x_1+74x_1^2-5x_2+14x_1x_2-74x_2^2+17x_3+126x_1x_3-74x_3^2-3x_4-146x_1x_4-74x_4^2+38x_5$
$+272x_1x_5-74x_5^2+12x_6+48x_1x_6-74x_6^2+18x_7-10x_1x_7-74x_7^2+18x_8+86x_1x_8-74x_8^2$
$f_6 = 1-17x_1-49x_1^2+19x_2+80x_1x_2+49x_2^2-9x_3-194x_1x_3+49x_3^2-7x_4-58x_1x_4+49x_4^2$
$-10x_5-32x_1x_5+49x_5^2+20x_6+116x_1x_6+49x_6^2-20x_7-134x_1x_7+49x_7^2-10x_8-102x_1x_8+49x_8^2$
$f_7 = -2+x_1+45x_1^2-5x_2+10x_1x_2-45x_2^2-7x_3-112x_1x_3-45x_3^2-45x_4-266x_1x_4-45x_4^2$
$-32x_5-158x_1x_5-45x_5^2+14x_6+110x_1x_6-45x_6^2+8x_7+120x_1x_7-45x_7^2-4x_8+32x_1x_8-45x_8^2$
$f_8 = 3-35x_1-89x_1^2-3x_2-2x_1x_2+89x_2^2+11x_3+66x_1x_3+89x_3^2-11x_4-100x_1x_4+89x_4^2$
$-26x_5-206x_1x_5+89x_5^2-6x_6+54x_1x_6+89x_6^2+10x_7+36x_1x_7+89x_7^2+28x_8+128x_1x_8+89x_8^2$
The command

$$\text{NSolve}[\{f_1 == 0, ..., f_8 == 0\}, \{x_1, ..., x_8\}]$$

produces sixteen solutions $(x_1, .., x_8)$ of which only two are real:
(-0.438302,-0.0917744,-0.0337274,-0.0556383,

$$0.00354174, 0.00138381, -0.00894543, 0.073778) \tag{3.5}$$

and
(0.120854,0.0392421,0.010805,0.0197584,

$$-0.0669689, -0.00859953, 0.0121543, -0.0312448) \tag{3.6}$$

To see that these are the only real solutions we can use Groebner bases. The command

$$\text{GroebnerBasis}[\{f_1, ..., f_8\}, \{x_1, ..., x_8\}]$$

produces eight polynomials $\{g_i = g_i(x_1, ..., x_8), i = 1, ..., 8\}$. The first polynomial g_1 is of degree 16 in x_8 only. The command NSolve gives sixteen complex roots x_8 of g_1 , in agreement with Gauss's Fundamental Theorem of Algebra, of which only two are real:

$$x_8 = -0.031244799138755847 \text{ and } x_8 = 0.07377797762985848$$

The rest of the polynomials $g_2, ..., g_8$ are of degree 1 in $x_7, ..., x_1$ respectively, and of degree 15 in x_8. Thus for each one of the two real values of x_8 found above we find the values of $x_1, ..., x_7$. Therefore there are exactly two real solutions $(x_1, ..., x_8)$ as listed in (3.5) and (3.6).

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