International Journal of Mathematics and
Computer Science, **11**(2016), no. 2, 133–149

Good variation theory: a Tauberian approach to the Riemann Hypothesis

(Dedicated to the memory of Jonathan M. Borwein)

Benoit Cloitre

19 rue Louise Michel
92300 Levallois-Perret, France

email: benoit7848c@yahoo.fr

(Received August 15, 2016, Accepted September 8, 2016)

Abstract

In this note, I present a Tauberian conjecture that I consider to be the simplest and the best Tauberian reformulation of the Riemann Hypothesis (RH) using good variation theory.

Notations and definitions

Let $(a(n))_{n\geq 1}$ be a sequence and let g be a function. We use the notations $A(n) = \sum_{k=1}^{n} a(k)$ and $A_g(n) = \sum_{k=1}^{n} a(k)g\left(\frac{k}{n}\right)$.

The little Mellin transform of g which is Riemann integrable on $]0,1]$ is the analytic continuation of the function $g^\star$ defined for $\Re z < 0$ by

$$g^\star(z) := \int_0^1 g(t)t^{-z-1}dt$$

Next, the analytic index of g is defined by $\eta(g) := \min\{\Re(\rho) \mid g^\star(\rho) = 0\}$.

Key words and phrases: Zeta and L functions, Operator theory, Tauberian theory, Multiplicative number theory, Riemann hypothesis, Ingham summation method.
AMS (MOS) Subject Classifications: 11A05, 26A06, 30D30,40A05,47G10,65Q10.
ISSN 1814-0432, 2016, http://ijmcs.future-in-tech.net

Introduction

Trying to better understand the Riemann hypothesis, I have developed during the past few years my own ideas via experiments. I am now suspecting that the problem (RH) can neither succumb to an analytic approach only nor to an arithmetic approach only. A mixture of arithmetic and analysis seems required and I think to have succeeded in capturing this duality via good variation theory which depends both on arithmetic and on complex analysis. Good variation theory emerged in 2010 when I came across the following two facts

$$\sum_{k=1}^{n} \lambda(k) \left\lfloor \frac{n}{k} \right\rfloor = \lfloor \sqrt{n} \rfloor \tag{1}$$

which is easy to prove, where $\lambda(k) = (-1)^{\Omega(k)}$ is the Liouville lambda function and

$$\sum_{k=1}^{n} \lambda(k) \ll n^{1/2+\varepsilon} \tag{2}$$

which is a statement equivalent to the Riemann hypothesis [Bor]. The appearance of the square root of n in the RHS of (1) and (2) is striking so that I asked myself naively whether we could have (1) $\Rightarrow$ (2)? It turned out that there was no immediate answer to this question and I tried to figure out what was going on. The first progress I made arose when I came across the following Tauberian theorem of Ingham ([Ing], [Kor]).

$$na(n) \geq -C \;\wedge\; \lim_{n\to\infty} \sum_{k=1}^{n} a(k) \frac{k}{n} \left\lfloor \frac{n}{k} \right\rfloor = \ell \Rightarrow \sum_{k=1}^{\infty} a(k) = \ell$$

Indeed, some experiments led me to formulate the following general Tauberian conjecture. Assuming that the condition $na(n) = O(1)$ is satisfied I claim that we have, letting $\Phi(x) := x \left\lfloor \frac{1}{x} \right\rfloor$ denote the Ingham function

$$A_\Phi(n) \sim n^{-\beta} \Rightarrow \begin{cases} 0 < \beta < \frac{1}{2} & A(n) \sim \frac{1-\beta^{-1}}{\zeta(1-\beta)} n^{-\beta} \\ \\ \beta \geq \frac{1}{2} & A(n) \ll n^{-1/2+\varepsilon} \end{cases} \qquad (n \to \infty) \tag{3}$$

which includes (1) $\Rightarrow$ (2) letting $a(n) = \frac{\lambda(n)}{n}$.

Afterwards, it was interesting to introduce the broader concept of functions of good variation as follows.

Function of good variation (primary definition)

A bounded function g which is Riemann integrable on $]0,1]$ is a function of good variation (FGV) of index $\alpha(g)$ if considering the discrete Volterra equation of linear type [1]

$$A_g(n) = n^{-\beta}$$

we have the following two Tauberian properties

1. $\beta < \alpha(g) \Rightarrow A(n) \sim C(\beta) n^{-\beta} \ \ (n \to \infty)$ where $C(\beta) \neq 0$
2. $\beta \geq \alpha(g) \Rightarrow A(n) \ll n^{-\alpha(g)} L(n)$ [2]

So the conjecture (3) is not directly obtained from this primary definition where we consider $A_g(n) = n^{-\beta}$ not the stronger condition $A_g(n) \sim n^{-\beta}$. For our purpose, however, this doesn't matter because this primary definition will suffice to formulate a conjecture interesting enough to write down a new equivalence of RH.

In section 1, I prove that FGV exist taking the simplest ones and I formulate a general conjecture for continuous functions.

In section 2, the Ingham function Φ is generalized to a wider class of similar discontinuous functions: the BHF (broken harmonic functions). This allows me to formulate a Tauberian conjecture for BHF in section 3. Next, in section 4, I prove that the Ingham function satisfies an important condition of the Tauberian conjecture so that RH would be true. In section 5, I prove that generalized Ingham functions satisfy this important condition as well so that the generalized Riemann hypothesis would be true. In section 6, I extend the method to a set of L functions with multiplicative coefficients so that the Grand Riemann Hypothesis would be true.

1 Functions of good variation exist

It seems important to exhibit examples of FGV since it is not obvious to see that they exist. Hereafter, I show that FGV is a consistent concept by proving that affine functions are FGV. In particular, the function $g(x) = \frac{x+1}{2}$ is a FGV of index $\frac{1}{2}$ which is not self evident at first glance.

[1] It is an equation of the type $x(n) = f(n) + \sum_{j=0}^{n} y(n,j)x(j)$ $(n \geq 0)$ where $y(.,.)$ and $f(.)$ are known functions and $x(.)$ is unknown. Cf. for instance [Dib] for some results related to these equations.

[2] L denotes a (Karamata) slowly varying function; i.e., $\forall x > 0$ $\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1$.

1.1 Theorem

Let g be the affine function $g(x) = c_1 x + c_0$, where $c_0, c_1 > 0$. Then g is a FGV of index $\alpha(g) = \frac{c_0}{c_1+c_0}$ according to the primary definition of FGV.

More precisely, if $A_g(n) = n^{-\beta}$ we have seven cases to consider summarized in the following table where $g^\star(z) = \frac{c_1}{1-z} - \frac{c_0}{z}$ is the little Mellin transform of g.

Condition on β	$A(n)$ (as $n \to \infty$)
$\beta < \alpha(g) - 1$	$\left(-\frac{1}{\beta g^\star(\beta)}\right) n^{-\beta} + O\left(n^{-1-\beta}\right)$
$\beta = \alpha(g) - 1$	$\left(-\frac{1}{\beta g^\star(\beta)}\right) n^{-\beta} + O\left(n^{-\alpha(g)} \log n\right)$
$\alpha(g) - 1 < \beta < 0$	$\left(-\frac{1}{\beta g^\star(\beta)}\right) n^{-\beta} + O\left(n^{-\alpha(g)}\right)$
$\beta = 0$	$\frac{1}{c_0} + O\left(n^{-\alpha(g)}\right)$
$0 < \beta < \alpha(g)$	$\left(-\frac{1}{\beta g^\star(\beta)}\right) n^{-\beta} + O\left(n^{-\alpha(g)}\right)$
$\beta = \alpha(g)$	$(1 - \alpha(g))\, n^{-\alpha(g)} \log n + O\left(n^{-\alpha(g)}\right)$
$\beta > \alpha(g)$	$O\left(n^{-\alpha(g)}\right)$

1.2 Proof of theorem 1.1

Without loss of generality, we take $c_1 = c_0 = 1/2$ so that $\alpha(g) = \frac{1}{2}$ and prove the formula of theorem 1.1 for the case $\beta = \alpha(g) - 1 = -1/2$. The other formulas are proved similarly for any $c_1, c_0 > 0$ and any β. So let

- $g(x) = \frac{x}{2} + \frac{1}{2}$
- $A_g(n) = n^{1/2}$
- $h(n) = n^{-1}\left(n^{3/2} - (n-1)^{3/2}\right)$

First, we get the exact formula for any $n \geq 2$ (details are omitted)

$$A_g(n) = n^{1/2} \Rightarrow A(n) = h(n) + \frac{(1/2)_n}{n!}\left(2 + \sum_{k=2}^{n-1} h(k) \frac{k!}{(1/2)_k}\right) \qquad (4)$$

where $(x)_n = x(x+1)...(x+n-1)$ and it is easy to see that we have the 3 asymptotic formulas as $n \to \infty$

$$h(n) = \frac{3}{2}n^{-1/2} - \frac{3}{8}n^{-3/2} + O\left(n^{-5/2}\right)$$

$$\Gamma\left(1/2\right)\frac{(1/2)_n}{n!} = n^{-1/2} - \frac{1}{8}n^{-3/2} + O\left(n^{-5/2}\right)$$

$$\frac{1}{\Gamma\left(1/2\right)}h(k)\frac{k!}{(1/2)_k} = \frac{3}{2} - \frac{3}{16}k^{-1} + O\left(k^{-2}\right)$$

hence plugging these three asymptotic formulas in (4) we get

$$A_g(n) = n^{1/2} \Rightarrow A(n) = \frac{3}{2}n^{1/2} - \frac{3}{16}n^{-1/2}\log n + O\left(n^{-1/2}\right)$$

□

1.3 A conjecture for continuous functions

Experiments show that much more is true. Indeed, the following conjecture is very well supported by experiments.

Conjecture

Let g be continuous on $[0,1]$ satisfying $g(0)g(1) \neq 0$. Then g is a FGV of index $\alpha(g) = \eta(g)$ according to the primary definition of FGV and supposing that $A_g(n) = n^{-\beta}$ we have:

- $\beta < \alpha(g) \Rightarrow A(n) \sim \left(-\frac{1}{\beta g^\star(\beta)}\right) n^{-\beta} \ \ (n \to \infty)$
- $\beta \geq \alpha(g) \Rightarrow A(n) \ll n^{-\alpha(g)} L(n),$

where L is slowly varying.

2 Broken harmonic functions

Almost all tested bounded continuous and discontinuous functions on $]0,1]$ seem to be FGV and often the conjecture 1.3 works yielding $\alpha(g) = \eta(g)$ but it is not always the case (see for instance [Clo] and example 20.1). Anyhow what makes good variation interesting from a number theoretic view point relies on the Ingham function. Hence the quest for a better and deeper understanding of the problem led me to consider a set of functions sharing the main properties of the Ingham function. Some thoughts and many experiments led me to the natural choice of the so called broken harmonic functions (BHF). I will only consider these specific functions in the sequel.

2.1 Broken harmonic functions (BHF)

A bounded function g is a BHF if there exists a real positive sequence $(u_n)_{n\geq 1}$ satisfying $1 = u_1 > u_2 > u_3 > ... > u_\infty = 0$ and such that for any $n \geq 1$ we have $u_{n+1} < x \leq u_n \Rightarrow g(x) = v_n x$, where $v_n > 0$ is an increasing sequence of reals such that $\forall i \geq 1$ we have $u_i v_i < M$ for a constant $M > 0$.

Examples of BHF

- The Ingham function $\Phi(x) = x\left\lfloor \frac{1}{x} \right\rfloor$ for which $u_i = \frac{1}{i}$ and $v_i = i$.
- For $\lambda > 1$ the functions $g_\lambda(x) = x\lambda^{\left\lfloor -\frac{\log x}{\log \lambda} \right\rfloor}$ for which $u_i = \frac{1}{\lambda^{i-1}}$ and $v_i = \lambda^{i-1}$

2.2 The Hardy-Littlewood-Ramanujan (HLR) criterion

A function g satisfies the HLR criterion if for any $\beta \geq 0$ we have the property

$$A_g(n) = n^{-\beta} \Rightarrow \forall \varepsilon > 0 \ \lim_{n\to\infty} a(n)n^{1-\varepsilon} = 0$$

This criterion was discovered after studying the functions $g_\lambda(x) = x\lambda^{\left\lfloor -\frac{\log x}{\log \lambda} \right\rfloor}$ which are BHF satisfying the HLR criterion and the extended conjecture 1.3 if and only if $\lambda \geq 2$ is an integer (for more details see[Clo]). The name comes from the Hardy-Littlewood condition in their first Tauberian theorems and from a conjecture of Ramanujan on the size of coefficients of Dirichlet series in the Selberg class. This criterion acts as a bridge between arithmetic and complex analysis and the following Tauberian conjecture illustrates this connection qualitatively. In this note, I won't formulate the quantitative Tauberian conjectures described in [Clo] since the goal is to keep this presentation as short as possible.

In the sequel "g is HLR" means that g satisfies the HLR criterion.

3 The anti HLR conjecture for BHF

Let g be a BHF such that:

- $\lim_{x\to 0} g(x) \neq 0$ exists
- $(1-z)g^\star(z)$ satisfies a Riemann functional equation

Then if $g^\star$ has a zero in the half-plane $\Re z < \frac{1}{2}$ g is not HLR.

3.1 Corollary of the conjecture 3

Suppose g is as above and is HLR. Then the non trivial zeros of $g^\star$ are on the critical line.

Proof of corollary 3.1 It is simply the contrapositive statement of the anti HLR conjecture.

3.2 Illustration of the anti HLR conjecture for BHF

The Heilbronn-Davenport zeta function

The Heilbronn-Davenport zeta function is a good example illustrating the anti HLR conjecture. Let $\xi = \frac{-2+\sqrt{10-2\sqrt{5}}}{\sqrt{5}-1} = 0.284079...$ Davenport and Heilbronn [Dav] considered the analytic continuation of the Dirichlet series $H(s) = \sum_{n\geq 1} \frac{h(n)}{n^s}$, where h is the 5-periodic sequence $1, \xi, -\xi, -1, 0, ...$ and showed that it has nontrivial zeros off the critical line despite the fact that H satisfies a Riemann functional equation. Then considering the BHF

$$g_H(x) := x \sum_{1\leq k\leq \left\lfloor \frac{1}{x} \right\rfloor} h(k) \left\lfloor \frac{1}{kx} \right\rfloor$$

we have $g_H^\star(z) = \frac{\zeta(1-z)H(1-z)}{1-z}$ (details are omitted) and experiments show clearly that g_H is not HLR (see for instance figure 1 in the concluding remarks). Therefore the conjecture 3 works since $g_H^\star$ has zeros in the half plane $\Re z < \frac{1}{2}$.

4 The Riemann Hypothesis

Here I prove that the Ingham function is HLR and the reader will see that it is pure arithmetic involving the fundamental theorem of arithmetic. Then the Riemann hypothesis follows.

4.1 Theorem

The Ingham function Φ is HLR and more precisely for any $\beta \geq 0$ we have

$$A_{\Phi}(n) = n^{-\beta} \Rightarrow \; a(n) = O(n^{-1})$$

Remark

Although the Ramanujan condition (the extra ε) is not necessary for Φ it will be necessary for the generalised Ingham functions (see section 5). In some way this means that the Hardy-Littlewood Tauberian condition is sufficient for RH but one needs the deepness of Ramanujan condition to generalize RH.

4.2 Proof of the theorem 4.1

We need the following lemma:

4.2.1 Lemma

Suppose that f is a multiplicative function satisfying $0 < f(n) \leq 1$ for $n \geq 1$. Then $\sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) = O(1)$.

Proof of lemma 4.2.1

It is easy to see that $b(n) := \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d)$ is the multiplicative function given by $b_{p^v} = f\left(p^v\right) - f\left(p^{v-1}\right)$. Next letting $n = \prod p_i^{\alpha_i}$ where p_i are the distinct primes in the factorization of n we have $-1 \leq f\left(p_i^{\alpha_i}\right) - f\left(p_i^{\alpha_i - 1}\right) \leq 1$ hence we get

$$b_n = \prod \left(f\left(p_i^{\alpha_i}\right) - f\left(p_i^{\alpha_i-1}\right)\right) = O(1)$$

□

4.2.2 Proof of the theorem 4.1

The theorem is true for the case $\beta = 0$ since we have trivially in this case $a_1 = 1$ and $a_n = 0$ for $n \geq 2$. So I consider $\beta > 0$. It is well known that we have

$$\sum_{k=1}^{n} k a_k \left\lfloor \frac{n}{k} \right\rfloor = \sum_{k=1}^{n} \sum_{d|k} d a_d \Rightarrow \sum_{d|n} d a_d = n^{1-\beta} - (n-1)^{1-\beta}$$

Therefore by Möbius inversion we get

$$n a_n = \sum_{d|n} \mu\left(\frac{n}{d}\right)\left(d^{1-\beta} - (d-1)^{1-\beta}\right) \tag{5}$$

next we have $d^{1-\beta} - (d-1)^{1-\beta} = (1-\beta)d^{-\beta} + O\left(d^{-1-\beta}\right)$ thus (5) becomes

$$n a_n = (1-\beta)\sum_{d|n} \mu\left(\frac{n}{d}\right) d^{-\beta} + \sum_{d|n} \mu\left(\frac{n}{d}\right) O(d^{-1-\beta}) \tag{6}$$

Now since $\beta > 0$ we have, on one hand, from the lemma 4.2.1 $\sum_{d|n} \mu\left(\frac{n}{d}\right) d^{-\beta} = O(1)$ and, on the other hand, $\sum_{n \geq 1} n^{-1-\beta}$ converges toward the finite value $\zeta(1+\beta)$. Consequently we get $\sum_{d|n} \mu\left(\frac{n}{d}\right) O(d^{-1-\beta}) = O(1)$.

As a result, (6) becomes $n a_n = O(1)$ and Φ is HLR.

□

4.3 Corollary

The conjecture 3.1 and the theorem 4.1 imply that RH is true.

Proof

We have

- $\lim_{x \to 0} \Phi(x) = 1 \neq 0$
- $(1-z)\Phi^{\star}(z) = \zeta(1-z)$ satisfies a Riemann functional equation
- Φ is HLR from theorem 4.1

hence the conjecture 3.1. tells us that RH is true for $\zeta(s)$.

□

5 The Generalized Riemann Hypothesis

In this section, I prove that the generalized Ingham functions are HLR so that the anti HLR conjecture implies that the generalized Riemann hypothesis is true.

5.1 Theorem 5.1

The generalized Ingham function $\Phi_\chi(x) = x\sum_{1\le k\le 1/x}\chi(k)\left\lfloor\frac{1}{kx}\right\rfloor$ is HLR, where χ is a Dirichlet character.

5.2 Proof of the theorem 5.1

The proof is based on the following two lemmas whose proofs are omitted:

5.2.1 Lemma

Suppose that w is a completely multiplicative function. Then w has a Dirichlet inverse given by $w^{-1}(n) = \mu(n)w(n)$.

5.2.2 Lemma

For $n \geq 1$ if $f(n)$ is defined by

$$f(n) = \sum_{k=1}^{n} u(k) \sum_{1\le i\le n/k} v(i)\left\lfloor\frac{n}{ik}\right\rfloor,$$

then we have, with the convention $f(0) = 0$,

$$\sum_{d|n} u(d)\, v\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right)(f(d) - f(d-1))$$

5.2.3 Proof of the theorem 5.1

From lemma 5.2.2 we have

$$A_{\Phi_\chi}(n) = n^{-\beta} \Rightarrow \sum_{d|n} da(d)\,\chi\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right)\left(d^{1-\beta} - (d-1)^{1-\beta}\right) \quad (7)$$

From 4.2.2. we know that for $\beta \geq 0$ we have

$$\sum_{d|n} \mu\left(\frac{n}{d}\right)\left(d^{1-\beta} - (d-1)^{1-\beta}\right) = O(1)$$

Letting $a'(n) = na(n)$ and $b(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right)\left(d^{1-\beta} - (d-1)^{1-\beta}\right)$ we get from (7) and lemma 5.2.1 (Dirichlet characters are completely multiplicative)

$$a' \star \chi(n) = b(n) \Rightarrow a'(n) = b \star \chi^{-1}(n) = \sum_{d|n} b\left(d\right) \chi\left(\frac{n}{d}\right) \mu\left(\frac{n}{d}\right)$$

then since $|b(n)| \leq C$ (for some $C > 0$) we get

$$|a'(n)| \leq \sum_{d|n} |b\left(d\right)| \left|\chi\left(\frac{n}{d}\right)\right| \left|\mu\left(\frac{n}{d}\right)\right| \leq C\tau(n) \ll n^{\varepsilon}$$

so that $na(n) = O(n^{\varepsilon})$ and Φ_χ is HLR.

$\square$

Remark

Here we can't have something like the Ingham function; i.e., $na(n) = O(1)$. Indeed, consider $\chi(n) = 1, 0, -1, 0, 1, 0, -1, 0, ...$ and

$$\sum_{k=1}^{n} a(k)\Phi_\chi\left(\frac{k}{n}\right) = n^{-1}$$

Let $p_1(n)$ denote the increasing sequence of primes congruent to 1 modulo 4; i.e., $p_1(1) = 5, p_1(2) = 13, p_1(3) = 17, p_1(4) = 29$, etc. Letting $P(n) = \prod_{i=1}^{n} p_1(i)$, it can be shown that we have for any integer $m \geq 1$

$$|P(m)a\left(P(m)\right)| = 2^m$$

therefore $na(n)$ is unbounded.

5.3 Corollary

The conjecture 3.1 and the theorem 5.1 imply that RH is true for $L(s,\chi)$, where χ is a Dirichlet character.

Proof

We have

- $\lim_{x\to 0}\Phi_\chi(x) = L\left(1,\chi\right) \neq 0$
- $(1-z)\Phi_\chi^\star(z) = \zeta(1-z)L(1-z,\chi)$ satisfies a Riemann functional equation
- Φ_χ is HLR from theorem 5.1

hence, from the conjecture 3.1, RH is true for both $\zeta(s)$ and $L(s,\chi)$.

□

6 The Grand Riemann Hypothesis

The previous method extends naturally to the Grand Riemann hypothesis. In order to do this I need to prove the following theorem.

6.1 Theorem

The generalised Ingham function $\Phi_u(x) = x\sum_{1\le k\le 1/x} u(k)\left\lfloor \frac{1}{kx}\right\rfloor$ is HLR where u is any multiplicative function satisfying the Ramanujan condition $u(n) = O\left(n^\varepsilon\right)$.

6.2 Proof of theorem 6.1

Before proving this theorem two lemmas are in order. I will prove only the lemma 6.2.2. the lemma 6.2.1 is classical.

6.2.1 Lemma

If u is multiplicative then the Dirichlet inverse u^{-1} is also multiplicative.

6.2.2 Lemma

If u is multiplicative with $u(n) = O(n^\varepsilon)$ then its Dirichlet inverse u^{-1} satisfies also $u^{-1}(n) = O\left(n^\varepsilon\right)$.

Proof of lemma 6.2.2

Let us fix $u(1) = 1$ so that $u^{-1}(1) = 1$ then it is well known that we have the recursive formula for $n \geq 2$

$$u^{-1}(n) = -\sum_{d|n, d<n} u^{-1}(d)u\left(\frac{n}{d}\right)$$

Since u is multiplicative so does u^{-1} from lemma 6.2.1. hence it suffices to evaluate $u^{-1}\left(p^n\right)$ for $p \geq 2$ prime and $n \geq 1$. From the recursive formula we get

$$u^{-1}\left(p^n\right) = -\sum_{k=0}^{n-1} u^{-1}\left(p^k\right) u\left(p^{n-k}\right) \tag{8}$$

We assume that $\forall \varepsilon > 0,\ u\left(p^{n-k}\right) = O\left(p^{\varepsilon(n-k)}\right)$. Suppose also that we have the recurrence hypothesis

- $\forall \varepsilon > 0,\ u^{-1}\left(p^k\right) = O\left(p^{\varepsilon k}\right)$ for any $p \geq 2$ prime and any $k \leq n-1$

Then (8) becomes

$$\forall \varepsilon > 0, \left|u^{-1}\left(p^n\right)\right| \leq \sum_{k=0}^{n-1} \left|u^{-1}\left(p^k\right)\right| \left|u\left(p^{n-k}\right)\right| \ll \sum_{k=0}^{n-1} p^{\frac{\varepsilon}{2}k} p^{\frac{\varepsilon}{2}(n-k)} = np^{\frac{\varepsilon}{2}n}$$

next $n \ll p^{\frac{\varepsilon}{2}n}$ for any $\varepsilon > 0$ and any $p \geq 2$ hence we get

$$\left|u^{-1}\left(p^n\right)\right| \ll p^{\varepsilon n}$$

and the recurrence hypothesis is true for all n. Thus letting $n = \prod p_i^{\alpha_i}$ we get

$$\left|u^{-1}\left(n\right)\right| \ll \prod p_i^{\varepsilon \alpha_i} = n^\varepsilon$$

□

6.2.3 Proof of theorem 6.1

From lemma 5.2.2 we have

$$A_{\Phi_u}(n) = n^{-\beta} \Rightarrow \sum_{d|n} da\left(d\right) u\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right)\left(d^{1-\beta} - (d-1)^{1-\beta}\right) \tag{9}$$

From 4.2.2. we know that for $\beta \geq 0$ we have

$$\sum_{d|n} \mu\left(\frac{n}{d}\right)\left(d^{1-\beta} - (d-1)^{1-\beta}\right) = O(1)$$

Hence letting $a'(n) = na(n)$ and $b(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right)\left(d^{1-\beta} - (d-1)^{1-\beta}\right)$ we get from (9)

$$a' \star u(n) = b(n) \Rightarrow a'(n) = b \star u^{-1}(n) = \sum_{d|n} b\left(n/d\right) u^{-1}(d)$$

whence since $b(n) = O(1)$ and $u^{-1}(n) = O\left(n^{\varepsilon}\right)$ from lemma 6.1.2 we get for any $\varepsilon > 0$

$$|na(n)| \leq \sum_{d|n} \left|b\left(n/d\right)\right| \left|u^{-1}\left(d\right)\right| \ll \sum_{d|n} d^{\varepsilon/2} \ll n^{\varepsilon/2}\tau(n) \ll n^{\varepsilon}$$

since $\tau(n) \ll n^{\varepsilon/2}$, consequently $na(n) = O(n^{\varepsilon})$ and Φ_u is HLR.

□

6.3 Corollary

Suppose that:

- u is multiplicative with $u(n) = O(n^{\varepsilon})$
- the analytic continuation of $U(s) = \sum_{n\geq 1} \frac{u(n)}{n^s}$ satisfies a Riemann functional equation
- $\sum_{n\geq 1} \frac{u(n)}{n}$ converges toward a non zero limit

Then the conjecture 3.1 and the theorem 6.1 imply that RH is true for $U(s)$.

Proof of corollary 6.3

We have

- $\lim_{x\to 0}\Phi_u(x)=\sum_{n\geq 1}\frac{u(n)}{n}\neq 0$
- $(1-z)\Phi_u^\star(z)=\zeta(1-z)U(1-z)$ satisfies a Riemann functional equation
- $\Phi_u(x)=x\sum_{1\leq k\leq 1/x}u(k)\left\lfloor\frac{1}{kx}\right\rfloor$ is HLR from theorem 6.1

hence, from the conjecture 3.1, RH is true for both $\zeta(s)$ and $U(s)$.

Example

Let $\tau_r(n)$ denote the Ramanujan tau numbers ([Apo], pp. 114, 131) defined by

$$\sum_{n\geq 1}\tau_r(n)x^n=x\prod_{n\geq 1}(1-x^n)^{24}$$

then $u(n)=\frac{\tau_r(n)}{n^{11/2}}$ satisfies the conditions of corollary 6.3 since letting $U(s)=\sum_{n\geq 1}\frac{\tau_r(n)}{n^{s+11/2}}$ we have:

- τ_r is multiplicative (conjectured by Ramanujan and proved soon after by Mordell in 1917 [Mor])
- $\tau_r(n)=O\left(n^{11/2+\varepsilon}\right)$ (conjectured by Ramanujan and proved much later by Deligne in 1974 [Del])
- $\lim_{x\to 0}\Phi_u(x)=\sum_{n\geq 1}\frac{\tau_r(n)}{n^{13/2}}=0.8...\neq 0$
- $(1-z)\Phi_u^\star(z)=\zeta(1-z)U(1-z)$ satisfies a Riemann functional equation

Hence, from corollary 6.3, $U(s)$ satisfies the Riemann hypothesis.

Concluding remarks

The Davenport-Heilbronn example described in 3.2. is interesting on its own. It is known (see for instance [Ivi]) that the non trivial zeros of $H(s)$ have real part dense in $]0,1[$ and it could be a general property.

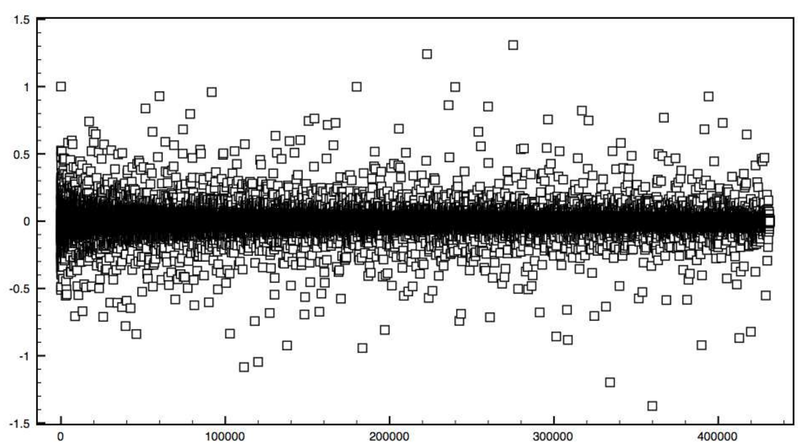
Namely I claim that if $F(s)=\sum_{n\geq 1}\frac{f(n)}{n^s}$ has an analytic continuation, satisfies a Riemann functional equation and has some zeros off the critical line then it has in fact infinitely many zeros off the critical line and the real part of these zeros are dense in $]0,1[$.

In this case the generalized Ingham function $g_f(x) = x\sum_{k\le x^{-1}} f(k)\left\lfloor \frac{1}{kx}\right\rfloor$ isn't HLR from the anti HLR conjecture and I claim that we have for any $\beta \geq 0$

$$A_{g_f}(n) = n^{-\beta} \Rightarrow \forall\varepsilon > 0 \ \lim_{n\to\infty} a_n n^{1/2-\varepsilon} = 0 \ \wedge \limsup_{n\to\infty} \left|a_n n^{1/2+\varepsilon}\right| = +\infty$$

which seems supported by experiments using the Heilbronn-Davenport counter-example and the associated BHF g_H defined in 3.2. as shown by the graphic below (fig. 1) which looks bounded by a slowly varying function like log.

Fig.1) Plot of $n^{1/2}a(n)$ where $A_{g_H}(n) = n^{-1/3}$



In particular if ζ has a zero off the critical line it should have in fact infinitely many zeros off the critical line and we could find zeta zeros as close as we wish from the line $x = 1$. In some way this is supported by the best zero free regions known to this day which are always asymptotically close to the line $x = 1$.

Hence one should explore this kind of property of values distribution of little Mellin transform of BHF like the above function g_f in order to understand better properties of non trivial zeros (simplicity,independence over the rationals, Montgomery's pair correlation conjecture,...). To prove the anti HLR conjecture however I suspect that an algebraic approach could be promising considering the space of FGV and the "subspace" of affine functions by parts and trying to find invariants.

With this approach we link zeros of L functions to multiplicative number theory. Thus the reader should note the consistency with what is suggested by experts in analytic number theory, i.e., RH is true for analytic continuation of Dirichlet series satisfying a Riemann functional equation if and only if there is an Euler product.

Acknowledgments. I warmly thank Doron Zeilberger for his ongoing support during this research and the referee for his time.

References

[Apo] Tom M. Apostol, Modular functions and Dirichlet series in number theory, second Edition, Springer, 1990.

[Bor] P. Borwein et al, The Riemann hypothesis: a resource for expert and afficionado alike, Springer, 2009.

[Clo] B. Cloitre, Good variation theory, http://bcmathematics.monsite-orange.fr/personal web page 2016.

[Dav] H. Davenport, H. Heilbronn, On the zeros of certain Dirichlet series I, II., J. London Math. Soc. **11,** 1936, 181-185, 307-312.

[Del] P. Deligne, La conjecture de Weil. I., Inst. Hautes Études Sci. Publ. Math., **43,** 1974, 273-307.

[Dib] J. Diblik et al, On the existence of solutions of linear Volterra difference equations asymptotically equivalent to a given sequence, Applied Mathematics and Computation, **218**, no. 18, 2012, 9310-9320.

[Ing] A. E. Ingham, Some Tauberian theorems connected with the prime number theorem, Journal of the London Mathematical Society, **20,** 1945, 171-180.

[Ivi] A. Ivic, The Riemann zeta function, Wiley, New York,1985.

[Kor] J. Koorevar, Tauberian theory: a century of developements, Springer, **329,** 2004.

[Mor] L. J. Mordell, On Mr. Ramanujan's Empirical Expansions of Modular Functions, Proc. Cambridge Phil. Soc., **19,** 1917, 117-124.