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Riemannian foliation with dense leaves on a compact manifold

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Abstract

In this paper, we show that if $\mathcal{G} = Lie(G)$ is the Lie structural algebra of a Riemannian foliation with dense leaves $(M, \mathcal{F})$ on a compact manifold M, there exists a representation $\rho : H_0 \to Diff(V)$ where V is an open subset of G such as:
(a) There exists a biunivocal correspondence between the Lie subalgebras of $\mathcal{G}$ invariant by $Ad_{(\rho_a(v))^{-1}.v}$ for every $(a, v) \in H_0 \times V$ and $\mathcal{F}$ extensions.
(b) An extension is a Lie foliation if the subalgebra corresponding is an ideal of $\mathcal{G}$.
(c) Every extension $\mathcal{F}'$ of $\mathcal{F}$ is a Riemannian foliation and there exists a common bundle-like metric for the foliations $\mathcal{F}$ and $\mathcal{F}'$.
(d) If $\mathcal{F}_{\mathcal{H}}$ is an extension of $\mathcal{F}$ corresponding to a subalgebra $\mathcal{H}$ of $\mathcal{G}$, then to isomorphism nearly of Lie algebras we have

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{u \in \mathcal{H}^{\perp} / \forall\, (h, a, v) \in \mathcal{H} \times H_0 \times V,\ [u, h] = 0 \text{ and } Ad_{(\rho_a(v))^{-1}.v}(u) = u\}.$$

1 Introduction

The purpose of this article is to generalize the following results in ([6], [8]) and [7] respectively to Riemannian foliations with dense leaves on a compact manifold:

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i) If $\mathcal{F}$ is a Lie G-foliation with dense leaves on a compact manifold, then there exists a biunivocal correspondence between the Lie subalgebras of $\mathcal{G} = lie\,(G)$ and $\mathcal{F}$ extensions,

ii) if $\mathcal{F}_{\mathcal{H}}$ is an extension of a Lie $\mathcal{G}$-foliation with dense leaves on a compact manifold corresponding to a subalgebra $\mathcal{H}$ of $\mathcal{G}$ and $\ell(M, \mathcal{F}_{\mathcal{H}})$ the Lie algebra of $\mathcal{F}_{\mathcal{H}}-$foliated transverse vectors fields, then we have

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{u \in \mathcal{H}^{\perp} / \, [u, h] = 0 \text{ for every } h \in \mathcal{H}\}.$$

To achieve that, we first establish that the closure of a leaf $F^{\natural}$ of lifted foliation $\mathcal{F}^{\natural}$ on the orthonormal transverse frame bundle $M^{\natural}$ of a Riemannian foliation $\mathcal{F}$ with dense leaves on a compact manifold is a covering of the manifold M.

We note that the fact that $\overline{F^{\natural}}$ is a covering of the manifold M entails that the dimension of the Lie structural algebra of a Riemannian foliation $\mathcal{F}$ on a compact manifold which is less than or equal to the codimension of the foliation $\mathcal{F}$. This allows us to say that we should limit the classification of Molino in codimension 1, 2 and 3 of Riemannian foliation on a compact manifold [12] to only a few cases of this classification.

We denote by H_0 the structure group of covering $\phi : \overline{F^{\natural}} \rightarrow M$. This group H_0 will be called the *discrete group of Riemannian foliation* $\mathcal{F}$ on a compact manifold with dense leaves.

That said, we show that if $\mathcal{G}$ is the Lie structural algebra of a Riemannian foliation with dense leaves $(M, \mathcal{F})$ on a compact manifold M, there exists a representation $\rho : H_0 \rightarrow Diff\,(V)$ where V is an open of G such as:

i) there exists a biunivocal correspondence between the Lie subalgebras of $\mathcal{G} = Lie\,(G)$ invariant by $Ad_{(\rho_a(v))^{-1}.v}$ for every $(a, v) \in H_0 \times V$ and $\mathcal{F}$ extensions,

ii) an extension is a Lie foliation if the subalgebra corresponding is an ideal of $\mathcal{G}$,

iii) every extension $\mathcal{F}'$ of $\mathcal{F}$ is a Riemannian foliation and there exists a common bundle-like metric for the foliations $\mathcal{F}$ and $\mathcal{F}'$,

iv) if $\mathcal{F}_{\mathcal{H}}$ is an extension of $\mathcal{F}$ corresponding to a subalgebra $\mathcal{H}$ of $\mathcal{G}$, then to isomorphism nearly of Lie algebras we have

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{u \in \mathcal{H}^{\perp} / \forall\,(h, a, v) \in \mathcal{H} \times H_0 \times V, \; [u, h] = 0 \text{ and } Ad_{(\rho_a(v))^{-1}.v}\,(u) = u \;\}.$$

In particular,

$$\ell(M, \mathcal{F}) = \{u \in \mathcal{G} / \forall\,(a, v) \in H_0 \times V, \; Ad_{(\rho_a(v))^{-1}.v}\,(u) = u\}.$$

Our paper is divided into two parts:

-the first part is devoted to reminders on Riemannian foliations and on extensions of foliations,

- the second part is devoted to the establishment of the already stated primary outcome.

In all that follows, the manifolds considered are supposed connected and differentiability is C^∞.

2 Definitions and Reminders

In this section, we reformulate some definitions and theorems in ([4], [6], [7], [8], [10] [11], [12], [13]).

Definition 2.1. *Let M be a manifold.*

An extension of a codimension q foliation $(M, \mathcal{F})$ is a codimension q' foliation $(M, \mathcal{F}')$ such that $0 < q' < q$ and $(M, \mathcal{F}')$ leaves are $(M, \mathcal{F})$ leaves meetings $(\mathcal{F} \subset \mathcal{F}')$.

We show that if $(M, \mathcal{F}')$ is a simple extension of a simple foliation $(M, \mathcal{F})$ and if $(M, \mathcal{F})$ and $(M, \mathcal{F}')$ are defined respectively by submersions $\pi : M \to T$ and $\pi' : M \to T'$, then there exists a submersion $\theta : T \to T'$ such that $\pi' = \theta \circ \pi$.

We say that the submersion θ is a bond between the foliation $(M, \mathcal{F})$ and its extension foliation $(M, \mathcal{F}')$.

In [6] we showed that if the foliation $(M, \mathcal{F})$ and its extension $(M, \mathcal{F}')$ are defined respectively by the cocycles $(U_i, f_i, T, \gamma_{ij})_{i \in I}$ and $(U_i, f'_i, T', \gamma'_{ij})_{i \in I}$, then we have

$$f'_i = \theta_i \circ f_i \quad \text{and} \quad \gamma'_{ij} \circ \theta_j = \theta_i \circ \gamma_{ij}$$

where θ_s is a bond between the foliation $(U_s, \mathcal{F})$ and its extension foliation $(U_s, \mathcal{F}')$.

Definition 2.2. *Let $\mathcal{F}_q$ be a codimension q foliation on a manifold M.*

A flag of extensions of $\mathcal{F}_q$ is a sequence $\mathcal{D}^k_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, ..., \mathcal{F}_k)$ of foliations on the manifold M such that $\mathcal{F}_q \subset \mathcal{F}_{q-1} \subset \mathcal{F}_{q-2} \subset ... \subset \mathcal{F}_k$ and each foliation $\mathcal{F}_s$ is a codimension s foliation.

For $k = 1$, the flag of extensions $\mathcal{D}^k_{\mathcal{F}_q}$ will be called complete and will be denoted $\mathcal{D}_{\mathcal{F}_q}$.

If each foliation $\mathcal{F}_s$ is a Riemannian foliation, the flag of extensions $\mathcal{D}^k_{\mathcal{F}_q}$ will be called flag of Riemannian extensions of $\mathcal{F}_q$.

The following theorem is the biunivocal correspondence theorem between Lie subalgebras of $\mathcal{G}$ =*Lie* (G) and the extensions of a Lie $G-$foliation with dense leaves existing in [8].

Theorem 2.3. [8] *Let $(M, \mathcal{F})$ be a Lie G-foliation with dense leaves on compact connected manifold and let $\mathcal{G}$ be the Lie algebra of G.*

Then:

1-There exists a biunivocal correspondence between the Lie subalgebras of $\mathcal{G}$ (or if you prefer between the connected Lie subgroups of G) and extensions of $\mathcal{F}$.

2- An extension of $\mathcal{F}$ is a Riemannian $\frac{\mathcal{G}}{\mathcal{H}}$-foliation having trivial normal bundle and defined by a 1-form with values in $\frac{\mathcal{G}}{\mathcal{H}}$.

3- An extension of $\mathcal{F}$ is transversely homogeneous (resp. Lie) if and only if the Lie subgroup of G corresponding is a closed subgroup (resp. Normal subgroup) in G.

In [7] was calculated $\ell(M, \mathcal{F}_{\mathcal{H}})$, where $\mathcal{F}_{\mathcal{H}}$ is the extension of a Lie $\mathcal{G}-$foliation with dense leaves on a compact manifold corresponding to a Lie subalgebra $\mathcal{H}$ of $\mathcal{G}$. This calculation gave us the following result:

Theorem 2.4. [7] *Let $\mathcal{H}$ be a Lie subalgebra of Lie algebra $\mathcal{G}$ of a Lie foliation $\mathcal{F}$ with dense leaves on a compact manifold. Let ω be a 1-form of Fedida defining $\mathcal{F}$ and let $\mathcal{F}_{\mathcal{H}}$ be the extension of $\mathcal{F}$ corresponding to $\mathcal{H}$.*

Then:

i)

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{<h>})$$

where $\mathcal{F}_{<h>}$ is the extension of $\mathcal{F}$ corresponding to the Lie subalgebra $< h >$ of $\mathcal{G}$ generated by h.

ii)

$$\omega\left(\ell(M, \mathcal{F}_{\mathcal{H}})\right) \subset \mathcal{H}^{\perp}$$

where $\mathcal{H}^{\perp}$ is the ortho-complementary of $\mathcal{H}$ in $\mathcal{G}$ by the transverse metric associated with $\mathcal{F}$.

iii) For $h \in \mathcal{G}$,

$$\omega\left(\ell(M, \mathcal{F}_{<h>})\right) = \{u \in < h >^{\perp} \; / \; [u, h] = 0 \; \}.$$

iv)

$$\omega\left(\ell(M, \mathcal{F}_{\mathcal{H}})\right) = \{u \in \mathcal{H}^{\perp} / \, [u, h] = 0 \;\; \textit{for every } h \in \mathcal{H}\}.$$

The theorem and the three following propositions allow us to give a generalization of the previous two theorems for Riemannian foliations with dense leaves on a compact manifold .

Theorem 2.5. [*7*] *Let $\mathcal{F}$ be a Lie $G-$foliation with dense leaves on a compact manifold M. Let λ be a metric on M which is bundle-like for $\mathcal{F}$ and which admits λ_T as its associated transverse metric and let X be a $\mathcal{F}-$transverse foliated vectors field.*

Then:

i) for every point $a \in M$ there exist an open $\mathcal{F}$-distinguished V_a of M containing the point a such that the restriction X_{V_a} of X at V_a is a local $\mathcal{F}$-transverse Killing vector field,

ii) any vectors field left invariant of G is a Killing vectors field for the $\mathcal{F}$-transverse metric λ_T left invariant,

iii) in the case where G is connected, the right translation R_a associated with the element $a \in G$ is an isometry for the metric λ_T left invariant.

Proposition 2.6. [*7*] *Let $\mathcal{F}''$ be an extension of a Riemannian foliation $\mathcal{F}'$ on a manifold M.*

Then $\ell(M, \mathcal{F}'') \subset \ell(M, \mathcal{F}')$.

Proposition 2.7. [*12*] *Let $\mathcal{F}$ be a codimension q Riemannian foliation on a compact connected manifold M. Let $\overline{F^\natural}$ be the closure of a leaf $F^\natural$ of lifted foliation $\mathcal{F}^\natural$ of $\mathcal{F}$ on the orthonormal transverse frame bundle $M^\natural$ and let $\phi{:}M^\natural \to M$ be the projection which to a frame at x associates x.*

Them:

i) $\phi\left(F^\natural\right)$ is a leaf of $\mathcal{F}$ and $\phi\left(\overline{F^\natural}\right) = \overline{\phi\left(F^\natural\right)}$,

ii) the map $\phi : \overline{F^\natural} \to \phi\left(\overline{F^\natural}\right)$ is a locally trivial fibration.

Proposition 2.8. [*12*] *Let (M, λ) be a Riemannian connected manifold and let $\mathcal{K}(M, \lambda)$ be a Lie algebra of Killing vector field on (M, λ).*

Then the orbits of $\mathcal{K}(M, \lambda)$ having maximal dimension form an open dense in M.

3 Riemannian foliation with dense leaves on a compact manifold

In what follows G is a Lie group of Lie algebra $\mathcal{G}$, X^r (resp. X^l) is the vectors field on G right (resp. left) invariant obtained from $X \in \mathcal{G}$ and L_a (resp. R_a) is the left (resp. right) translation associated with $a \in G$.

Proposition 3.1. *Let $\mathcal{K}(M,\lambda)$ be a Lie algebra of Killing vector fields on a Riemannian connected manifold (M,λ). Let $i_x = \{X \in \mathcal{K}(M,\lambda) / X_x = 0\}$ be the isotropy of $\mathcal{K}(M,\lambda)$ at the point $x \in (M,\lambda)$. Let $\mathcal{O}_x$ be the orbit at the point x of $\mathcal{K}(M,\lambda)$ and let $\mathcal{O}_{(x_0,x)}$ be the orbit at the point $x \in (M,\lambda)$ of the isotropy i_{x_0} where $x_0 \in (M,\lambda)$.*

If for $x \in (M,\lambda)$, all orbits $\mathcal{O}_x$ have the same dimension, then $\mathcal{O}_{(x_0,x)} = \{x\}$ for every $(x_0,x) \in (M,\lambda)^2$.

Proof . Let $x \in (M,\lambda)$. We have $\dim i_x = \dim \mathcal{K}(M,\lambda) - \dim \mathcal{O}_x$.

As all orbits $\mathcal{O}_x$ have the same dimension then $\dim i_x = \dim i_y$ for every $y \in (M,\lambda)$.

We note that i_x is a Lie subalgebra of Killing vector fields of $\mathcal{K}(M,\lambda)$.

Let $x_0 \in (M,\lambda)$ and let $i_{(x_0,x)}$ be the isotropy at the point x of i_{x_0}.

We know from [12] that there exists an open neighborhood U_{x_0} of x_0 such as if $x' \in U_{x_0}$, then $i_{(x_0,x')} = i_{x'}$; that is to say, the isotropy at the point x' of i_{x_0} is equal to the isotropy at the point x' of $\mathcal{K}(M,\lambda)$ for every $x' \in U_{x_0}$. For the proof we assume that $(k_1, k_2, ..., k_r, k'_1, k'_2, ..., k'_s)$ is a base of $\mathcal{K}(M,\lambda)$ such as $(k_1, k_2, ..., k_r)$ is a base of i_{x_0}.

For every $j \in \{1, ..., s\}$, $k'_j(x_0) \neq 0$. Hence there exists an open U_{x_0} containing x_0 such as for every $x' \in U_{x_0}$ and for every $j \in \{1, ..., s\}$, $k'_j(x') \neq 0$.

Let $x \in U_{x_0}$ and let $Y \in i_x$. Then there exist two finite sequences $(y_j)_{1\leq j\leq r}$ and $\left(y'_j\right)_{1\leq j\leq s}$ of real numbers such that $Y = \sum_{j=1}^{r} y_j.k_j + \sum_{j=1}^{s} y'_j.k'_j$.

We have $\sum_{j=1}^{r} y_j.k_j(x) + \sum_{j=1}^{s} y'_j.k'_j(x) = 0$ because $Y \in i_x$.

Reordering the base of vectors fields $(k_1, k_2, ..., k_r)$ of i_{x_0} we can assume that there exists $r' \in \{0, 1, ..., r\}$ such as $k_j(x) \neq 0$ for $j \leq r'$ and $k_j(x) = 0$ for $r'+1 \leq j \leq r$.

Thus, the fact that $Y \in i_x$ implies that $\sum_{j=1}^{r'} y_j.k_j(x) + \sum_{j=1}^{s} y'_j.k'_j(x) = 0 \quad (*)$.

As $k_j(x) \neq 0$ for $j \leq r'$ and $k'_j(x) \neq 0$ and $(k_1, k_2, ..., k_{r'}, k'_1, k'_2, ..., k'_s)$ is a free system of vectors fields of $\mathcal{K}(M,\lambda)$ then $(k_1(x), ..., k_{r'}(x), k'_1(x), ..., k'_s(x))$ form a free system. Therefore the equality $(*)$ shows that $y_j = 0$ for $j \leq r'$ and $y'_j = 0$ for every j.

It follows that the equality $Y = \sum_{j=1}^{r} y_j.k_j + \sum_{j=1}^{s} y'_j.k'_j$ implies that $Y = \sum_{j=r'+1}^{r} y_j.k_j$. Therefore $Y \in i_{x_0}$ and $i_x \subset i_{x_0}$.

The fact that the dimension of i_y for all $y \in (M, \lambda)$ is constant implies that $i_x = i_{x_0}$.

Thus for any $x \in U_{x_0}$, $i_{(x_0,x)} = i_x = i_{x_0}$ and the orbit $\mathcal{O}_{(x_0,x)}$ of i_{x_0} at the point x is the point x.

We know from ([14], [12]) the orbits of Lie algebra of Killing vector fields on Riemannian connected manifold V having maximal dimension form an open dense in V. Thus, using the fact that the maximal dimension of the orbits of i_{x_0} in the open U_{x_0} is zero, $\mathcal{O}_{(x_0,x)} = \{x\}$ for all $x \in (M, \lambda)$. □

This proposition means that when a Lie algebra of Killing vector fields $\mathcal{K}(M, \lambda)$ on a connected Riemannian manifold (M, λ) has orbits having the same dimension, then the isotropy i_x of $\mathcal{K}(M, \lambda)$ at any point $x \in (M, \lambda)$ *induces* on (M, λ) a null Lie algebra of Killing vector fields.

That said, the previous proposition allows us to establish the following result:

Proposition 3.2. *Let H be a closed Lie subgroup of a connected Lie group G, Let λ be a metric on G left invariant. Let $\mathcal{G}$ =Lie(G) and let $\mathcal{H}$ =Lie(H).*

If λ is invariant by right translations obtained from the elements of H, then the Lie subalgebra $\mathcal{H}$ is an ideal of $\mathcal{G}$.

Proof . If the Lie group H is discrete, then $\mathcal{H} = \{0\}$. In this case, $\mathcal{H}$ is an ideal of $\mathcal{G}$.

In what follows we assume that H is not discrete.

The Lie subgroup H is closed in the Lie group G. From where $\pi : G \to \frac{G}{H}$ is a principal fibration having H for structure group. We note in passing that π is a Riemannian submersion because the metric λ left invariant on G is invariant by right translations obtained from the elements of H.

Let α be the left Maurer-Cartan form of G. Let $p_{\mathcal{H}} : \mathcal{G} \to \mathcal{H}$ be the orthogonal projection on $\mathcal{H}$ and let $\alpha_{\mathcal{H}} = p_{\mathcal{H}} \circ \alpha$.

It is easy to see that $\alpha_{\mathcal{H}}$ is a differential form on G with values in $\mathcal{H}$.

Let $a \in H$, Suppose $X \in TG$ and let $p_{\mathcal{H}^\perp} : \mathcal{G} \to \mathcal{H}^\perp$ be the orthogonal projection on $\mathcal{H}^\perp$, where $\mathcal{H}^\perp$ is the ortho-complementary of $\mathcal{H}$ in $\mathcal{G}$.

We have

$$\begin{aligned}
(R_a^*\alpha_{\mathcal{H}})(X) &= \alpha_{\mathcal{H}}(R_{a*}(X)) \\
&= p_{\mathcal{H}} \circ \alpha(R_{a*}(X)) \\
&= p_{\mathcal{H}} \circ Ad_{a^{-1}}(\alpha(X)) \\
&= p_{\mathcal{H}} \circ Ad_{a^{-1}}(p_{\mathcal{H}}(\alpha(X)) + p_{\mathcal{H}^\perp}(\alpha(X))) \\
&= p_{\mathcal{H}} \circ Ad_{a^{-1}} \circ p_{\mathcal{H}}(\alpha(X)) + p_{\mathcal{H}} \circ Ad_{a^{-1}} \circ p_{\mathcal{H}^\perp}(\alpha(X)).
\end{aligned}$$

We note that for all $a \in H$, $\mathcal{H}$ and $\mathcal{H}^{\perp}$ are invariant by $Ad_{a^{-1}}$ because the left invariant metric λ is right invariant by translations obtained from elements of H.

It follows that

$$p_{\mathcal{H}} \circ Ad_{a^{-1}} \circ p_{\mathcal{H}} (\alpha (X)) = Ad_{a^{-1}} \circ p_{\mathcal{H}} (\alpha (X)) \text{ and } p_{\mathcal{H}} \circ Ad_{a^{-1}} \circ p_{\mathcal{H}^{\perp}} (\alpha (X)) = 0.$$

This is true for all $a \in H$ and for all $X \in TG$,

$$(R_a^* \alpha_{\mathcal{H}}) (X) = Ad_{a^{-1}} \circ p_{\mathcal{H}} (\alpha (X)) = Ad_{a^{-1}} \circ \alpha_{\mathcal{H}} (X).$$

This means that $\alpha_{\mathcal{H}}$ is a connection on the principal bundle $\pi : G \to \frac{G}{H}$.

Let $\mathcal{F}_{G,H}$ be the foliation obtained by the left translations of H and let $(T\mathcal{F}_{G,H})^{\perp}$ be the orthogonal bundle of $T\mathcal{F}_{G,H}$.

It is clear that $(T\mathcal{F}_{G,H})^{\perp} \subset \ker \alpha_{\mathcal{H}}$. But $\dim \ker \alpha_{\mathcal{H}} = \dim \frac{G}{H} = \dim (T\mathcal{F}_{G,H})^{\perp}$ so $\ker \alpha_{\mathcal{H}} = (T\mathcal{F}_{G,H})^{\perp}$.

In what follows the transverse bundle $\mathcal{V}(\mathcal{F}_{G,H})$ is identified to $(T\mathcal{F}_{G,H})^{\perp}$.

Above all, we note that H is not discrete so that foliation $\mathcal{F}_{G,H}$ is not a foliation by points.

Let $\mathcal{G}^r$ be the Lie algebra of right invariant vectors fields of G and let $\mathcal{H}^r$ be the Lie subalgebra of right invariant vectors fields obtained from the vectors of $\mathcal{H}$.

Any vector field $u^r \in \mathcal{G}^r$ associate with the vector $u \in \mathcal{G}$ commutes with every vectors field left invariant. Thus u^r is a $\mathcal{F}_{G,H}-$foliated vector field.

As the submersion $\pi : G \to \frac{G}{H}$ defined the foliation $\mathcal{F}_{G,H}$, u^r is projected by π on $\frac{G}{H}$ following a vector field notes that $\underline{u}^r$.

Let $\mathcal{X}\left(\frac{G}{H}\right)$ be the Lie algebra of vector fields tangent to $\frac{G}{H}$. Let $\mathcal{X}(G)$ be the Lie algebra of vector fields tangent to G. Let $w \in \mathcal{X}\left(\frac{G}{H}\right)$. Let $X \in \mathcal{X}(G)$. Let $\widetilde{w}$ be the horizontal lift of w. Let X^h be the horizontal component of X and let X^v be the vertical component of X.

We have ([1], [13]) for $w_1 \in \mathcal{X}\left(\frac{G}{H}\right)$ and $w_2 \in \mathcal{X}\left(\frac{G}{H}\right)$,

$$\begin{aligned} \pi_* ([\widetilde{w}_1, \widetilde{w}_2]) &= \pi_* \left([\widetilde{w}_1, \widetilde{w}_2]^h\right) \\ &= \pi_* \left(\widetilde{[w_1, w_2]}\right) \\ &= [w_1, w_2] \\ &= [\pi_* (\widetilde{w}_1), \pi_* (\widetilde{w}_2)]. \end{aligned}$$

Moreover, for all for all $(u_1, u_2) \in \mathcal{G}^2$ as u_1^r and u_2^r are $\mathcal{F}_{G,H}-$foliated vector fields then $\left[(u_1^r)^h, (u_2^r)^v\right]$ is a section of $T\mathcal{F}_{G,H}$ because $(u_1^r)^v$ and

$(u_2^r)^v$ are sections of $T\mathcal{F}_{G,H}$ and

$$\left[(u_1^r)^h, (u_2^r)^v\right] = [u_1^r, (u_2^r)^v] - [(u_1^r)^v, (u_2^r)^v].$$

It follows that $\pi_*([u_1^r, u_2^r]) = [\pi_*(u_1^r), \pi_*(u_2^r)]$ for all $(u_1, u_2) \in \mathcal{G}^2$ because

$$\begin{aligned}
\pi_*([u_1^r, u_2^r]) &= \pi_*\left(\left[(u_1^r)^v + (u_1^r)^h, (u_2^r)^v + (u_2^r)^h\right]\right) \\
&= \pi_*([(u_1^r)^v, u_2^r]) + \pi_*\left(\left[(u_1^r)^h, (u_2^r)^v\right]\right) + \pi_*\left(\left[(u_1^r)^h, (u_2^r)^h\right]\right) \\
&= \pi_*\left(\left[(u_1^r)^h, (u_2^r)^h\right]\right) \\
&= \pi_*\left(\left[\widetilde{\pi_*(u_1^r)}, \widetilde{\pi_*(u_2^r)}\right]\right) \\
&= \left[\pi_*\left(\widetilde{\pi_*(u_1^r)}\right), \pi_*\left(\widetilde{\pi_*(u_2^r)}\right)\right] \\
&= [\pi_*(u_1^r), \pi_*(u_2^r)]
\end{aligned}$$

The equality $\pi_*([u_1^r, u_2^r]) = [\pi_*(u_1^r), \pi_*(u_2^r)]$ for all $(u_1, u_2) \in \mathcal{G}^2$ show that $\underline{\mathcal{G}}^r = \pi_*(\mathcal{G}^r)$ is a Lie algebra because $\mathcal{G}^r$ is a Lie algebra.

As the right invariant vector fields u^r for all $u \in \mathcal{G}$ are Killing fields for the metric λ, then the fact that π is a Riemannian submersion implies that the vector fields $\underline{u}^r$ are also Killing vector fields. Thus $\underline{\mathcal{G}}^r$ is a Lie algebra of Killing fields on $\frac{G}{H}$.

We note that $(u^r)^h$ is the $\mathcal{F}_{G,H}-$ foliated transverse vectors field associated with u^r since $\ker \alpha_{\mathcal{H}} = (T\mathcal{F}_{G,H})^\perp$ and we have identified $(T\mathcal{F}_{G,H})^\perp$ and the transverse bundle $\mathcal{V}(\mathcal{F}_{G,H})$.

We note also that the horizontal lift $\widetilde{\underline{u}}^r$ of $\underline{u}^r = \pi_*(u^r)$ checks

$$\widetilde{\underline{u}}^r = (u^r)^h.$$

Let $(u_1, u_2...u_s, u_{s+1}, ..., u_q)$ be an orthonormal base of $\mathcal{G}$ such as $(u_{s+1}, ..., u_q)$ is a base of $\mathcal{H}$ and u_i^r the right invariant vector field obtained from u_i.

From the equality $\widetilde{\underline{u}}_i^r = (u_i^r)^h$ it follows that $(u_i^r)^h$ is invariant by right translations obtained from the elements of H. Therefore, these right translations being isometric, we have for all $i \le s$ and for all $a \in H$, $\left((u_1^r)^h(a), (u_2^r)^h(a), ..., (u_s^r)^h(a)\right)$ is an orthonormal base of $(T_aH)^\perp$ and

$$\widetilde{\underline{u}}_i^r(a) = (u_i^r)^h(a) = u_i^r(a).$$

Thus the fact that π is a Riemannian submersion implies that $\left(\underline{u}_1^r\left(\dot{e}\right), \underline{u}_2^r\left(\dot{e}\right), ..., \underline{u}_s^r\left(\dot{e}\right)\right)$ is an orthonormal base of $T_{\dot{e}}\frac{G}{H}$, where $\dot{e}$ is the class of the identity element e of G in $\frac{G}{H}$. This entails that $\underline{\mathcal{G}}^r$ is a Lie algebra of Killing fields on $\frac{G}{H}$ having $\dim\left(\frac{G}{H}\right)$ for maximal dimension of its orbits.

We know that ([14], [12]) the orbits of maximal dimension of a Lie algebra of Killing fields on a connected manifold V form an open dense in V. So there is an open U_r of $\frac{G}{H}$ dense in $\frac{G}{H}$ and containing $\overset{\bullet}{e}$ on which the orbits of $\underline{\mathcal{G}}^r$ are the dimension of $\frac{G}{H}$.

We note that there exists an open $U_r' \subset U_r$ such as U_r' is connected and $\overset{\bullet}{e} \in U_r'$.

We also note that $\frac{G}{H}$ is connected because G is connected and the map $\pi : G \to \frac{G}{H}$ is continuous.

By Proposition 3.1, the dimension of any orbit $\mathcal{O}_{\left(\overset{\bullet}{e},\overset{\bullet}{x}\right)}$ of isotropy $i_{\overset{\bullet}{e}}$ of $\underline{\mathcal{G}}^r$ at every point $\overset{\bullet}{x}$ of U_r' is null. Now the the orbits of maximal dimension of the Lie algebra of Killing fields $i_{\overset{\bullet}{e}}$ on the manifold $\frac{G}{H}$ form a dense open of $\frac{G}{H}$. So the fact that $\mathcal{O}_{\left(\overset{\bullet}{e},\overset{\bullet}{x}\right)} = \left\{\overset{\bullet}{x}\right\}$ for every $\overset{\bullet}{x} \in U_r'$ implies that $\mathcal{O}_{\left(\overset{\bullet}{e},\overset{\bullet}{x}\right)} = \left\{\overset{\bullet}{x}\right\}$ for every $\overset{\bullet}{x} \in \frac{G}{H}$.

It is easy to see that $\underline{\mathcal{H}}^r = \pi_*\left(\mathcal{H}^r\right)$ is a Lie subalgebra of $i_{\overset{\bullet}{e}}$. Therefore the fact that $\mathcal{O}_{\left(\overset{\bullet}{e},\overset{\bullet}{x}\right)} = \left\{\overset{\bullet}{x}\right\}$ for every $\overset{\bullet}{x} \in \frac{G}{H}$ shows us that $\underline{\mathcal{H}}^r = \pi_*\left(\mathcal{H}^r\right)$ is null. This means that any vector field right invariant obtained from the vectors of $\mathcal{H}$ is tangent to the foliation $\mathcal{F}_{G,H}$.

Thus, for every $a \in G$, we have $aH = Ha$. In other words, $\mathcal{H}$ is an ideal of $\mathcal{G}$. □

The following result is another consequence of the proposition 3.1. This result is the basis of the generalization that we do in this paper. It allows us to look Riemannian foliations with dense leaves on a compact manifold with a new look. With this result we can for example associated with a Riemannian foliation $\mathcal{F}$ with dense leaves on a compact manifold a finished group whose properties depend on the nature of $\mathcal{F}$.

Proposition 3.3. *Let $\mathcal{F}$ be a Riemannian foliation with dense leaves on a compact connected manifold M and let $\overline{F^\natural}$ be the closure of a leaf $F^\natural$ of lifted foliation $\mathcal{F}^\natural$ of $\mathcal{F}$ on the orthonormal transverse frame bundle $M^\natural$.*

Then $\overline{F^\natural}$ is a compact covering of M.

Proof . Let $\mathcal{F}$ be a Riemannian foliation with dense leaves on a compact connected manifold M. Let $\phi : \overline{F^\natural} \to M$ be the projection which to a frame at x associates x. Let $f : U \to \overline{U}$ be a Riemannian submersion defining Riemannian foliation $\mathcal{F}$ on a distinguished connected open U and let $f^\natural$:

$\phi^{-1}(U) \to E^{\natural}\left(\overline{U}\right)$ be the projection of $\phi^{-1}(U)$ on the orthonormal frame bundle of the local $\mathcal{F}$-quotient manifold $\overline{U}$.

We know from [12] that $\phi : \overline{F^{\natural}} \to M$ is a principal fibration and the submersion $f^{\natural}$ defined the lifted foliation $\mathcal{F}^{\natural}$ on $\phi^{-1}(U)$ and there exists a submersion $\overline{\phi} : E^{\natural}\left(\overline{U}\right) \to \overline{U}$ making the diagram

$$\begin{array}{ccc} \phi^{-1}(U) & \overset{f^{\natural}}{\to} & E^{\natural}\left(\overline{U}\right) \\ \phi\downarrow & & \downarrow\overline{\phi} \\ U & \overset{f}{\to} & \overline{U} \end{array}$$

commutative because ϕ sends the fibers of $f^{\natural}$ on the fibers of f. According to Molino [12], $\overline{\phi} : E^{\natural}\left(\overline{U}\right) \to \overline{U}$ is the orthonormal transverse frame bundle above the local $\mathcal{F}$-quotient manifold $\overline{U}$.

Let X be a Killing vector fields on $\overline{U}$.

We recall that if $\left(\varphi_t^X\right)_{|t|<\varepsilon}$ is the local 1 parameter group associated to X, then the local 1 parameter group $\left(\varphi_t^X\right)_* \circ x^{\natural}$, where $x^{\natural} \in E^{\natural}\left(\overline{U}\right)$ defined a vector field $X^{\natural}$ on $E^{\natural}\left(\overline{U}\right)$ that we call the lifted vectors field of X on $E^{\natural}\left(\overline{U}\right)$ [12]. This lifted vectors field commutes with the canonical parallelism [12] of $E^{\natural}\left(\overline{U}\right)$.

We also recall that [12] any vector field $Y^{\natural}$ of $E^{\natural}\left(\overline{U}\right)$ coincides in a neighborhood of each point of $E^{\natural}\left(\overline{U}\right)$ with the lift of a local Killing vector field on $\overline{U}$ if and only if $Y^{\natural}$ commutes with the canonical parallelism of $E^{\natural}\left(\overline{U}\right)$.

Since $\mathcal{F}$ is a Riemannian foliation with dense leaves, the Lie algebra $\mathcal{G}^r$ of right invariant vectors fields obtained from the structural Lie algebra $\mathcal{G}$ of $\mathcal{F}$ is a Lie algebra of Killing vectors fields operating transitively [12] respectively on each connected component of $E^{\natural}\left(\overline{U}\right)$ and on $\overline{U}$. To be specific, it is the Lie algebra of Killing fields $\overline{\phi}_*(\mathcal{G}^r)$ isomorphic ([12], [11]) to $\mathcal{G}^r$ which operates transitively on $\overline{U}$ and the lifted Lie algebra of $\overline{\phi}_*(\mathcal{G}^r)$ on $E^{\natural}\left(\overline{U}\right)$ is $\mathcal{G}^r$.

Let $x_0 \in \overline{U}$. By proposition 3.1, the dimension of any orbit $\mathcal{O}_{(x_0,x)}$ of the isotropy i_{x_0} of $\overline{\phi}_*(\mathcal{G}^r)$ at every point x of $\overline{U}$ is null.

Thus, the isotropy i_x of $\overline{\phi}_*(\mathcal{G}^r)$ for every $x \in \overline{U}$ is null.

As

$$\dim i_x = \dim\left(\overline{\phi}_*(\mathcal{G}^r)\right) - \dim \mathcal{O}_x = \dim\left(\overline{\phi}_*(\mathcal{G}^r)\right) - \dim \overline{U},$$

where $\mathcal{O}_x$ is the orbit at the point $x \in \overline{U}$ of $\overline{\phi}_*(\mathcal{G}^r)$, then

$$\dim \mathcal{G} = \dim \mathcal{G}^r = \dim\left(\overline{\phi}_*(\mathcal{G}^r)\right) = \dim \overline{U} = co\dim \mathcal{F}.$$

Let H be the structure group of the principal bundle $\phi : \overline{F^\natural} \to M$.

Using the fact that the restriction $\mathcal{F}^\natural_{\overline{F^\natural}}$ of $\mathcal{F}^\natural$ at $\overline{F^\natural}$ is a Lie $\mathcal{G}$-foliation with dense leaves is obtained that:

$$\dim M + \dim H = \dim \overline{F^\natural} = \dim \mathcal{F} + \dim \mathcal{G} = \dim \mathcal{F} + co\dim \mathcal{F} = \dim M.$$

Equality $\dim M + \dim H = \dim M$ shows that $\dim H = 0$ which means that H is discrete.

It follows that the principal bundle $\phi : \overline{F^\natural} \to M$ is a covering.

Before concluding, we note that H is finished because it is a structure group of a compact covering. □

We note that this proposition allows us to say that the dimension of the Lie structural algebra of a Riemannian foliation on a compact connected manifold is less than or equal to the codimension of this foliation.

Corollary 3.4. *All Riemannian foliation $\mathcal{F}$ with dense leaves on a compact connected simply connected manifold M is an abelian $\mathcal{G}$-foliation with dense leaves.*

Proof. Let $\mathcal{F}$ be a Riemannian foliation with dense leaves on a compact connected simply connected manifold M and let $\overline{F^\natural}$ be the closure of a leaf $F^\natural$ of lifted foliation $\mathcal{F}^\natural$ of $\mathcal{F}$ on the orthonormal transverse frame bundle $M^\natural$.

By the previous proposition, $\overline{F^\natural}$ is a covering of M. As M is a simply connected manifold then M is diffeomorphic to $\overline{F^\natural}$. This implies that $\mathcal{F}$ is a Lie $\mathcal{G}$-foliation with dense leaves.

As the Lie structural algebra of a Riemannian foliation on a simply connected compact connected manifold is an abelian Lie algebra [12], we obtain $\mathcal{F}$ is an abelian Lie $\mathcal{G}$-foliation with dense leaves. □

The following theorem is the main result of this article. This is a generalization of theorems 2.3 and 2.4 to Riemannian foliations with dense leaves on a compact manifold.

In what follows the structure group of covering $\phi : \overline{F^\natural} \to M$ will be noted H_0 and we will call it the *discrete group of Riemannian foliation* $\mathcal{F}$ with dense leaves.

Theorem 3.5. *Let $\mathcal{G} = Lie\,(G)$ be the Lie structural algebra of a Riemannian foliation with dense leaves $(M, \mathcal{F})$ on a compact manifold M. Let H_0 be the discrete group of $\mathcal{F}$ and let λ be a metric on M which is bundle-like for $\mathcal{F}$ and which admits λ_T as its associated transverse metric.*

Then there exists a representation $\rho : H_0 \to Diff\,(V)$ where V is an open of G such as:

i) there exists a biunivocal correspondence between the Lie subalgebras of $\mathcal{G} = Lie\,(G)$ invariant by $Ad_{(\rho_a(v))^{-1}.v}$ for every $(a, v) \in H_0 \times V$ and $\mathcal{F}$ extensions,

ii) an extension is a Lie foliation if the subalgebra corresponding is an ideal of $\mathcal{G}$,

iii) every extension $\mathcal{F}'$ of $\mathcal{F}$ is a Riemannian foliation and there exists a common bundle-like metric for the foliations $\mathcal{F}$ and $\mathcal{F}'$,

iv) if $\mathcal{F}_{\mathcal{H}}$ is an extension of $\mathcal{F}$ corresponding to a subalgebra $\mathcal{H}$ of $\mathcal{G}$ then to Lie algebra isomorphism nearly we have

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{u \in \mathcal{H}^{\perp} / \forall\,(h, a, v) \in \mathcal{H} \times H_0 \times V,\ [u, h] = 0 \ \ and\ Ad_{(\rho_a(v))^{-1}.v}\,(u) = u\ \}.$$

In particular

$$\ell(M, \mathcal{F}) = \{u \in \mathcal{G} / \forall\,(a, v) \in H_0 \times V,\ Ad_{(\rho_a(v))^{-1}.v}\,(u) = u\ \}.$$

Proof. Let $\mathcal{F}$ be a Riemannian foliation on a compact connected manifold M, let $\overline{F^\natural}$ be the closure of a leaf $F^\natural$ of lifted foliation $\mathcal{F}^\natural$ of $\mathcal{F}$ on the orthonormal transverse frame bundle $M^\natural$. L et ϕ:$\overline{F^\natural} \to M$ be the projection which to a frame at x associates x and let $\mathcal{F}^\natural_{\overline{F^\natural}}$ be the restriction of $\mathcal{F}^\natural$ at $\overline{F^\natural}$.

We know that $\phi : \overline{F^\natural} \to M$ is a covering having H_0 for structure group (*cf. Proposition* 3.3).

In what follows, $(U_i, f_i, T, \gamma_{ij})_{i \in I}$ denotes a foliated cocycle defining the Riemannian foliation $\mathcal{F}$ such as open U_i are open of local trivialization of covering $\phi : \overline{F^\natural} \to M$ and $f_i^\natural : \phi^{-1}(U_i) \to E^\natural\left(\overline{U}_i\right)$ denotes the projection of $\phi^{-1}(U_i)$ on the orthonormal transverse frame bundle $\overline{\phi}_i : E^\natural\left(\overline{U}_i\right) \to \overline{U}_i$ above the local $\mathcal{F}$-quotient manifold $\overline{U}_i$ of U_i.

The fact that $\phi : \overline{F^\natural} \to M$ is a covering of M implies that for every open U_i of foliated cocycle $(U_i, f_i, T, \gamma_{ij})_{i \in I}$ of $\mathcal{F}$ there are open $U^\natural_{ia}$ of $\overline{F^\natural}$ where $a \in H_0$ such that:

- $\phi^{-1}(U_i) = \bigcup\limits_{a \in H_0} U^\natural_{ia}$ and $\phi : U^\natural_{ia} \to U_i$ is a local diffeomorphism,

- for every $(a, b) \in H_0 \times H_0$, $R^\natural_b\left(U^\natural_{ia}\right) = U^\natural_{iab}$ where $R^\natural_b$ is the right translation on $\overline{F^\natural}$ associated to b. Moreover, $\left(U^\natural_{ia}\right)_{(i,a) \in I \times H_0}$ is an open cover of $\overline{F^\natural}$.

According to Molino in [12], each submersion $f_i^\natural : \phi^{-1}(U_i) \to E^\natural\left(\overline{U}_i\right)$ defined the Lie $G-$foliation $\mathcal{F}^\natural_{\overline{F^\natural}}$ on $\phi^{-1}(U_i)$. Thus $E^\natural\left(\overline{U}_i\right) \subset G$.

We also note that according to Molino [12] for all $a \in H_0$ and all $x^\natural \in \phi^{-1}(U_i)$ we have

$$f_i^\natural \circ R_a^\natural\left(x^\natural\right) = \rho_a \circ f_i^\natural\left(x^\natural\right)$$

where ρ_a is a diffeomorphism on the open $V = E^\natural\left(\overline{U}_i\right)$ of G induce by $R_a^\natural$ because $\mathcal{F}^\natural_{\overline{F^\natural}}$ is invariant by $R_a^\natural$.

We easily verify that $\rho : H_0 \to Diff(V)$ is a representation and $\rho(H_0) \backsimeq H_0$.

Let λ be a $\mathcal{F}-$bundle-like metric on M.

As $\phi : \overline{F^\natural} \to M$ is a covering and $\mathcal{F}^\natural_{\overline{F^\natural}} = \phi^*\mathcal{F}$, the metric $\widetilde{\lambda} = \phi^*\lambda$ is a $\mathcal{F}^\natural_{\overline{F^\natural}}-$bundle-like metric on $\phi : \overline{F^\natural} \to M$ and the right translation $R_a^\natural$ on $\overline{F^\natural}$ associated to $a \in H_0$ is an isometry for the metric $\widetilde{\lambda}$.

The equality $f_i^\natural \circ R_a^\natural = \rho_a \circ f_i^\natural$ implies that ρ_a is an isometry because $R_a^\natural$ is an isometry and $f_i^\natural$ is a Riemannian submersion defining $\mathcal{F}^\natural_{\overline{F^\natural}}$ on $\phi^{-1}(U_i)$.

Note that [12] for all $i \in I$ the diagram

$$\begin{array}{ccc} \phi^{-1}(U_i) & \overset{f_i^\natural}{\to} & E^\natural\left(\overline{U}_i\right) \\ \phi\downarrow & & \downarrow\overline{\phi}_i \\ U_i & \overset{f_i}{\to} & \overline{U}_i \end{array}$$

is commutative and the $f_i^\natural \circ R_a^\natural = \rho_a \circ f_i^\natural$ implies again that $\overline{\phi}_i : E^\natural\left(\overline{U}_i\right) \to \overline{U}_i$ is a covering having $\rho(H_0) \backsimeq H_0$ for structure group.

As $\mathcal{F}$ is a Riemannian foliation with dense leaves, the Lie algebra $\mathcal{G}^r$ of right invariant vectors fields obtained from the structural Lie algebra $\mathcal{G}$ of $\mathcal{F}$ is a Lie algebra of Killing vectors fields operating transitively [12] on each connected component of $E^\natural\left(\overline{U}\right)$ and for each $X^r \in \mathcal{G}^r$, $\left(\overline{\phi}_i\right)_*(X^r)$ is a vector fields on $\overline{U}$ then for all $(v,a) \in E^\natural\left(\overline{U}_i\right) \times H_0$ and for all $X^r \in \mathcal{G}^r$,

$$(\rho_a)_*(X_v^r) = X^r_{\rho_a(v)} = R_{v^{-1}.\rho_a(v)}(X_v^r).$$

The fact $\mathcal{G}^r$ operate transitively [12] on each connected component of $E^\natural\left(\overline{U}\right)$ implies that for all $(v, X_v) \in E^\natural\left(\overline{U}_i\right) \times T_vE^\natural\left(\overline{U}\right)$ there exists $X^r \in \mathcal{G}^r$ such as $X_v = X_v^r$.

Thus, for all $\left(v, x^\natural, a\right) \in E^\natural\left(\overline{U}_i\right) \times \phi^{-1}(U_i) \times H_0$ and for all $\left(X_v, Y^\natural_{x^\natural}\right) \in T_vE^\natural\left(\overline{U}\right) \times T_{x^\natural}\phi^{-1}(U_i)$,

$$(\rho_a)_*(X_v) = R_{v^{-1}.\rho_a(v)}(X_v^r) = R_{v^{-1}.\rho_a(v)}(X_v)$$

and

$$\left(f_i^\natural \circ R_a^\natural\right)_* \left(Y_{x^\natural}^\natural\right) = \left(\rho_a \circ f_i^\natural\right)_* \left(Y_{x^\natural}^\natural\right) = R_{v^{-1}.\rho_a(v)} \left(\left(f_i^\natural\right)_* \left(Y_{x^\natural}^\natural\right)\right) \text{ for } v = f_i^\natural \left(x^\natural\right).$$

In what follows, $\omega^\natural$ denotes the 1-form of Fedida of Lie $G-$foliation $\mathcal{F}_{\overline{F^\natural}}^\natural$ whose restriction $\omega_i^\natural$ at each open $U_i^\natural = \phi^{-1}\left(U_i\right)$ is such that

$$\omega_i^\natural = \left(f_i^\natural\right)^* \alpha_i$$

where α_i is the restriction of left Maurer–Cartan form α of G at $E^\natural\left(\overline{U}_i\right)$.

For all $\left(x^\natural, a\right) \in \phi^{-1}\left(U_i\right) \times H_0$ and for all $Y_{x^\natural}^\natural \in T_z\phi^{-1}\left(U_i\right)$ we have

$$\begin{aligned}
\left(R_a^\natural\right)^* \omega^\natural \left(Y_{x^\natural}^\natural\right) &= \omega_i^\natural \circ \left(R_a^\natural\right)_* \left(Y_{x^\natural}^\natural\right) \\
&= \alpha_i \left(\left(f_i^\natural \circ R_a^\natural\right)_* \left(Y_{x^\natural}^\natural\right)\right) \\
&= \alpha_i \left(\left(\rho_a\right)_* \left(\left(f_i^\natural\right)_* \left(Y_{x^\natural}^\natural\right)\right)\right) \\
&= \alpha_i \left(R_{v^{-1}.\rho_a(v)} \left(\left(f_i^\natural\right)_* \left(Y_{x^\natural}^\natural\right)\right)\right) \text{ for } v = f_i^\natural\left(x^\natural\right) \\
&= Ad_{(\rho_a(v))^{-1}.v} \circ \omega^\natural \left(Y_{x^\natural}^\natural\right).
\end{aligned}$$

i) Let $\mathcal{F}'$ be an extension of $\mathcal{F}$ and let $\phi^*\mathcal{F}'$ be the inverse image of $\mathcal{F}'$.

We easily verify that $\mathcal{F}_{\overline{F^\natural}}^\natural \subset \phi^*\mathcal{F}'$. This implies that (*cf. theoreme 2.3*) there exists a Lie subalgebra of $\mathcal{G} = Lie\left(G\right)$ corresponding to $\phi^*\mathcal{F}'$. It will be noted $\mathcal{G}_{\mathcal{F}'}$.

We know that [6] $\phi^*\mathcal{F}'$ is defined by the differential system $\mathcal{P}^\natural$ defined on $\overline{F^\natural}$ by

$$\mathcal{P}^\natural\left(x^\natural\right) = T_{x^\natural}\mathcal{F}_{\overline{F^\natural}}^\natural \oplus ev_{x^\natural}\left(\mathcal{G}_{\mathcal{F}'}^\natural\right)$$

where $\mathcal{G}_{\mathcal{F}'}^\natural$ is the Lie algebra of $\mathcal{F}_{\overline{F^\natural}}^\natural-$foliated transverse vectors fields associated to $\mathcal{G}_{\mathcal{F}'}$ and $ev_{x^\natural}\left(X^\natural\right) = X_{x^\natural}^\natural$ for every $X^\natural \in \mathcal{G}_{\mathcal{F}'}^\natural$.

Let $a \in H_0$. The foliation $\phi^*\mathcal{F}'$ is invariant by the right translation $R_a^\natural$ on $\overline{F^\natural}$ associated to a. Hence the differential system $x^\natural \mapsto \mathcal{P}^\natural\left(x^\natural\right)$ is invariant by $R_a^\natural$.

As $T\mathcal{F}_{\overline{F^\natural}}^\natural$ and the differential system $x^\natural \mapsto \mathcal{P}^\natural\left(x^\natural\right)$ are invariant by the isometry $R_a^\natural$ and as the ortho-complementary of $T_{x^\natural}\mathcal{F}_{\overline{F^\natural}}^\natural$ in $\mathcal{P}^\natural\left(x^\natural\right)$ is $ev_{x^\natural}\left(\mathcal{G}_{\mathcal{F}'}^\natural\right)$ then for all $x^\natural \in \phi^{-1}\left(U_i\right)$,

$$\left(R_a^\natural\right)_* \left(ev_{x^\natural}\left(\mathcal{G}_{\mathcal{F}'}^\natural\right)\right) = ev_{R_a^\natural\left(x^\natural\right)}\left(\mathcal{G}_{\mathcal{F}'}^\natural\right).$$

But [6] for all $x^\natural \in \phi^{-1}(U_i)$

$$\omega_i^\natural\left(ev_{x^\natural}\left(\mathcal{G}^\natural_{\mathcal{F}'}\right)\right) = \mathcal{G}_{\mathcal{F}'}$$

so for all $v = f_i^\natural\left(x^\natural\right) \in E^\natural\left(\overline{U}_i\right)$ we have

$$\begin{aligned}
\mathcal{G}_{\mathcal{F}'} &= \omega_i^\natural\left(ev_{R_a^\natural(x^\natural)}\left(\mathcal{G}^\natural_{\mathcal{F}'}\right)\right) \\
&= \omega_i^\natural\left(\left(R_a^\natural\right)_*\left(ev_{x^\natural}\left(\mathcal{G}^\natural_{\mathcal{F}'}\right)\right)\right) \\
&= \left(f_i^\natural\right)^*\alpha_i\left(R_{a*}^\natural\left(ev_{x^\natural}\left(\mathcal{G}^\natural_{\mathcal{F}'}\right)\right)\right) \\
&= \alpha\left(\left(f_i^\natural\right)_*\circ R_{a*}^\natural\left(ev_{x^\natural}\left(\mathcal{G}^\natural_{\mathcal{F}'}\right)\right)\right) \\
&= \alpha\left(R_{v^{-1}.\rho_a(v)}\left(\left(f_i^\natural\right)_*\right)\left(ev_{x^\natural}\left(\mathcal{G}^\natural_{\mathcal{F}'}\right)\right)\right) \\
&= Ad_{(\rho_a(v))^{-1}.v}\circ\alpha\left(\left(f_i^\natural\right)_*\left(ev_{x^\natural}\left(\mathcal{G}^\natural_{\mathcal{F}'}\right)\right)\right) \\
&= Ad_{(\rho_a(v))^{-1}.v}\circ\left(\left(f_i^\natural\right)^*\alpha_i\right)\left(ev_{x^\natural}\left(\mathcal{G}^\natural_{\mathcal{F}'}\right)\right) \\
&= Ad_{(\rho_a(v))^{-1}.v}\circ\omega_i^\natural\left(ev_{x^\natural}\left(\mathcal{G}^\natural_{\mathcal{F}'}\right)\right) \\
&= Ad_{(\rho_a(v))^{-1}.v}\left(\mathcal{G}_{\mathcal{F}'}\right)
\end{aligned}$$

Conversely, suppose there is a Lie subalgebra $\mathcal{G}'$ of $\mathcal{G}$ such as $Ad_{(\rho_a(v))^{-1}.v}\left(\mathcal{G}'\right) = \mathcal{G}'$ for all $v = f_i^\natural\left(x^\natural\right) \in E^\natural\left(\overline{U}_i\right)$ and for all $a \in H_0$.

As $Ad_{(\rho_a(v))^{-1}.v}\left(\mathcal{G}'\right) = \mathcal{G}'$ for all $a \in H_0$ and for all $v = f_i^\natural\left(x^\natural\right) \in E^\natural\left(\overline{U}_i\right)$ then the differential system $\mathcal{S}^\natural$ defined on $\overline{F^\natural}$ by $\mathcal{S}^\natural\left(x^\natural\right) = ev_{x^\natural}\left(\mathcal{G}'^\natural\right)$ where $\mathcal{G}'^\natural$ is the Lie algebra of $\mathcal{F}^\natural_{\overline{F^\natural}}-$foliated transverse vector fields associated to $\mathcal{G}'$, is invariant by $R_a^\natural$ for all $a \in H_0$.

Indeed, for $a \in H_0$ we have:

$$\begin{aligned}
\omega_i^\natural\left(ev_{R_a^\natural(x^\natural)}\left(\mathcal{G}'^\natural\right)\right) &= \mathcal{G}' \\
&= Ad_{(\rho_a(v))^{-1}.v}\left(\mathcal{G}'\right) \quad \text{for } v = f_i^\natural\left(x^\natural\right) \\
&= Ad_{(\rho_a(v))^{-1}.v}\left(ev_{x^\natural}\left(\mathcal{G}'^\natural\right)\right) \\
&= \left(R_a^\natural\right)^*\omega_i^\natural\left(ev_{x^\natural}\left(\mathcal{G}'^\natural\right)\right) \\
&= \omega_i^\natural\left(\left(R_a^\natural\right)_*\left(ev_{x^\natural}\left(\mathcal{G}'^\natural\right)\right)\right).
\end{aligned}$$

The fact that $\omega_i^\natural : \left(T_{R_a^\natural(x^\natural)}\mathcal{F}^\natural_{\overline{F^\natural}}\right)^\perp \to \mathcal{G}$ is an isomorphism, because $\mathcal{F}^\natural_{\overline{F^\natural}}$ is a $\mathcal{G}-$Lie foliation with dense leaves, implies that

$$\left(R_a^\natural\right)_{*x^\natural}\left(ev_{x^\natural}\left(\mathcal{G}'^\natural\right)\right) = \left(ev_{R_a^\natural(x^\natural)}\left(\mathcal{G}'^\natural\right)\right)$$

for all $a \in H_0$ and $x^\natural \in \overline{F^\natural}$.

It follows that the extension $\mathcal{F}^\natural_{\mathcal{G}'}$ of $\mathcal{F}^\natural_{\overline{F^\natural}}$ corresponding to $\mathcal{G}'$ is invariant by the right translation of the elements of H_0 because ([6], [8])

$$T_{x^\natural}\mathcal{F}^\natural_{\mathcal{G}'} = T_{x^\natural}\mathcal{F}^\natural_{\overline{F^\natural}} \oplus ev_{x^\natural}\left(\mathcal{G}'^\natural\right)$$

for all $x^\natural \in \overline{F^\natural}$ and $T\mathcal{F}^\natural_{\overline{F^\natural}}$ is invariant by $R_a^\natural$ for all $a \in H_0$.

Thus $\mathcal{F}^\natural_{\mathcal{G}'}$ is projected by ϕ into an extension $\mathcal{F}'$ of $\mathcal{F}$.

ii) Let $\mathcal{F}'$ be an extension of Lie of Riemannian foliation $(M, \mathcal{F})$ with dense leaves on a compact manifold M. Let $\phi^*\mathcal{F}'$ be the lifted foliation of $\mathcal{F}'$ on the covering $\phi : \overline{F^\natural} \to M$. Let $\ell(M, \mathcal{F}')$ be the Lie algebra of $\mathcal{F}'-$foliated transverse vectors fields and let $\ell^\natural(M, \mathcal{F}')$ be the lifted of $\ell(M, \mathcal{F}')$ on the covering $\overline{F^\natural}$.

For all $X^\natural \in \ell^\natural(M, \mathcal{F}')$ and for all $Y^\natural \in \mathcal{X}\left(\overline{F^\natural}, \phi^*\mathcal{F}'\right)$ where $\mathcal{X}\left(\overline{F^\natural}, \phi^*\mathcal{F}'\right)$ is the Lie algebra of vector fields tangent to $\phi^*\mathcal{F}'$, we have $\forall\,(i,a) \in I \times H_0$,

$$\phi_*\left(\left[X^\natural, Y^\natural\right]_{/U_{ia}^\natural}\right) = \left[\phi_*\left(X^\natural_{/U_{ia}^\natural}\right), \phi_*\left(Y^\natural_{/U_{ia}^\natural}\right)\right] \quad (**)\,.$$

We note that $\forall\,(i,a) \in I \times H_0$, $\left[\phi_*\left(X^\natural_{/U_{ia}^\natural}\right), \phi_*\left(Y^\natural_{/U_{ia}^\natural}\right)\right]$ is tangent to $\mathcal{F}'$ because $\phi_*\left(X^\natural_{/U_{ia}^\natural}\right)$ is $\mathcal{F}'-$foliated and $\phi_*\left(Y^\natural_{/U_{ia}^\natural}\right)$ is tangent to $\mathcal{F}'$. It follows from this that $\phi_*\left(\left[X^\natural, Y^\natural\right]_{/U_{ia}^\natural}\right)$ is tangent to $\mathcal{F}'$ for all $(i,a) \in I \times H_0$. Therefore, $\left[X^\natural, Y^\natural\right]$ is tangent to $\phi^*\mathcal{F}'$.

Thus, all vectors fields of $\ell^\natural(M, \mathcal{F}')$ is $\phi^*\mathcal{F}'-$foliated.

Using the fact that for any $(i,a) \in I \times H_0$, $\phi : U_{ia}^\natural \to U_i$ is an isometry (relatively to the metrics λ and $\widetilde{\lambda}$), it is easily verified that all vectors field of $\ell^\natural(M, \mathcal{F}')$ is $\phi^*\mathcal{F}'-$tranverse since we identify the orthogonal bundle $(T\phi^*\mathcal{F}')^\perp$ of $T\phi^*\mathcal{F}'$ and the transverse bundle $\mathcal{V}\left(\phi^*\mathcal{F}'\right)$.

We note that $\ell^\natural(M, \mathcal{F}')$ is stable for the Lie bracket of two vector fields.

Indeed, for all $X^\natural \in \ell^\natural(M, \mathcal{F}')$ and $Y^\natural \in \ell^\natural(M, \mathcal{F}')$, there exists $X \in \ell(M, \mathcal{F}')$ and $Y \in \ell(M, \mathcal{F}')$ such as $\phi_*\left(X^\natural\right) = X$ and $\phi_*\left(Y^\natural\right) = Y$.

The lifted vector fields $X^{\natural}$ and $Y^{\natural}$ are invariant by the right translations obtained from the elements of H_0. It follows that $\left[X^{\natural}, Y^{\natural}\right]$ is invariant by the right translations obtained from elements of H_0.

Thus, $\left[X^{\natural}, Y^{\natural}\right]$ is projected by ϕ along a vector field on M. We have, for all $(i,a) \in I \times H_0$,

$$\begin{aligned}
\left(\phi_*\left(\left[X^{\natural}, Y^{\natural}\right]\right)\right)_{/U_i} &= \phi_*\left(\left[X^{\natural}, Y^{\natural}\right]_{/U^{\natural}_{ia}}\right) \\
&= \left[\phi_*\left(X^{\natural}_{/U^{\natural}_{ia}}\right), \phi_*\left(Y^{\natural}_{/U^{\natural}_{ia}}\right)\right] \\
&= \left[\left(\phi_*\left(X^{\natural}\right)\right)_{/U_i}, \left(\phi_*\left(Y^{\natural}\right)\right)_{/U_i}\right] \\
&= \left(\left[\left(\phi_*\left(X^{\natural}\right)\right), \left(\phi_*\left(Y^{\natural}\right)\right)\right]\right)_{/U_i} \\
&= [X, Y]_{/U_i}.
\end{aligned}$$

Thus $\phi_*\left(\left[X^{\natural}, Y^{\natural}\right]\right) = [X, Y]$.

Since $[X, Y] \in \ell(M, \mathcal{F}')$, $\left[X^{\natural}, Y^{\natural}\right] \in \ell^{\natural}(M, \mathcal{F}')$; that is, $\ell^{\natural}(M, \mathcal{F}')$ is stable for the Lie bracket.

It follows that $\ell^{\natural}(M, \mathcal{F}')$ is a Lie algebra of $\phi^*\mathcal{F}'-$foliated transverse vector fields.

We note that $\ell^{\natural}(M, \mathcal{F}')$ has the same dimension as $\ell(M, \mathcal{F}')$.

The fact that $\phi^*\mathcal{F}'$ is an extension of the Lie foliation $\mathcal{F}^{\natural}_{\overline{F^{\natural}}}$ ensures, by Proposition 2.6, that $\ell^{\natural}(M, \mathcal{F}')$ is a Lie algebra of vectors fields $\mathcal{F}^{\natural}_{\overline{F^{\natural}}}-$foliated transverse.

Let $\mathcal{G}^{\natural}_{\mathcal{F}'}$ be the Lie algebra of $\mathcal{F}^{\natural}_{\overline{F^{\natural}}}-$foliated transverse vectors field associated to the Lie subalgebra $\mathcal{G}_{\mathcal{F}'}$ corresponding of the extension $\phi^*\mathcal{F}'$ of $\mathcal{F}^{\natural}_{\overline{F^{\natural}}}$.

We signal that $\ell^{\natural}(M, \mathcal{F}')$ is ortho- complementary to $\mathcal{G}^{\natural}_{\mathcal{F}'}$ in $\ell(\overline{F^{\natural}}, \mathcal{F}^{\natural}_{\overline{F^{\natural}}})$.

Indeed, $\dim\left(\ell^{\natural}(M, \mathcal{F}')\right) = \dim\left(\ell(M, \mathcal{F}')\right) = co\dim\left(\phi^*\mathcal{F}'\right)$ and [6]

$$\dim\left(\mathcal{G}^{\natural}_{\mathcal{F}'}\right) = co\dim\left(\mathcal{F}^{\natural}_{\overline{F^{\natural}}}\right) - co\dim\left(\phi^*\mathcal{F}'\right) = \dim\left(\ell(\overline{F^{\natural}}, \mathcal{F}^{\natural}_{\overline{F^{\natural}}})\right) - \dim\left(\ell^{\natural}(M, \mathcal{F}')\right)$$

and $\ell^{\natural}(M, \mathcal{F}')$ is orthogonal to $\mathcal{G}^{\natural}_{\mathcal{F}'}$ because $\ell^{\natural}(M, \mathcal{F}')$ is a Lie algebra of $\phi^*\mathcal{F}'-$foliated transverse vector fields.

For $X^{\natural} \in \ell^{\natural}(M, \mathcal{F}')$ and $Y^{\natural} \in \mathcal{G}^{\natural}_{\mathcal{F}'}$ we have $\left[X^{\natural}, Y^{\natural}\right] \in \ell(\overline{F^{\natural}}, \mathcal{F}^{\natural}_{\overline{F^{\natural}}})$ and the equality $(**)$ shows that $\left[X^{\natural}, Y^{\natural}\right]$ is tangent to $\phi^*\mathcal{F}'$ that is to say $\left[X^{\natural}, Y^{\natural}\right] \in \mathcal{X}\left(\overline{F^{\natural}}, \phi^*\mathcal{F}'\right)$.

Thus, for all $X^\natural \in \ell^\natural(M, \mathcal{F}')$ and $Y^\natural \in \mathcal{G}^\natural_{\mathcal{F}'}$ we have $\left[X^\natural, Y^\natural\right] \in \mathcal{G}^\natural_{\mathcal{F}'}$ because [6]

$$\ell(\overline{F^\natural}, \mathcal{F}^\natural_{\overline{F^\natural}}) \cap \mathcal{X}\left(\overline{F^\natural}, \phi^*\mathcal{F}'\right) = \mathcal{G}^\natural_{\mathcal{F}'}.$$

It follows that $\mathcal{G}^\natural_{\mathcal{F}'}$ is an ideal of $\ell(\overline{F^\natural}, \mathcal{F}^\natural_{\overline{F^\natural}})$. This implies that $\mathcal{G}_{\mathcal{F}'}$ is an ideal of $\mathcal{G}$.

iii) Let $\mathcal{F}'$ be an extension of $\mathcal{F}$ and let λ be a $\mathcal{F}$-bundle-like metric on M.

To reduce the size of open sets, we distinguish each open set U_i for $\mathcal{F}$ and $\mathcal{F}'$.

The local isometry $\phi : U^\natural_{ia} \to U_i$ sends the leaves of $\left(U^\natural_{ia}, \mathcal{F}^\natural_{\overline{F^\natural}}\right)$ on leaves of $(U_i, \mathcal{F})$ and leaves of $\left(U^\natural_{ia}, \phi^*\mathcal{F}'\right)$ on leaves of $(U_i, \mathcal{F}')$ for all $(i, a) \in I \times H_0$.

Since the metric $\widetilde{\lambda} = \phi^*\lambda$ is bundle-like for $\left(U^\natural_{ia}, \mathcal{F}^\natural_{\overline{F^\natural}}\right)$ and $\left(U^\natural_{ia}, \phi^*\mathcal{F}'\right)$, the metric λ is bundle-like for $(U_i, \mathcal{F})$ and $(U_i, \mathcal{F}')$. This implies that the metric λ is a common bundle-like metric for the foliations $\mathcal{F}$ and $\mathcal{F}'$.

iv) Let $\mathcal{F}_{\mathcal{H}}$ be the extension of $\mathcal{F}$ corresponding to a Lie subalgebra $\mathcal{H}$ of $\mathcal{G}$. Let $\mathcal{F}^\natural_{\mathcal{H}}$ be the extension of $\mathcal{F}^\natural_{\overline{F^\natural}}$ corresponding to $\mathcal{H}$, let $\ell(\overline{F^\natural}, \mathcal{F}^\natural_{\mathcal{H}})$ be the Lie algebra of $\mathcal{F}^\natural_{\mathcal{H}}-$foliated transverse vector fields and let $\ell_{inv}(\overline{F^\natural}, \mathcal{F}^\natural_{\mathcal{H}})$ be the Lie subalgebra of $\ell(\overline{F^\natural}, \mathcal{F}^\natural_{\mathcal{H}})$ of invariant vectors fields by the right translations $R^\natural_a$ for all $a \in H_0$.

The following equality

$$\mathcal{F}^\natural_{\mathcal{H}} = \phi^*\mathcal{F}_{\mathcal{H}}$$

holds true due to the fact that $\forall\, (i, a) \in I \times H_0$, $\phi : U^\natural_{ia} \to U_i$ is a local isometry that $\phi_* : \ell_{inv}(\overline{F^\natural}, \mathcal{F}^\natural_{\mathcal{H}}) \to \ell(M, \mathcal{F}_{\mathcal{H}})$ is an isomorphism of Lie algebras.

Using theorem 2.4 and the fact that for all $a \in H_0$ and $Y^\natural_{x^\natural} \in T_{x^\natural}\phi^{-1}(U_i)$

$$\left(R^\natural_a\right)^* \omega^\natural\left(Y^\natural_{x^\natural}\right) = Ad_{(\rho_a(v))^{-1}.v} \circ \omega^\natural\left(Y^\natural_{x^\natural}\right) \text{ for } v = f^\natural_i\left(x^\natural\right)$$

one easily gets equality between $\omega^\natural\left(\ell_{inv}(\overline{F^\natural}, \mathcal{F}^\natural_{\mathcal{H}})\right)$ and

$$\{u \in \mathcal{H}^\perp / \ \forall h \in \mathcal{H},\ [u, h] = 0 \ \text{ and } \forall\, (a, v) \in H_0 \times V,\ Ad_{(\rho_a(v))^{-1}.v}(u) = u\ \}.$$

But $\omega^\natural : \ell(\overline{F^\natural}, \mathcal{F}^\natural_{\overline{F^\natural}}) \to \mathcal{G}$ is an isomorphism of Lie algebras because the Lie foliation $\mathcal{F}^\natural_{\overline{F^\natural}}$ has leaves that are dense, so $\omega^\natural \circ (\phi_*)^{-1}$ is an isomorphism of Lie algebras between $\ell(M, \mathcal{F}_{\mathcal{H}})$ and

$$\{u \in \mathcal{H}^\perp / \ \forall h \in \mathcal{H},\ [u, h] = 0 \ \text{ and } \forall\, (a, v) \in H_0 \times V,\ Ad_{(\rho_a(v))^{-1}.v}(u) = u\ \}.$$

□

Corollary 3.6. *Let $\mathcal{G}$ be the structural Lie algebra of a Riemannian foliation with dense leaves $\mathcal{F}$ on a compact manifold M.*

Then $\mathcal{F}$ admits a complete flag of extensions if and only if $\mathcal{F}$ is an abelian Lie $\mathcal{G}-$ foliation with dense leaves.

Proof. Let q be the codimension of $\mathcal{F}$ and let λ be a $\mathcal{F}$-bundle-like metric on M.

Suppose that $\mathcal{F}$ admits a complete flag of extensions
$\mathcal{D}_{\mathcal{F}} = (\mathcal{F}_{q-1},\ \mathcal{F}_{q-2},\ ...,\ \mathcal{F}_1)$.

For $i \in \{1, 2, ..., q\ \}$, it is easy to verify that the dimension of the bundle $(T\mathcal{F}_i)^{\perp} \cap T\mathcal{F}_{i-1}$ is 1. We denote by X_i the unified field that orients $(T\mathcal{F}_i)^{\perp} \cap T\mathcal{F}_{i-1}$.

We note in passing that the foliation $\mathcal{F}_0$ has one leaf and this leaf is the manifold M. We note also that

$$\begin{aligned} T\mathcal{F}_{i-1} &= T\mathcal{F}_i \oplus (T\mathcal{F}_i)^{\perp} \cap T\mathcal{F}_{i-1} \\ &= T\mathcal{F}_i \oplus < X_i > \end{aligned}$$

So X_i is a $\mathcal{F}_i-$foliated transverse vector field.

Indeed, $\mathcal{F}$ is a Riemannian foliation with dense leaves. So, by theorem 3.7, $\mathcal{D}_{\mathcal{F}}$ is a complete flag of Riemannian extensions and the metric λ is a common bundle-like metric for the foliations $\mathcal{F}_i$.

Since X_i is tangent to $(T\mathcal{F}_i)^{\perp}$, we have for all $Y \in \mathcal{X}(\mathcal{F}_i)$,

$$0 = Y\lambda(X_i, X_i) = 2\lambda([Y, X_i], X_i).$$

Therefore $[Y, X_i]$ is orthogonal to X_i. But $[Y, X_i] \in \mathcal{X}(\mathcal{F}_{i-1})$ and

$$T\mathcal{F}_{i-1} = T\mathcal{F}_i \oplus < X_i >$$

so $[Y, X_i]$ is tangent to $T\mathcal{F}_i$. It follows that $X_i \in \ell(M, \mathcal{F}_i)$ because X_i is tangent to $(T\mathcal{F}_i)^{\perp}$.

In what follows, we denote $\mathcal{F}$ by $\mathcal{F}_q$. By proposition 2.6 for all i and j such as $1 \leq i < j \leq q$ we have $\ell(M, \mathcal{F}_i) \subset \ell(M, \mathcal{F}_j)$.

It follows that for all $i \leq j$ we have $X_i \in \ell(M, \mathcal{F}_j)$.

The fact that the unified vectors fields X_i are orthogonal two by two implies that $\mathcal{F}_j$ is a Lie G_j-foliation for all j and

$$\ell(M, \mathcal{F}_j) =< X_j, X_{j-1}, ...X_1 > .$$

Indeed, for all j the codimension of $\mathcal{F}_j$ is j and the leaves of $\mathcal{F}_j$ are dense.

Let U be a $\mathcal{F}_q-$distinguished open. Let $f : U \to \overline{U}$ be a Riemannian submersion defining $\mathcal{F}_q$ on U. Let ω_q be a Fedida 1-form of $\mathcal{F}_q$ such as $\omega_{q/U} = (f)^* \alpha_q$, where α_q is the left Maurer-Cartan form of G_q. Let $\omega_q (X_i) = \overline{X}_i$ and let $\mathcal{G}_q = Lie (G_q)$.

We note in passing that ω_q:$\ell(M, \mathcal{F}_q) \to \mathcal{G}_q$ is an isomorphism of Lie algebra because $\mathcal{F}_q$ is a Lie G_q-foliation with dense leaves.

It is easily verified using the equality $\omega_{q/U} = (f)^* \alpha_q$ and the fact that the unified vector fields X_i are orthogonal two by two that $\left(\overline{X}_q, \overline{X}_{q-1}, ...\overline{X}_1\right)$ is an orthonormal base for $\mathcal{G}_q$.

Let $\mathcal{H}_j$ be the Lie subalgebra of $\mathcal{G}_q$ corresponding to the extension $\mathcal{F}_j$ of $\mathcal{F}_q$.

We note [6] that $\dim (\mathcal{H}_j) = q-j$. And that implies that $\dim \left((\mathcal{H}_j)^\perp\right) = j$.

By Theorem 2.4 we have $\omega_q (\ell(M, \mathcal{F}_j)) \subset (\mathcal{H}_j)^\perp$.

Consequently the fact that $\dim \left((\mathcal{H}_j)^\perp\right) = j$ and

$$\omega_q (\ell(M, \mathcal{F}_j)) = \omega_q (< X_j, X_{j-1}, ...X_1 >) =< \overline{X}_j, \overline{X}_{j-1}, ...\overline{X}_1 >$$

implies that

$$(\mathcal{H}_j)^\perp =< \overline{X}_j, \overline{X}_{j-1}, ...\overline{X}_1 > .$$

Thus we have

$$\mathcal{H}_j =< \overline{X}_q, \overline{X}_{q-1}, ...\overline{X}_{j+1} > .$$

As

$$\omega_q (\ell(M, \mathcal{F}_j)) =< \overline{X}_j, \overline{X}_{j-1}, ...\overline{X}_1 > \text{ and } \mathcal{H}_j =< \overline{X}_q, \overline{X}_{q-1}, ...\overline{X}_{j+1} >$$

then according to the theorem 2.4 for all i and k such as $1 \leq i \leq j < k \leq q$ we have $\left[\overline{X}_i, \overline{X}_k\right] = 0$.

Thus, for all i and k such that $i < k$, $\left[\overline{X}_i, \overline{X}_k\right] = 0$. So $\mathcal{G}_q$ is an abelian Lie algebra. Consequently, $\mathcal{F}$ is an abelian Lie $\mathcal{G}-$foliation because $\mathcal{G} = \mathcal{G}_q$.

Conversely, suppose that $\mathcal{F}$ is an abelian Lie $\mathcal{G}-$foliation.

In this case, theorem 2.3 shows that $\mathcal{F}$ admits a complete flag of extensions. □

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