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A Mixed-Ramsey Result

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Abstract

We show that if the complete graph on n vertices, $n \geq 3$, is edge-colored with m colors appearing so that no K_3 subgraph is either monochromatic or rainbow and some color appears exactly once, then K_{2n-2} can be edge-colored with m colors appearing so that no K_3 subgraph is either monochromatic or rainbow.

1 Introduction

In this paper, all graphs will be finite and simple. The complete graph on n vertices will be denoted K_n.

For the following definitions, let G be a graph whose edges are colored.

Definition 1.1. A subgraph H of G is *monochromatic* with respect to the given coloring if and only if every edge of H bears the same color.

Definition 1.2. A subgraph H of G is *rainbow* with respect to the given coloring if and only if no two edges of H bear the same color.

A standard type of hypergraph problem that is associated with Ramsey's Theorem (see [4]) is: Given graphs G and H, find the smallest integer m

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such that the edges of G can be colored with m colors so that no subgraph of G isomorphic to H is monochromatic with respect to the given coloring. (It is said that the coloring *forbids* monochromatic H.)

Over the past 20 years, *anti-Ramsey* theory has emerged, inspired by the following type of problem: Given G and H, find the largest integer m such that there is an edge-coloring of G with m colors appearing, which forbids rainbow copies of H.

Recently, there has been an efflorescence of *mixed-Ramsey* problems: Given G and H, find all positive integers m such that the edges of G can be colored with m colors appearing so that every copy of H in G is neither monochromatic nor rainbow. See [1].

Currently in the study of mixed-Ramsey theory, G and H are almost always complete graphs; however, in a small but significant portion of cases H is a cycle. The case $H = K_3$ is of special interest for obvious reasons. If G is edge-colored so that monochromatic and/or rainbow K_3's are forbidden, then monochromatic and/or rainbow, complete subgraphs of G for all orders greater or equal to three are forbidden. Also, if $K_n, n \geq 3$ is edge-colored so that rainbow K_3's are forbidden, then rainbow cycles of all orders less than or equal to n are forbidden [3]. Thus, all rainbow subgraphs of K_n other than forests are forbidden.

Our main result here is a modest contribution to the world of mixed-Ramsey theory with $G = K_n$, for $n \geq 3$, and $H = K_3$.

2 A Left-Over Question

An edge-coloring of K_n which forbids both monochromatic and rainbow K_3 subgraphs will be called an MR3 coloring (MR: Mixed Ramsey). If *no more than* k colors appear, such a coloring is an MR3 k-coloring. We define, for $k \geq 1$, the set

$$\mathrm{SPMR3}(k) = \{n \mid K_n \text{ has a MR3 } k\text{-coloring}\}.$$

Following [2], let

$$f(k) = \max(\mathrm{SPMR3}(k)).$$

Then the main result of [2] is:

Theorem 2.1. *For a positive integer k,*

$$f(k) = \begin{cases} 5^{k/2} & \text{if } k \text{ is even} \\ 2 \cdot 5^{(k-1)/2} & \text{if } k \text{ is odd.} \end{cases}$$

Since the restriction of any MR3 k-coloring of K_n to any complete subgraph is an MR3 k-coloring of that subgraph, we have $\text{SPMR3}(k) = \{1, \dots, f(k)\}$.

Theorem 2.2. *If $n \geq 1$ and the edges of K_n are colored with n or more colors appearing, then K_n contains a rainbow K_3. However, K_n can be edge-colored with $n-1$ colors appearing so that there are no rainbow K_3 subgraphs, and every such coloring also forbids monochromatic K_3's.*

The first claim is folkloric, although probably first due to Erdős. Proofs of both claims can be found in [3].

An $\text{MR3}'$ k-coloring of K_n is an MR3 k-coloring of K_n in which exactly k colors appear, and

$$\text{SPMR3}' = \{n \mid \text{there is an MR3}' \ k\text{-coloring of } K_n\}.$$

By Theorem 2.2 and the definition of $f(k)$, for $k \geq 1$ we have

$$\{k+1\} \cup \{f(k-1)+1, \dots, f(k)\} \subset \text{SPMR3}'(k) \subset \{k+1, \dots, f(k)\}.$$

The question remains: is $\text{SPMR3}'(k) = \{k+1, \dots, f(k)\}$? The author does not know; however, the result in the next section bears on the question.

3 The Main Result

Theorem 3.1. *For $n \geq 3$, if K_n is edge-colored with m colors actually appearing so that no K_3 subgraph is either monochromatic or rainbow and some color appears on exactly one edge of K_n, then K_{2n-2} can be edge-colored with m colors appearing so that no K_3 subgraph is either monochromatic or rainbow.*

Proof. Denote the vertices of K_n by $v_1, \dots, v_n$, and WLOG, suppose that the color c_1 appears only on the edge v_1v_n. Introduce $n-2$ new vertices, $u_2, \dots, u_{n-1}$. Color the edges of K_{2n-2} on vertices $v_1, \dots, v_n, u_2, \dots, u_{n-1}$ by the following rules:

1. Let the colors on edges among $v_1, \dots, v_n$ be as in the original coloring of K_n.

2. Let u_iu_j be colored with the color on v_iv_j for $2 \leq i \neq j \leq n-1$.

3. Let u_iv_j be colored with the color on v_iv_j for $1 \leq i \leq n$, $2 \leq j \leq n-1$, and $i \neq j$.

4. Let u_iv_i be colored c_1 for $i = 2, \ldots, n-1$.

To finish this proof, it remains to be shown that for any three distinct elements x, y, z of $\{v_1, \ldots, v_n, u_2, \ldots, u_{n-1}\}$ that two of the edges xy, xz, yz bear the same color, which is different from the color on the remaining edge. By assumption about the original coloring of K_n and the way that the coloring of K_{2n-2} is defined, this certainly holds when $\{x, y, z\}$ is one of: $\{v_i, v_j, v_k\}, \{u_i, v_j, v_k\}, \{u_i, u_j, v_k\}$, or $\{u_i, u_j, u_k\}$, for distinct indices i, j, k. This leaves the cases $\{u_i, v_i, v_j\}$ and $\{u_i, v_i, u_j\}$ for distinct indices i, j. Since the colors of $v_iv_j, v_iu_j, u_iv_j, u_iu_j$ are the same for distinct $i, j \in \{2, \ldots, n-1\}$, it suffices to prove the first case: $\{x, y, z\} = \{u_i, v_i, v_j\}$ with i, j distinct and $2 \leq i \leq n-1$, $1 \leq j \leq n$. In this case, u_iv_i is colored c_1 and u_iv_j and v_iv_j bear the same color, which must be different from c_1 because $2 \leq i \leq n-1$ and v_rv_s is colored c_1 only when $\{r, s\} = \{1, n\}$.

■

Corollary 3.2. *For all $t \geq 2$, K_{2t} can be edge-colored with t colors appearing so that no K_3 subgraph is either monochromatic or rainbow.*

Proof. In [3], there is a characterization of MR3$'$ $(n-1)$-colorings of K_n which shows that, for $n \geq 3$, for every such coloring at least one of the $n-1$ colors appears exactly once. Putting $t = n-1 \geq 2$ and applying the above theorem with $m = n-1$, we obtain the conclusion of this corollary. ■

Corollary 3.3. *Let f be as in Theorem 2.1, and k, n be positive integers. If $2n-2 > f(k)$, then there is no MR3 k-coloring of K_n such that at least one of the k colors appears on exactly one edge.*

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