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Common Fixed Point For Generalized Five Self Maps In Cone Metric Spaces

K. Rauf, S. M. Alata, O. T. Wahab

Department of Mathematics
University of Ilorin
Ilorin, Nigeria

email: krauf@unilorin.edu.ng, balk_r@yahoo, wahab4math@gmail.com

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Abstract

In this paper, we establish and prove some common fixed points for self-maps in a complete cone metric space that generalize and improve earlier results in the literature.

1 Introduction

In 2007, Huang and Zhang introduced the notion of cone metric space. They have generalized the concept of metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying contractive conditions and a complete cone metric space with the assumption of normality of a cone. These authors also described the convergence of sequence in the cone metric space and introduced the corresponding notion of complete cone metric space. Abbas and Rhoades (2008) obtained fixed and periodic point results in cone metric space. Beiranvand et al (2009) introduced new classes of contractive functions called f-contraction and f-contractive mapping and then established and extended the Banach contraction mapping. Sumitra et al (2010) proved common fixed point theorem for a Banach pair of mappings satisfying f-Hardy Rogers type contraction condition in cone metric spaces. Guangxing et. al. (2010) proved a new common fixed point theorems for maps on cone metric space. *Ö*zturk and Metin

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(2011) proved some common fixed point theorems for f-contraction mappings in cone metric spaces. Rangamma and Pruhdvi (2012) proved common fixed points under contraction conditions for three maps in cone metric space. Kutbi (2013) derived some common fixed point for Banach operator pairs in strongly M-star shaped metric space. Dubey et al (2014) proved common fixed point theorem for generalized T-Hardy-Rogers contraction mapping in a cone metric space. Vigaya and Sucharitha (2014) proved fixed points for four self-mapping in cone metric spaces. Alata et al (2015) proved some results on common fixed point for generalized f-contraction mapping.

In this paper, we establish new fixed point theorems for five self-mappings in a complete cone metric space to obtain a common fixed point. These theorems generalized the results of Dubey, Vigaya and Alata.

2 Preliminaries

In this section, we give some definitions and notations on cone metric spaces and their properties.

Definition 2.1. *Öztürk and Metin (2011) Let E be a real Banach space and K be a subset of E. K is called a cone if and only if*

C1: K is closed, nonempty and $K \neq \{0\}$

C2: $ax + by \in K$ for $x, y \in K$ and $a, b \geq 0$

C3: $x \in K$ and $-x \in K \Rightarrow x = 0 \iff K \cap (-K) = \{0\}$

Consider a cone $K \subset E$. We define a partial ordering "$\leq$" with respect to K by $x \leq y$ if and only if $y - x \in K$, we write $x < y$ to indicate that $x \leq y$ but $x \neq y$ and $x << y$ to imply $y - x \in intK$. The $int[K]$ denotes the interior of K.

A cone K is called a normal if for a given number $M > 0$ such that for all $x, y \in K$

$$0 \leq x \leq y \Longrightarrow \|x\| \leq M \|y\| \tag{1}$$

The least positive number of M satisfying (1) is called the normal constant of K (Best possible constant).

The cone K is said to be regular if for every $n \in \mathbb{N}$ such that $x_n \leq x_{n+1} \leq y$ for some $y \in E$, then there is $x \in E$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0$$

that is. K is called regular if every monotone increasing sequence bounded from above is convergent.
Similarly, K is called regular if every monotone decreasing sequence bounded from below is convergent.

Remark: Every regular cone is a normal cone.

Definition 2.2. *Öztürk and Metin (2011)*
Let X be a nonempty set and $E \supset K$ be a real Banach spaces. Suppose the metric mapping $d : X \times X \to E$ satisfies

CM1: $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$.

CM2: $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

CM3: $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then, d is called a cone metric on X and (X, d) is called a cone metric space

Example 2.3. *Let $E = R^3$, define $K = \{(x, y, z) \in E : x, y, z \geq 0\} \subset R^3$. Let $X = R$ and define $d : X \times X \to E$ such that*

$$d(x_1, x_2) = (\alpha |x_1 - x_2|, \beta |x_1 - x_2|, \gamma |x_1 - x_2|)$$

where, $x_1, x_2 \in X$ and α, β, γ are nonnegative.
Then, d is a cone metric on X and (X, d) is a cone metric space.

Lemma 2.4. *Dubey et. al. (2013)*
Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence in X, if for every $c \in E$ with $0 << c$

i $\{x_n\}$ *converges to x, then given $n(c) \in \mathbb{N}$ such that $\|d(x_n, x)\| << c$ for all $n > n(c)$.*

ii $\{x_n\}$ *converges to x and $\{x_n\}$ converges to y, then $\|d(x, y)\| << c \Rightarrow x = y$.*

iii $\{x_n\}$ *is Cauchy sequence, then there is $n(c) \in \mathbb{N}$ such that $\|d(x_n, x_m)\| << c$ for $n, m > n(c)$.*

iv $\{x_n\}$ *and $\{y_n\}$ converge to x and y respectively, then for given $n(c) \in \mathbb{N}$ and $n > n(c)$, $\|d(x_n, y_n) - d(x, y)\| \to 0$.*

v $\{x_n\}$ *is convergent, then $\{x_n\}$ is a Cauchy sequence.*

vi the Cauchy sequence converges to a point in X, then (X, d) is complete.

Lemma 2.5. *Dubey et. al. (2014)*
Let (X, d) be a cone metric space and $u, v, w \in X$.

i if $u << v$ and $v << w$, then $u << w$

ii if $u \leq v$ and $v << w$, then $u << w$

iii if $0 \leq u << c$ for each $c \in intK$, then $u = 0$

iv if $0 \leq u << \alpha c$ for $\alpha > o$ and each $c \in intK$, then $u = 0$

v if $c \in intK$, $0 \leq u_n$ and $u_n \to 0$, then there is $n(c)$ such that $u_n << c$ for all $n > n(c)$.

Definition 2.6. *Banach (1922)*
A self mapping T of a metric space (X, d) is said to be a contraction mapping if there exists a number $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y), \quad \text{for all} \quad x, y \in X. \tag{2}$$

A number of definitions exist, in the literature, which define a contractive type condition between a pair of functions $f, g : X \to X$ [Rohades (1977)].

Definition 2.7. *Beiranvand et. al. (2009)*
Let (X, d) be a metric space and $f, T : X \longrightarrow X$ be two self mappings. The mapping T is called f-contraction if there is a number $k \in [0, 1)$ such that

$$d(fTx, fTy) \leq kd(fx, fy), \quad \text{for all} \quad x, y \in X. \tag{3}$$

If f is an identity mapping, then (3) reduces to (2).

Definition 2.8. *Alata et. al. (2015)*
Let (X, d) be a metric space and $f, T, S : X \longrightarrow X$ be three self mappings. The pair (T, S) is called f-contraction if there is a number $k \in [0, 1)$ such that

$$d(fTx, fSy) \leq kd(fx, fy), \quad \text{for all} \quad x, y \in X. \tag{4}$$

The following example is given to justify our claim in Definition 2.8

Example 2.9. *Let $X = [0, \infty)$ be endowed with the usual metric and let the mappings be defined by*

$$fx = e^{-x}, \quad Tx = 2x + \ln 2^{\beta-1}, \quad Sy = 2y + \ln 2^{(\beta-1)}$$

where, x, y are partially ordered in X and $\beta > 1$

Clearly, the pair (T, S) is not a contraction. But

$$\begin{aligned} d(fTx, fSy) &= \left|2^{-(\beta-1)}e^{-2x} - 2^{-(\beta-1)}e^{-2y}\right| \\ &= |2^{-(\beta-1)}| \left|e^{-2x} - e^{-2y}\right| \\ &\leq 2^{-(\beta-1)} \left|e^{-x} - e^{-y}\right| \\ &\equiv k \left|fx - fy\right|, \quad k \equiv 2^{-(\beta-1)} < 1 \end{aligned}$$

Therefore, the pair (T, S) is f-contraction. We end this section with the following crucial lemma:

Lemma 2.10. *Öztürk and Metin (2011)*
Let (X, d) be a cone metric space, then

- *i a mapping $T : X \longrightarrow X$ is continuous if for $n \in \mathbb{N}$ $\lim_{n\to\infty} x_n \longrightarrow x$ implies that $\lim_{n\to\infty} Tx_n \longrightarrow Tx$ for $\{x_n\}$ and x are in X.*
- *ii the mapping $f, T : X \longrightarrow X$ or $f, S : X \longrightarrow X$ is said to be sequentially convergent, if the sequence $\{y_n\}$ in X is convergent whenever $\{Ty_n\}$ is convergent.*
- *iii the mapping $f, T : X \longrightarrow X$ or $f, S : X \longrightarrow X$ is said to be sub-sequentially convergent, if the sequence $\{y_n\}$ in X has a convergent subsequence whenever $\{Ty_n\}$ is convergent.*

Definition 2.11. *Singh et al (2009) Let $f; g : X \longrightarrow X$, then the pair $(f; g)$ is said to be (IT)-Commuting at $z \in X$, if $f(g(z)) = g(f(z))$ with $f(z) = g(z)$. The pair (f, g) and (h, k) are (IT)-Commuting if $f(g(h(k(z))) = g(f(k(h(z)))$.*

3 Main Results

In this section, we present our main results as follows:

Theorem 3.1. *Let S, T, f, g and h be five continuous self mappings of a complete cone metric space (X, d), such that $T(X) \subset f(X), S(X) \subset g(X)$ and $S(X) = T(X)$. Assume h is an injective mapping and M is a normal constant with normal cone K. If $S(X)$ or $T(X)$ is a complete subspace of X and satisfy*

$$\begin{aligned} d(hSx, hTy) \leq &\alpha d(hfx, hgy) + \beta[d(hSx, hfx) + d(hTy, hgy)] \\ &+ \gamma[d(hSx, hgy) + d(hTy, hfx)] \end{aligned} \tag{5}$$

for all $x, y \in X$, where $\alpha + 2\beta + 2\gamma < 1$, $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$.
Then, the maps (S, f, h) and (T, g, h) have a coincidence point in X.
More so, S, T, f, g and h have a unique common fixed point.

Proof

Let $x_0 \in X$ be fixed, since $S \subset g$, then there exists a point $x_1 \in X$ such that $Sx_0 = gx_1$ and for $T \subset f$ there is $x_2 \in X$ such that $Tx_1 = fx_2$.
By induction,

$$y_{2n} = hSx_{2n} = hgx_{2n+1}$$

and

$$y_{2n+1} = hTx_{2n+1} = hfx_{2n+2}$$

Now,

$$\begin{aligned} d\left(y_{2n}, y_{2n+1}\right) &= d\left(hSx_{2n}, hTx_{2n+1}\right) \\ &\leq \alpha d\left(hfx_{2n}, hgx_{2n+1}\right) + \beta[d\left(hSx_{2n}, hfx_{2n}\right) + d\left(hTx_{2n+1}, hgx_{2n+1}\right)] \\ &\quad + \gamma[d\left(hSx_{2n}, hgx_{2n+1}\right) + d\left(hTx_{2n+1}, hfx_{2n}\right)] \\ &= \alpha d\left(y_{2n-1}, y_{2n}\right) + \beta[d\left(y_{2n}, y_{2n-1}\right) + d\left(y_{2n+1}, y_{2n}\right)] \\ &\quad + \gamma[d\left(y_{2n}, y_{2n}\right) + d\left(y_{2n+1}, y_{2n-1}\right)] \\ &\leq \alpha d\left(y_{2n-1}, y_{2n}\right) + \beta d\left(y_{2n}, y_{2n-1}\right) + \beta d\left(y_{2n+1}, y_{2n}\right)] \\ &\quad + \gamma d\left(y_{2n+1}, y_{2n}\right) + \gamma d\left(y_{2n}, y_{2n-1}\right) \\ &\leq \alpha d\left(y_{2n-1}, y_{2n}\right) + \beta d\left(y_{2n}, y_{2n-1}\right) + \beta d\left(y_{2n+1}, y_{2n}\right) \\ &\quad + \gamma d\left(y_{2n+1}, y_{2n}\right) + \gamma d\left(y_{2n}, y_{2n-1}\right) \\ &= (\alpha + \beta + \gamma) d(y_{2n}, y_{2n-1}) + (\beta + \gamma) d(y_{2n}, y_{2n+1}) \end{aligned} \tag{6}$$

This implies that

$$(1 - \beta - \gamma) d(y_{2n}, y_{2n+1}) \leq (\alpha + \beta + \gamma) d(y_{2n}, y_{2n-1})$$

and

$$d(y_{2n}, y_{2n+1}) \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(y_{2n-1}, y_{2n}) \tag{7}$$

Let $\lambda = \dfrac{\alpha + \beta + \gamma}{1 - \beta - \gamma}$ and for all cases of $\lambda < 1$.
we have,

$$d(y_{2n}, y_{2n+1}) \leq \lambda d(y_{2n-1}, y_{2n}) \tag{8}$$

In general,

$$d(y_{2n}, y_{2n+1}) \leq \lambda^{2n} d(y_0, y_1) \tag{9}$$

For $m, n \in \mathbb{N}$; $m > n$, we have

$$\begin{aligned} d(y_{2n}, y_{2m}) &\leq d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + \ldots + d(y_{2m-1}, y_{2m}) \\ &\leq \lambda^{2n} d(y_0, y_1) + \lambda^{2n+1} d(y_0, y_1) + \ldots + \lambda^{2m-1} d(y_0, y_1) \\ &= (\lambda^{2n} + \lambda^{2n+1} + \cdots \lambda^{2m-1}) d(y_0, y_1) \\ &= \lambda^{2n}(1 + \lambda + \lambda^2 \ldots \lambda^{2m-2n-1}) d(y_0, y_1) \\ &= \frac{\lambda^{2n}}{1-\lambda} d(y_0, y_1) \end{aligned} \tag{10}$$

By the normality assumption,

$$\|d(y_{2n}, y_{2m})\| \leq \left| \frac{\lambda^{2n}}{1-\lambda} \right| M \, \|d(y_0, y_1)\| \tag{11}$$

which implies that $d(y_{2n}, y_{2m}) \to 0$ as $n, m \to \infty$.
Hence, $\{y_{2n}\}$ is a Cauchy sequence in $T(X)$. Since $T(X)$ is complete then $\{y_{2n}\}$ converges to a point (say z) in $T(X)$.
Since $T(X) \subset f(X)$, there exists a point $b \in X$ such that

$$z = fb \quad \text{and} \quad hz = hfb$$

We need to prove that (hSb=hz).
By condition of metric space, we have

$$\begin{aligned} d(hSb, hz) &\leq d(hSb, hTy_{2n-1}) + d(hTy_{2n-1} hz) \\ &\leq \alpha d(hfb, hgy_{2n-1}) + \beta d[(hSb, hfb) + d(hTy_{2n-1} hgy_{2n-1})] \\ &\quad + \gamma d[(hSb, hgy_{2n-1}) + d(hTy_{2n-1}, hfb) + d(hTy_{2n-1}, hz)] \\ &= \alpha d(hz, hgy_{2n-1}) + \beta d(hSb, hz) + \beta d(hTy_{2n-1}, hgy_{2n-1}) \\ &\quad + \gamma[d(hSb, hgy_{2n-1}) + d(hTy_{2n-1}, hz) + d(hTy_{2n-1}, hz) \\ &= (\beta + \gamma) d(hSb, hz) + \beta d(hz, hz) + \gamma d(hz, hz) + \gamma d(hz, hz) \\ &\leq (\alpha + 2\beta + 2\gamma) d(hSb, hz) < d(hSb, hz) \end{aligned} \tag{12}$$

This is a contradiction.
Therefore,

$$d(hSb, hz) = 0 \quad \text{if and only if} \quad hSb = hz \tag{13}$$

Also, since $S(X) \subset g(X)$, then there exists a point $q \in X$ such that $hz = hgq$, we want to show that $hTq = hz$

$$\begin{aligned} d(hz, hTq) &= d(hSb, hTq) \\ &\leq \alpha(hfb, hgq) + \beta[d(hSb, hfb) + d(hTq, hgq)] \\ &\quad + \gamma[d(hSb, hgq) + d(hSb, hfb)] \\ &= \alpha d(hz, hz) + \beta d(hz, hz) + \beta d(hTq, hz) \\ &\quad + \gamma d(hz, hz) + \gamma(hz, hz) \\ &= \beta d(hTq, hz) \end{aligned} \tag{14}$$

This implies that,

$$(1-\beta)d(hz, hTq) \leq 0$$

which is a contradiction and hence

$$d(hz, hTq) = 0 \quad \text{if and only if} \quad hz = hTq \tag{15}$$

from (13) and (15), we have,

$$hz = hSb = hfb = hTq = hgq \tag{16}$$

Since (S, f) and (T, g) are (IT)-Commuting,

$$\begin{aligned} d(SShb, Shb) &= d(SShb, fhb) \\ &= d(SShb, Thq) \\ &\leq \alpha d(fShb, ghq) + \beta[d(SShb, fShb) + d(Thq, ghq)] \\ &\quad + \gamma\left[d(SShb, ghq) + d(Thq, fShb)\right] \\ &= \alpha d(SShb, Shb) + \beta[d(SShb, SShb) + d(hSb, hSb)] \\ &\quad + \gamma\left[d(SShb, Shb) + d(SShb, Shb)\right] \\ &= (\alpha + 2\gamma)d(SShb, Shb) \\ &\Rightarrow (1 - \alpha - 2\gamma)d(SShb, Shb) \leq 0 \end{aligned} \tag{17}$$

which is a contradiction and hence,

$$d(SShb, Shb) = 0 \text{ and } SShb = Shb(= hz) \tag{18}$$

Therefore, $Shb = hz$ is a fixed point of (f, S)

Also,

$$\begin{aligned}
&d(TThq, Thq) = d(TThq, ghq) = d(SShb, TThq) \\
&\leq \alpha d(fhSb, hgTq) + \beta\left[d(hSSb, hfSb) + d(hTTq, hgTq)\right] \\
&+ \gamma\left[d(hSSb, hgTq) + d(hTTq, hfSb)\right] \\
&= \alpha d(fhSb, hTTq) + \beta\left[d(hSSb, hSSb) + d(hTTq, hTTq)\right] \\
&+ \gamma\left[d(hSSb, hTTq) + d(hTTq, hSSb)\right] \\
&= [1 - (\alpha + 2\gamma)]d(hSSb, hTTq) \leq 0
\end{aligned} \tag{19}$$

This is a contradiction.
Hence,

$$d(TThq, Thq) = 0 \;\text{ if and only if }\; TThq = Thq \tag{20}$$

Therefore, hz is a common fixed point of (T, g).
Also,

$$\begin{aligned}
d(gghq, ghq) &\leq d(gghq, y_{2n}) + d(y_{2n}, ghq) \\
&= d(gghq, y_{2n}) + d(hSx_{2n}, Thq) \\
&\leq d(gghq, y_{2n}) + \alpha d(fhx_{2n}, ghq) + \beta[d(hSx_{2n}, fhx_{2n}) + d(Thq, ghq)] \\
&\quad + \gamma[d(hSx_{2n}, ghq) + d(Thq, fhx_{2n})] \\
&= d(gghq, hz) + \alpha d(hz, hz) + \beta d(hz, hz) + \beta d(hz, hz) \\
&\quad + \gamma d(hz, hz) + \gamma d(hz, hz)
\end{aligned} \tag{21}$$

By the normality assumption,

$$\|d(gghq, ghq)\| \leq M\,\|d(gghq, hz)\| = M\,\|d(gghq, ghq)\| \tag{22}$$

Therefore,

$$(1 - M)d(gghq, ghq) \leq 0$$

This implies that,

$$d(gghq, ghq) = 0 \;\text{ and }\; gghq = ghq \tag{23}$$

Therefore, hz is a fixed point of ghq. Also,

$$\begin{aligned}
d(ffhb, fhb) &\leq d(ffhb, y_{2n+1}) + d(y_{_{2n+1}}, fhb) \\
&= d(ffhb, y_{2n+1}) + d(hTx_{2n+1}, Shb) \\
&= d(ffhb, y_{2n+1}) + d(Shb, hTx_{2n+1}) \\
&\leq d(ffhb, y_{2n+1}) + \alpha d(fhb, ghx_{2n+1}) + \beta[d(Shb, fhb) + d(hTx_{2n+1}, ghx_{2n+1})] \\
&+ \gamma[d(Shb, ghx_{2n+1}) + d(hTx_{2n+1}, fhb)] \\
&= d(ffhb, hz) + \alpha d(fhb, hz) + \beta d(hz, hz) + \beta d(hz, hz) \\
&+ \gamma d(hz, hz) + \gamma(hz, hz) \\
&= d(ffhb, fhb) + \alpha d(hz, hz) + 2\beta d(hz, hz) + 2\gamma d(hz, hz)
\end{aligned} \tag{24}$$

Therefore,

$$d(ffhb, fhb) \leq M \left\| d(ffhb, fhb) \right\|$$
$$\Rightarrow (1 - M)d(ffhb, fhb) \leq 0$$

Hence,

$$d(ffhb, fhb) = 0 \text{ if and only if } ffhb = fhb. \tag{25}$$

This implies that hz is a fixed point of fhb.
And finally,

$$\begin{aligned}
d(hhz, hz) &= d(hhz, y_{_{2n}}) + d(y_{_{2n}}, hz) \\
&d(hhz, y_{_{2n}}) + d(hSx_{2n}, Thq) \\
&\leq d(hhz, y_{_{2n}}) + \alpha d(hfx_{2n}, ghq) + \beta[d(hSx_{2n}, fhx_{2n}) + d(Thq, ghq)] \\
&+ \gamma[d(hSx_{2n}, ghq) + d(Thq, fhx_{2n})]
\end{aligned} \tag{26}$$

This becomes

$$\left\| d(hhz, hz) \right\| \leq M \left\| d(hhz, hz) + (\alpha + 2\beta + 2\gamma)d(hz, hz) \right\|$$

$$\Rightarrow (1 - M)d(hhz, hz) \leq 0$$

which is a contradiction and therefore

$$d(hhz, hz) = 0 \;\text{ if and only if }\; hhz = hz \tag{27}$$

Hence, hz is a fixed point of hhz.
From (18), (20), (23), (25) and (27) we have

$hz(=z)$ is a common fixed of h, g, f, T and S
For Uniqueness, let w be another common fixed point of S, T, f, g and h, then

$$\begin{aligned}
d(z,w) &= d(Sb,Tw)\\
&\leq \alpha d(fb,gw) + \beta d(Sb,fb) + \beta d(Tw,gw) + \gamma d(Sb,gw) + \gamma d(Tw,fb)\\
&= \alpha d(z,w) + \beta d(z,z) + \beta d(w,w) + \gamma d(z,w) + \gamma d(w,z)\\
&\leq (\alpha + 2\gamma) d(z,w) \leq (\alpha + 2\beta + 2\gamma) d(z,w) < d(z,w)
\end{aligned}$$

This contradicts the fact that w is another fixed point. Hence, $z = w$.

Theorem 3.2. *Let S, T, f, g and h be five continuous self-maps of a complete cone metric space (X,d), such that $T(X) \subset f(X), S(X) \subset g(X)$ and $S(X) = T(X)$. Assume h is an injective mapping and M is a normal constant with normal cone K. Suppose $T(X)$ and $S(X)$ are complete subspaces of X satisfying*

$$\begin{aligned}
d(hSx,hTy) \leq \alpha_1 d(hfx,hgy) + \alpha_2 d(hSx,hfy) + \alpha_3 d(hTy,hgy)\\
+ \alpha_4 d(hSx,hgy) + \alpha_5 d(hTy,hfx) \qquad (28)
\end{aligned}$$

for all $x, y \in X$, where

$$\sum_{i=1}^{5} \alpha_i < 1, \qquad \alpha_i \geq 0$$

If (S,f) and (T,g) are Banach pairs, then S, T, f, g and h have a unique common fixed point.

Proof Let x_0 be fixed in X, since $T \subset f$ and $S \subset g$ then there exists $x_1, x_2 \in X$ such that $Sx_0 = gx_1$ and $Tx_1 = fx_2$ respectively.
Define

$$y_{2n} = hSx_{2n} = hgx_{2n+1}$$
$$y_{2n+1} = hTx_{2n+1} = hfx_{2n+2}$$

Now,

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(hSx_{2n}, hTx_{2n+1})\\
&\leq \alpha_1 d(hfx_{2n}, hgx_{2n+1}) + \alpha_2 d(hSx_{2n}, hfx_{2n}) + \alpha_3 d(hTx_{2n+1}, hgx_{2n+1})\\
&\quad + \alpha_4 d(hSx_{2n}, hgx_{2n+1}) + \alpha_5 d(hTx_{2n+1}, hfx_{2n})\\
&= \alpha_1 d(y_{2n-1}, y_{2n}) + \alpha_2 d(y_{2n}, y_{2n-1}) + \alpha_3 d(y_{2n+1}, y_{2n})\\
&\quad + \alpha_4 d(y_{2n}, y_{2n}) + \alpha_5 d(y_{2n+1}, y_{2n-1})\\
&\leq \alpha_1 d(y_{2n-1}, y_{2n}) + \alpha_2 d(y_{2n}, y_{2n-1}) + \alpha_3 d(y_{2n+1}, y_{2n})\\
&\quad + \alpha_5 d(y_{2n+1}, y_{2n}) + \alpha_5 d(y_{2n}, y_{2n-1})\\
&= (\alpha_1 + \alpha_2 + \alpha_5)\, d(y_{2n}, y_{2n-1}) + (\alpha_3 + \alpha_5) d(y_{2n+1}, y_{2n}) \qquad (29)
\end{aligned}$$

This implies,

$$d(y_{2n}, y_{2n+1}) \leq \frac{\alpha_1 + \alpha_2 + \alpha_5}{1 - (\alpha_3 + \alpha_5)} d(y_{2n-1}, y_{2n})$$

Let $\lambda_1 = \dfrac{\alpha_1 + \alpha_2 + \alpha_5}{1 - (\alpha_3 + \alpha_5)} < 1$. Then,

$$d(y_{2n}, y_{2n+1}) \leq \lambda_1 d(y_{2n-1}, y_{2n}) \tag{30}$$

By comparing (30) with (8), it can be concluded that the rest of the proof follows from Theorem 3.1.

Corollary 3.3. (see [15])
Suppose S, T, f and g be four continuous self-maps of a complete cone metric space (X, d), such that $T(X) \subset f(X), S(X) \subset g(X)$ and $S(X) = T(X)$ with a normal constant M of cone K. Suppose $T(X)$ and $S(X)$ are complete subspaces of X satisfying

$$\begin{aligned} d(Sx, Ty) \leq \alpha_1 d(fx, gy) + \alpha_2 d(Sx, fy) + \alpha_3 d(Ty, gy) \\ + \alpha_4 d(Sx, gy) + \alpha_5 d(Ty, fx) \end{aligned} \tag{31}$$

for all $x, y \in X$, where

$$\sum_{i=1}^{5} \alpha_i < 1, \qquad \alpha_i \geq 0$$

If (S, f) and (T, g) are Banach pair, then S, T, f and g have a unique common fixed point.

Proof

The proof of the Corollary is immediate if h is an identity map in Theorem 3.2 above, that is $h = I_X$.

Corollary 3.4. (see [2])
Let S, T and h be three self maps of a complete cone metric space (X, d) and M is a normal constant with normal cone K. Assume h is an injective mapping satisfying

$$\begin{aligned} d(fSx, fTy) \leq & \alpha d(fx, hy) + \beta[d(fSx, hx) + d(fTy, hy)] \\ & + \gamma[d(fSx, hy) + d(fTy, hx)] \end{aligned} \tag{32}$$

where, $\alpha + 2\beta + 2\gamma < 1$, $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ and for all $x, y \in X$.
Then, the pair (S, h) and (T, h) have a unique common fixed point.

Proof The proof is immediate by setting $S \equiv g$, $T \equiv f$ and $h = f$ in Theorem 3.1 above.

Corollary 3.5. (see [10])
Let (X, d) be a nonempty complete cone metric space and let $h,\ T,\ S$ be three continuous self mappings of X.Assume h is an injective mapping satisfying and M is a normal cone with normal constant K. Suppose $h,\ T,\ S$ satisfy

$$d(hTx, hSy) \leq \lambda d(hx, hy) + \mu[d(hTx, hy) + d(hSy, hx)] \tag{33}$$

where, $\lambda + 2\mu < 1$, $\lambda, \mu \in [0, 1)$ and $x, y \in X$.
Then, the pair (S, h) and (T, h) have a unique common fixed point in X.

Proof This is Corollary 3.4 with $\alpha = \lambda$, $\beta = 0$ and $\gamma = \mu$.

Corollary 3.6. (see [14])
Let (X, d) be a complete cone metric space and let $h,\ T$ be two continuous self mapings of X.Assume h is an injective mapping satisfying and M is a normal cone with normal constant K. If the mapping $h,\ T$ satisfy

$$\begin{aligned} d(hTx, hTy) \leq& \alpha d(hx, hy) + \beta[d(hTx, hx) + d(hTy, hy)] \\ &+ \gamma[d(hTx, hy) + d(hTy, hx)] \end{aligned} \tag{34}$$

where, $\alpha + 2\beta + 2\gamma < 1$, for $\alpha, \beta, \gamma \in [0, 1)$ and $x, y \in X$.
Then, h and T have a unique common fixed point in X.

Proof This follows directly from Corollary 3.4 by setting $S = T$.

Corollary 3.7. (see [6])
Let S, T and h be three continuous self maps of a complete cone metric space (X, d). Assume h is an injective mapping and M is a normal constant with normal cone K. Suppose

$$\begin{aligned} d(hSx, hTy) \leq& \alpha_1 d(hx, hy) + \alpha_2 d(hSx, hx) + \alpha_3 d(hTy, hy) \\ &+ \alpha_4 d(hSx, hy) + \alpha_5 d(hTy, hx) \end{aligned} \tag{35}$$

for all $x, y \in X$, where

$$\sum_{i=1}^{5} \alpha_i < 1, \quad \alpha_i \geq 0$$

Then, if (S, h) and (T, h) are Banach pairs, then S, T and h have a unique common fixed point.

Proof This becomes Theorem 3.2 by setting $S = f$ and $T = g$.

Corollary 3.8. (see [2])
Let S, T and h be three self maps of a complete cone metric space (X, d) which satisfy (4). Then, S and T have a unique common fixed point in X.

Proof The proof is immediate from Corollary 3.4 if $h \equiv f$ in (4) and $\alpha = k$, $\beta = \gamma = 0$.

4 Conclusion

In conclusion, we have established some theorems for generalized contraction mappings in complete cone metric space, these results generalized the results of Sumitra et. al. (2010), Öztürk and Metin (2011), Dubey et. al. (2014), Vijaya and Sucharitha (2014) and Alata et. al. (2015).

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