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On decomposing regular graphs into star forests

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Abstract

Let G be a forest with n edges. El-Zanati conjectures that G necessarily decomposes every $2n$-regular graph and every n-regular bipartite graph. We confirm these conjectures in the case when G consists of two stars.

1 Introduction

For integers a and b with $a \leq b$, let $[a,b] = \{a, a+1, \ldots, b\}$. For a positive integer n, let $\mathbb{Z}_n$ denote the group of integers modulo n. For a graph G with vertex set $V(G)$ and edge set $E(G)$, the *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. The graph $K_{1,k}$ is known as a *k-star* and is denoted by S_k. A *double-star* is a tree with exactly two vertices of degree greater than 1. The

two vertices of degree greater than 1 are called the *centers* of the double-star and the edge joining them is called the *central-edge*. If T is a double-star where the two centers have degrees $k_1 + 1$ and $k_2 + 1$, then T is denoted by S_{k_1,k_2}. Note that S_{k_1,k_2} has $k_1 + k_2 + 1$ edges and is isomorphic to S_{k_2,k_1}. For a graph G and a positive integer t, let tG denote the vertex disjoint union of t copies of G.

Let H and G be graphs with G a subgraph of H. A *G-decomposition* of H is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of H each of which is isomorphic to G and such that each edge of H appears in exactly one G_i. If there exists a G-decomposition of H, then we say G *decomposes* H.

A large amount of research has been done on the topic of graph decompositions over the last five decades (see [2] and [3] for surveys). Much investigation has been motivated by a conjecture of Ringel [15] on decomposing complete graphs into trees.

Conjecture 1.1. *Every tree T with n edges decomposes the complete graph K_{2n+1}.*

A folklore conjecture similar to Ringel's relates to decomposing complete bipartite graphs into trees.

Conjecture 1.2. *Every tree T with n edges decomposes the complete bipartite graph $K_{n,n}$.*

Both of the above conjectures are special cases of conjectures due to Graham and Häggkvist (see [9]).

Conjecture 1.3. *Every tree T with n edges decomposes every $2n$-regular graph H.*

Conjecture 1.4. *Every tree T with n edges decomposes every n-regular bipartite graph H.*

Despite persistent attacks over the last 40 years, Ringel's conjecture and variations thereof, such as the Graceful Tree Conjecture (see [8]), still stand today. Much less work has been done on the Graham and Häggkvist conjectures.

Results confirming Conjecture 1.3, in certain cases, can be found in [9] by Häggkvist, in [4], and in Snevily's Ph.D. thesis [17]. Some recent extensions of Snevily's results can be found in a paper by Jao, Kostochka, and West [14]. In [13], Jacobson, Truszczyński, and Tuza confirm Conjecture 1.4 for double-stars and for the path with 4 edges. Fink [7] confirms Conjecture

1.4 when H is the n-cube. Also, it is easy to see that S_n decomposes every $2n$-regular graph as well as every n-regular bipartite graph.

El-Zanati proposes that the conjectures by Graham and Häggkvist hold for forests with n edges.

Conjecture 1.5. *Every forest G with n edges decomposes every $2n$-regular graph H.*

Conjecture 1.6. *Every forest G with n edges decomposes every n-regular bipartite graph H.*

In this note, we provide some evidence in support of Conjectures 1.5 and 1.6. In particular, we show that the conjectures hold when G is the vertex-disjoint union of two stars.

2   Known Results

We begin by defining three graph labelings introduced by Rosa [16] as means for attacking problems like Ringel's Conjecture. Let G be a graph with n edges and let $f\colon V(G) \to [0, 2n]$ and $g\colon V(G) \to [0, n]$ be one-to-one functions. Then f is a σ-*labeling* of G if $\{|f(v) - f(u)|\colon \{u, v\} \in E(G)\} = [1, n]$ and g is a β-*labeling* if $\{|g(v) - g(u)|\colon \{u, v\} \in E(G)\} = [1, n]$. If in addition G is bipartite with vertex bipartition $\{A, B\}$, then a β-labeling g of G is an α-*labeling* if $\max\{g(u)\colon u \in A\} < \min\{g(v)\colon v \in B\}$. Thus an α-labeling of G is also a β-labeling which is also a σ-labeling of G. We have the following results (see [16] and [5]).

Theorem 2.1. *Let G be a graph with n edges. If G admits a σ-labeling, then there exists a G-decomposition of K_{2n+1} and of $K_{2n+2} - I$, where I is a 1-factor. If in addition, G is bipartite and G admits an α-labeling, then there also exists a G-decomposition of $K_{n,n}$.*

It is known that paths, stars, and all caterpillars in general admit α-labelings (see [16]). It is also known that trees with up to 35 edges admit β-labelings (see [8]). We also have the following result from [10].

Theorem 2.2. *The disjoint union of a graph with a β-labeling, together with a collection of graphs with α-labelings, has a σ-labeling.*

An example of a σ-labeling of a star forest with 7 components and 15 edges is given in Figure 1.
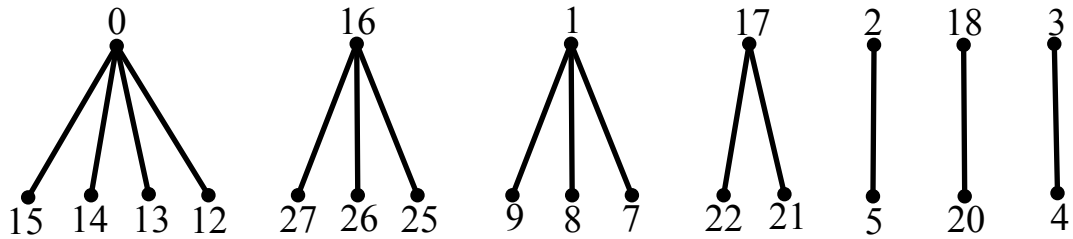
Figure 1: A σ-labeling of a star forest.

In the context of providing evidence in support of Conjecture 1.5, we have the following corollary to Theorem 2.2.

Corollary 2.3. *Let G be a forest with n edges. If one component of G is a tree on up to 36 vertices and all other components are caterpillars, then G decomposes K_{2n+1} and $K_{2n+2} - I$, where I is a 1-factor.*

As for Conjecture 1.6, a consequence of a result by Horak, Širáň, and Wallis [11] ensures that every forest with n edges decomposes the n-cube.

Also, Conjectures 1.5 and 1.6 hold when $G = nK_2$ as a consequence of a result by Alon [1].

Lemma 2.4. *For every graph G and every $t \geq 1$, tK_2 decomposes G if and only if t divides $|E(G)|$ and $\chi'(G) \leq |E(G)|/t$.*

Corollary 2.5. *Let $G = nK_2$ and suppose H is either n-regular and bipartite or $2n$-regular. Then G decomposes H.*

3   Main Results

We give some additional definitions before proceeding with our main results. An *orientation* of a graph H is an assignment of directions to the edges of H. An *Eulerian orientation* of H is a orientation where the indegree at each vertex is equal to the outdegree. It is simple to see that a graph with all even degrees has an Eulerian orientation.

Theorem 3.1. *Let G be a double-star with n edges and let H^+ be a $2n$-regular multigraph with edge-multiplicity at most 2. Suppose the subgraph M of H^+ consisting of the edges of multiplicity 2 is either empty or 2-regular. Let F be a 2-factor of H^+ that contains every component of M. Then there exists a G-decomposition Δ of H^+ with the property that the edges of F are the center-edges of the double-stars in Δ.*

Proof. Let G be the double-star S_{k_1,k_2} with center vertices a and b, where the degree of a is $k_1 + 1$ and the degree of b is $k_2 + 1$. Let H^+, D, and F be as in the hypothesis.

Orient the edges of G so that each leaf has indegree 1. Orient edge $\{a, b\}$ from a to b. Since F is a 2-factor in H^+, it has an Eulerian orientation. Since $H^+ - F$ is $(2n - 2)$-regular, it has an Eulerian orientation. Consider any cycle C in F, and let D_C denote the digraph consisting of all arcs with tail in $V(C)$. Thus every vertex in D_C will have outdegree either $k_1 + k_2 + 1$ (if the vertex is in C) or 0. The proof will be complete if we can show that each such subgraph D_C has a G-decomposition.

Let cycle C have length p and consist of alternating vertices and arcs labeled $v_0, e_1, v_1, e_2, \ldots, v_{p-1}, e_p, v_p = v_0$.

For the first copy G_1 of G in the decomposition, we use e_1 as the central arc, and identify v_0 with a and v_1 with b. Choose k_2 arcs other than e_2 with tail at v_1; label as X the set of endvertices of these k_2 arcs. The remaining k_1 arcs with tail at v_0 in G_1 in this construction will be determined at the end.

We construct the remaining copies $G_2, G_3, \ldots, G_p$ sequentially. After G_{i-1} is determined we construct G_i as follows:
The central arc of G_i is e_i, with v_{i-1} identified with a from G, and v_i identified with b. The remaining arcs with tail at v_{i-1} are all such arcs of $D_C - C$ that were not chosen to be in G_{i-1}. From the remaining $k_1 + k_2$ arcs with tail at v_i, we choose k_2 so that:

i) no arc is chosen that is adjacent with an arc chosen at this step to have tail v_{i-1} (to avoid an immediate triangle), and

ii) we include in the pool all arcs with head a vertex in X.

The selection process above can always be implemented because in G_{i-1} we chose all possible arcs with tail at v_{i-1} and head at a vertex in X, so no such arc appears in G_i.

It remains only to complete the construction of G_1. After G_p has been constructed, k_1 arcs with tail at v_0 have yet to be assigned; we include these arcs in G_1. Because of the pattern noted above, none of these arcs has as a head a vertex in X. Thus G_1 also has no triangles and is therefore isomorphic to G. ∎

Theorem 3.2. *Let k_1 and k_2 be positive integers and let $G = S_{k_1} \cup S_{k_2}$. Let $n = k_1 + k_2$ and suppose that H is a $2n$-regular graph. Then G decomposes H.*

Proof. Let H have order p. If H is the complete graph K_p, the result is covered by Corollary 2.3. Hereafter, we assume that H is not complete.

If H has odd order, then H^c, the complement of H, is even regular and thus contains a 2-factor F. Let H^+ denote the graph with vertex set $V(H)$ and edge set $E(H) \cup E(F)$. By Theorem 3.1, there is an S_{k_1,k_2}-decomposition Δ of H^+ with the property that the edges of F are the center-edges of the double-stars in Δ. By removing the center edges from the double-stars in Δ, we obtain a G-decomposition of H.

If H has even order, then H^c is odd regular. Let $^2H^c$ be the multigraph obtained from H^c by doubling all its edges. Let F be a 2-factor in $^2H^c$ and let H^+ be as in the previous case. Note that H^+ and F satisfy the conditions of Theorem 3.1. We proceed as in the previous case. ∎

Horsley [12] recently proved that Conjecture 1.6 holds when G is a star forest. We provide proofs of two results subsumed by Horsley's result because they parallel our results for $2n$-regular graphs.

Theorem 3.3. *Every star forest G with n edges decomposes $K_{n,n}$.*

Proof. Let $k_1, k_2, \ldots, k_t$ be positive integers with sum n and let G be a star forest with t components where component i has size k_i for $i \in [1, t]$. Let (A, B) be a bipartition of $V(K_{n,n})$, where $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \mathbb{Z}_n$. Let $k'_0 = 0$ and for each $j \in [1, t]$, let $k'_j = \sum_{i=1}^{j} k_i$.

Let $G_1, G_2, \ldots, G_n$ be copies of G in $K_{n,n}$ constructed as follows. For $i \in [1, t]$, component i of G_1 is centered at vertex $c_{i,1} = i - 1$ in B and has leaves $a_{k'_{i-1}+1}, a_{k'_{i-1}+2}, \ldots, a_{k'_i}$ in A. Thus the last component of G_1 has center $c_{t,1} = t - 1$ and leaves $a_{k'_{t-1}+1}, a_{k'_{t-1}+2}, \ldots, a_n$. For $i \in [1, t]$ and $j \in [2, n]$, we will let $c_{i,j}$ denote the center of component i in G_j. For $j \in [2, n]$, let G_j be the copy of G where each component has the same set of leaves as in G_1, but is centered at $c_{i,j} + 1 \pmod{n}$ in B. It is easy to verify that the n copies of G are edge-disjoint and thus $\Delta = \{G_i \colon i \in [1, n]\}$ is a G-decomposition of $K_{n,n}$. ∎

Theorem 3.4. *Let k_1 and k_2 be positive integers and let $n = k_1 + k_2 + 1$. Suppose that H is an n-regular bipartite graph and let I be a 1-factor in H. Then S_{k_1,k_2} decomposes H with the edges of I as the center edges of the double-stars in the decomposition.*

Proof. Let (A, B) be a bipartition of $V(H)$, where $A = \{x_1, x_2, \ldots, x_t\}$ and $B = \{y_1, y_2, \ldots, y_t\}$. Without loss of generality, let $E(I) = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_t, y_t\}\}$. Let $H' = H - I$. Let $I_1, I_2, \ldots, I_{k_1}$

be k_1 edge-disjoint 1-factors in H' and let $F = \bigcup_{i=1}^{k_1} I_i$. For each $i \in [1,t]$, let $S(x_i; k_1)$ be the k_1-star with center x_i induced by the edges in F incident with x_i. For each $j \in [1,t]$, let $S(y_j; k_2)$ be the k_2-star with center y_j induced by the edges in $H' - F$ incident with y_j. For each $\ell \in [1,t]$, let $G_\ell = S(x_\ell; k_1) \cup \{x_\ell, y_\ell\} \cup S(y_\ell; k_2)$. Each G_ℓ is isomorphic to S_{k_1,k_2} and $\Delta = \{G_1, G_2, \ldots, G_t\}$ is an S_{k_1,k_2}-decomposition of H with the edges of I as the center edges of the double-stars in the decomposition. ∎

Theorem 3.5. *Let k_1, k_2 be positive integers and let $G = S_{k_1} \cup S_{k_2}$. Let $n = k_1 + k_2$ and suppose that H is an n-regular bipartite graph. Then G decomposes H.*

Proof. Let (A, B) be a bipartition of $V(H)$, where $A = \{x_1, x_2, \ldots, x_t\}$ and $B = \{y_1, y_2, \ldots, y_t\}$. If H is the complete bipartite graph $K_{n,n}$, then the result is covered by Theorem 3.3. Otherwise, let I be a 1-factor in $K_{A,B} - H$ and let $H^* = H \cup I$. By Theorem 3.4, there exists an S_{k_1,k_2}-decomposition Δ of H^* with the edges of I as the center edges of the double-stars in the decomposition. By removing the center edges from the double-stars in Δ, we obtain a G-decomposition of H. ∎

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