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On the Secure-Domination Number of the Full Balanced Binary Tree with 2^n Leafs

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Abstract

The secure-domination number $\gamma_s(G)$ of a finite simple graph G is the smallest size of a set of vertices in G which is both secure and dominating in G. It is elementary that $\frac{\gamma_s(G)}{|V(G)|} \geq \frac{1}{2}$. A currently unsolved problem: Find the largest number $g \in (\frac{1}{2}, 1]$ such that there is a sequence (H_n) of distinct connected graphs such that $\lim_{n\to\infty} \frac{\gamma_s(H_n)}{|V(H_n)|} = g$. In early work on this problem the full balanced binary trees G_n (which has 2^n leafs, each a distance n from the root vertex) were of interest. They are no longer, but the unexpected difficulty of determining $\gamma_s(G_n)$ has made this determination of interest in itself. In this paper we show that $\liminf_{n\to\infty} \frac{\gamma_s(G_n)}{|V(G_n)|} \geq \frac{8}{15}$ and give a construction of a secure-dominating set $S_n \subseteq V(G_n)$ such that $\lim_{n\to\infty} \frac{|S_n|}{V(G_n)} = \frac{11}{20}$. However, we find that S_n is not a minimum secure-dominating set in G_n for $n \geq 6$.

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1 Introduction to Secure-Domination

All graphs will be finite and simple. Notation and terminology will be as in [3]. For instance, if $S \subseteq V(G)$ is a set of vertices in a graph G, the open and closed neighbor sets of S in G will be denoted $N_G(S)$ and $N_G[S] = S \cup N_G(S)$, respectively, or $N(S)$ and $N[S]$, if the graph G is obvious from the context.

A set $S \subseteq V(G)$ is *dominating* in G if $N_G[S] = V(G)$. In other words, S is dominating in G if and only if each vertex of G which is not in S has a neighbor in S.

In an *attack* on a set $S \subseteq V(G)$, each vertex in $N(S) \backslash S$ *attacks* one of its neighbors in S. In a *defense* of S, each $v \in S$ *defends* either itself or one of its neighbors. A defense of S *thwarts* an attack on S if each $v \in S$ has at least as many defenders as attackers. If every attack on S can be thwarted by a shrewdly chosen defense, then S is *secure* in G.

The secure-domination number of G, denoted $\gamma_s(G)$, is
$\gamma_s(G) = \min\{|S| : S \subseteq V(G)$ is both secure and dominating in $G\}$.

As noted in [2], if $S \subseteq V(G)$ is dominating in G and $|S| < \frac{|V(G)|}{2}$, then $|V(G)| - |S| = |V(G) \backslash S| = |N_G(S) \backslash S| > \frac{|V(G)|}{2} > |S|$; clearly, since S has more possible attackers outside of S than elements of S, such an S is not secure. Therefore,

$$\frac{\gamma_s(G)}{|V(G)|} \geq \frac{1}{2}.$$

In anticipation of future applications, the questions of how big a fraction of the vertex set of a graph is required to form a secure-dominating set, and of how to form minimum secure-dominating sets, are of interest. (We also find them intriguing as purely combinatorial problems.) For any graph G, let $q_s(G) = \frac{\gamma_s(G)}{|V(G)|}$, which we will call the *secure-domination quotient* of G.

Because every dominating set in a graph must contain every isolated vertex of the graph, arrangements can easily be made for $q_s(G)$ to be any rational number in the interval $[\frac{1}{2}, 1]$. Therefore, we confine our attention to connected graphs with two or more vertices. So far, the largest value of q_s known on these graphs is $\frac{2}{3}$, and there are only three small connected graphs known so far at which the value of q_s is $\frac{2}{3}$: P_3, the path on three vertices, $C_3 = K_3$, the complete graph on three vertices, and C_6, the cycle on six vertices.

Getting away from possibly anomalous values of q_s on small graphs, the outstanding problem is to determine what is defined in [1] as $\limsup_{G \text{ connected}} q_s(G)$, the largest number $g \in [\frac{1}{2}, 1]$ such that there exists a sequence (H_n) of distinct connected graphs such that $\lim_{n\to\infty} q_s(H_n) = g$.

In the early going, it was not even obvious that $g > \frac{1}{2}$. The balanced binary trees G_n on which we shall concentrate here were of great interest to the co-authors of [1], who eventually abandoned their study due to lack of success. By the results here and in [1],

$$\limsup_{n\to\infty} q_s(G_n) \le \frac{11}{20} < \frac{4}{7} \le \limsup_{G \text{ connected}} q_s(G),$$

so the G_n are no longer in the running as a possible winning sequence in the $\limsup_{G \text{ connected}} q_s(G)$ problem. But the problem of determining $\gamma_s(G_n)$ as a function of n may be of comparable status. It appears to lack generality, but often work on specific structures can be generalized. We expect that the lemma-sized discoveries and one construction that we will present here will point the way to efficient production of economical secure-dominating sets in arbitrary trees.

2 Balanced Full Binary Trees

A full binary tree is a tree (a connected graph with no cycles) in which one vertex, the *root*, has degree 2, and all other vertices have degrees 3 or 1. A vertex of degree 1 in any tree is called a *leaf* (pl. *leafs*).

In any tree, any two distinct vertices are joined by a unique path, the number of edges on which is the *distance* between the vertices, in the tree. For a vertex of degree 3 in a full binary tree, one of its incident edges belongs to the unique path joining it to the root, and the other two edges join it to the *children* of which it is the *parent*. Those two children are *siblings*.

It is elementary that every full binary tree with n leafs has $2n-1$ vertices. It is often convenient to consider these vertices as partitioned into *levels*; the vertices on the kth level are a distance k from the root. Obviously the children, if any, of the vertices on the level k are the vertices of the $(k+1)$st level. Any vertex on level $k \geq 1$ has exactly one neighbor on level $k-1$: its parent.

The *balanced* full binary trees are the full binary trees $G_1, G_2, \ldots$ such that G_n has 2^n leafs, each on level n of the graph.

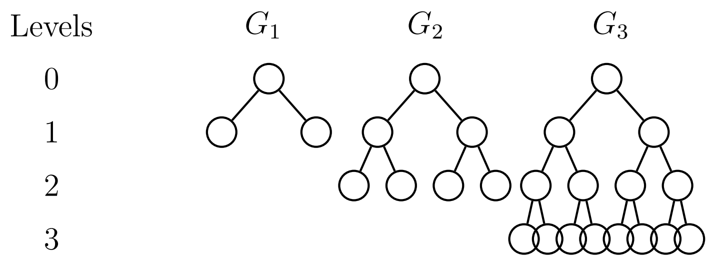

Figure 1: G_1, G_2, and G_3

There are 2^k vertices on level k of G_n, $0 \leq k \leq n$. Therefore, for $n \geq 3$, there are 2^{n-2} G_2's occupying levels $n-2$, $n-1$, and n of G_n, each rooted in level $n-2$. We will refer to these as *terminal* G_2's of G_n.

Lemma 2.1. *If $n \geq 3$ and $S \subseteq V(G_n)$ is secure-dominating in G_n, then each terminal G_2 contains at least 4 vertices of S.*

Proof. For this proof, ignore the darkened vertices in Figure 2. Let G be the terminal G_2 rooted at w depicted in Figure 2. If $w \notin S$, $S \cap V(G_1)$ must be secure-dominating in each of the G_1's induced by u, v, a and by b, x, c, respectively. Since $\gamma_s(G_1) = 2$, it follows that $|S \cap V(G)| \geq 2 + 2 = 4$. So, suppose that $w \in S$. Then $S \cap V(G)$ must be dominating in G. It is easy to check that the only three-element subset of $V(G)$ containing w which is dominating in G is $\{v, w, x\}$. But if $S \cap V(G) = \{v, w, x\}$, then any attack on S in which u and a attack v, and b and c attack x, cannot be thwarted, and this would contradict the presumed secureness of S. $\square$

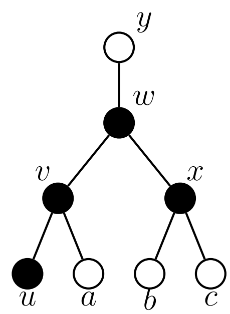

Figure 2: A terminal G_2 with the parent of its root and a particular secure-dominating set $\{u, v, w, x\}$ in that graph

Lemma 2.2. *In the graph depicted in Figure 2, $S' = \{u, v, w, x\}$ is secure and dominating.*

Proof. There is only one possible attack on S' that utilizes every attacker, and the following defense assignments thwarts that attack: u defends v, v defends w, and w and x defend x. $\square$

Later, when we construct secure-dominating sets in G_n, the terminal G_2's will always have the four set elements depicted in Figure 2, whether the root parent (y, in Figure 2) is in the set being constructed or not.

The lemmas following are about arbitrary subgraphs of G_n isomorphic to G_2 or G_3. In the application of these lemmas the G_2's and G_3's will be in "standard position" in G_n, with the children of the root on level $t+1$ of G_n if the root is on level t of G_n. These lemmas hold for arbitrary G_2's and G_3's. Perhaps this will be useful one day.

Lemma 2.3. *Suppose that $n \geq 3$, and S is a secure-dominating set in G_n. Suppose that $H \simeq G_2$ is a subgraph of G_n with its root, w, adjacent to a vertex $y \notin V(H)$. Then $|S \cap V(H)| \geq 1$, with equality only if $y \in S$. If $y \notin S$ then $|S \cap V(H)| \geq 3$. If $y \in S$ and $w \notin S$ then $|S \cap V(H)| \geq 2$.*

Proof. Let the vertices of H be named as in Figure 2. (Ignore the darkening of u, v, w, and x in that figure.) If $S \cap V(H) = \varnothing$ then v and x are not dominated by S. Therefore $|S \cap V(H)| \geq 1$. Suppose $|S \cap V(H)| = 1$. If the single vertex of $S \cap V(H)$ were any vertex in S other than w, then S is not dominating in H. Therefore the lone vertex in $S \cap V(H)$ must be w. If $y \notin S$ then S is not secure. Therefore, $|S \cap V(H)| = 1$ implies $y \in S$.

Suppose that $y \notin S$. If $w \in S$, then at least one of v, x must be in S; otherwise, S would not be secure. Let us suppose without loss of generality that $v \in S$ and $x \notin S$. Then at least one of u, a must be in S; if not, then $\{v, w\}$ has four potential attackers and no defenders except v, w, so S is not secure. Thus, in any case, if $y \notin S$ and $w \in S$ then $|S \cap V(H)| \geq 3$.

Now suppose that $w, y \notin S$. Since S is dominating in G, at least one of v, x must be in S. Suppose that $v \in S$. Because S is secure, at least one of u, a must be in S. If $x \in S$ also, then we are done (in fact, $|S \cap V(H)| \geq 4$ in this case), so assume that $x \notin S$. Because $w \notin S$ and S is dominating, at least one of b, c is in S. Thus $|S \cap V(H)| \geq 3$ if $y \notin S$.

Finally, suppose that $y \in S$ and $w \notin S$. Since S is dominating, for each of v, x, either the vertex itself or one of its children, in $H \simeq G_2$, is in S. Thus $|S \cap V(H)| \geq 2$ in this case. $\square$

Lemma 2.4. *Suppose that $n \geq 3$, and S is a secure-dominating set in G_n. Suppose that $H \simeq G_3$ is a subgraph of G_n. Then $|S \cap V(H)| \geq 4$.*

Proof. Let y be the root of $H \simeq G_3$. Each of y's children is a root of a G_2 in H. If $y \notin S$, by Lemma 2.3 $|S \cap V(H)| \geq 6$. If $y \in S$, then either both children of y are in S, or, by Lemma 2.3, $|S \cap V(H)| \geq 4$. If y and both its children in H are in S, then at least one grandchild of y in H must be in S; otherwise, an attack on S in which each grandchild attacks its parent cannot be thwarted. Thus, in any case, $|S \cap V(H)| \geq 4$. $\square$

Theorem 2.5. $\liminf\limits_{n \to \infty} q_s(G_n) \geq \frac{8}{15}$.

Proof. Suppose that $n \geq 6$ and that S is a minimum secure-dominating set in G_n. Let $t \in \{0, 1, 2, 3\}$ and $k \geq 0$ be such that $n - 2 - t = 4k$. Then $[t, t+1, \ldots, n-3]$ can be partitioned into k disjoint blocks of four consecutive integers, $B_0 = \{t, t+1, t+2, t+3\}, \ldots, B_{k-1} = \{n-6, n-5, n-4, n-3\}$. The four levels in G_n corresponding to $B_j = \{t+4j, t+4j+1, t+4j+2, t+4j+3\}$ contain 2^{t+4j} disjoint G_3's, each rooted at level $t+4j$ in G_n. By Lemmas 2.1

and 2.4,

$$\begin{aligned}|S| &\geq 4 \cdot 2^{n-2} + 4\sum_{j=0}^{k-1} 2^{t+4j} \\ &= 2^n + 4 \cdot 2^t \frac{16^k - 1}{15} \\ &= 2^n + \frac{4}{15}(2^{4k+t} - 2^t) = 2^n + \frac{2^n - 2^{t+2}}{15}.\end{aligned}$$

Recollect that, although t depends on n, $0 \leq t \leq 3$ for every n. Therefore,

$$\begin{aligned}q_s(G_n) &= \frac{\gamma_s(G_n)}{2^{n+1} - 1} = \frac{|S|}{2^{n+1} - 1} \\ &\geq \frac{1}{2} + \frac{2^n - 2^{t+2}}{15 \cdot 2^{n+1}} \\ &\to \frac{1}{2} + \frac{1}{30} = \frac{8}{15} \text{ as } n \to \infty. \blacksquare\end{aligned}$$

Remark: The proof of Theorem 2.5 was needlessly complicated for the sake of clarity. If a certain amount of reckless abandon is permitted, we could argue as follows: In the graph G_n, about half of the vertices are at level n. Therefore, about $\frac{7}{8}$ of the vertices are on the last three levels, occupied by those terminal G_2's. If S is a minimum secure-dominating set in G_n, and $n \geq 3$, at least $\frac{4}{7}$ of the vertices of the last three levels of G_n are in S (Lemma 2.1), and, except for no more than seven vertices in levels 0, 1, and 2 of G_n, at least $\frac{4}{15}$ of the approximately $\frac{1}{8}$ of the vertices of G_n not in the last three levels are in S. Therefore,

$$\liminf_{n\to\infty} \frac{\gamma_s(G_n)}{|V(G_n)|} \geq \frac{4}{7} \cdot \frac{7}{8} + \frac{4}{15} \cdot \frac{1}{8} = \frac{8}{15}.$$

3 An Easy Construction

The construction referred to will produce a secure-dominating set S_n of G_n such that $\frac{|S_n|}{|V(G_n)|} \to \frac{11}{20}$ as $n \to \infty$. Because, by Lemma 2.1, there is not much choice in picking a small secure-dominating set in G_n when it comes to the terminal G_2's, the construction starts by taking care of the last three levels of G_n, and then attacks the approximately $\frac{1}{8}$ of the graph remaining. The basic idea behind this attack is simple and greedy (a desirable attribute in an algorithm) and could be adapted to find a secure-dominating set in any tree. The algorithm starts: choose a vertex of the tree and make it the root. With the root chosen, work through the levels of the tree, putting vertices in or leaving them out of the set under construction, guided by the principle that a vertex goes in the set only if it must go in, either for domination or for security. So, by this principle, the root does not go into the set; then, because the root must be dominated, exactly one of its neighbors on level 1 goes into the set, that is, if none of those neighbors is a leaf of the tree. If there are leafs at level 1, then for both domination and security, they all go into the set, and it is on to level 2. Right away, we see that this algorithm can do badly; if the tree is $K_{1,t}$, $t > 2$, and we take the vertex of degree t as the root, the algorithm gives the secure-dominating set S consisting of all the leafs, $|S| = t$, whereas $\gamma_s(K_{1,t}) = \lceil \frac{t+1}{2} \rceil$ ([2]). If we ran the variant of the algorithm in which the root is put in the set, and thereafter the greedy principles are applied, then starting with the vertex of degree t as the root, we get a minimum secure-dominating set in $K_{1,t}$. We get the same result with the original algorithm if we take one of the leafs as a root. Perhaps an algorithmist should look into this matter.

Instructions for Forming a Secure-Dominating Set S_n in G_n, $n \geq 3$

1. Let each terminal G_2 contribute four vertices to S_n, the four indicated in Figure 2.

2. The root of G_n is not in S_n. If $n = 3$, stop.

3. Having determined membership in S_n for vertices on levels $0, \ldots, k$, with $k \leq n - 4$, determine S_n membership for the children of each vertex v on level k as follows.

    - If $v \in S_n$ then one child of v is in S_n and the other is not.

- If $v \notin S_n$ and the parent of v is in S_n, then neither child of v is in S_n.
- Otherwise (i.e., if $k = 0$, or $k > 0$ and neither v nor its parent is in S_n), one child of v is in S_n and the other one is not.

Figure 3 depicts S_7 in levels $0, \ldots, 4$.

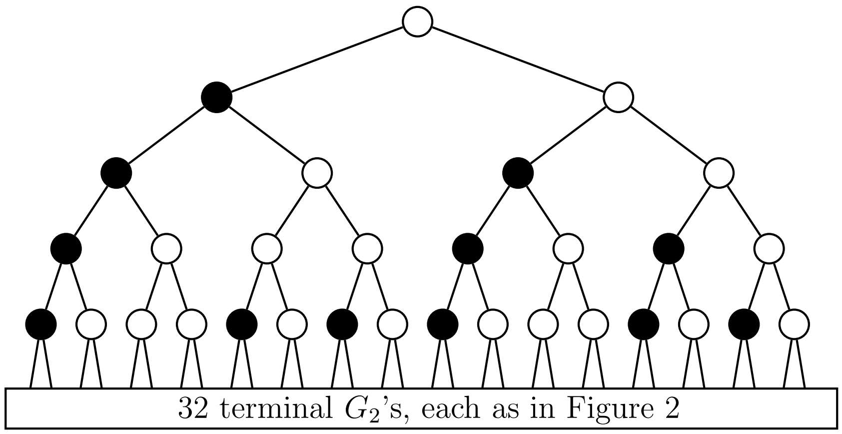

Figure 3: S_7

It is clear from the instructions that S_n is dominating in G_n - the root is dominated in G, and every vertex on level $k \geq 1$ which is not dominated by S_n intersected with levels k, $k-1$ will have one of its children in S_n.

To see that S_n is secure it suffices to see that each connected component of $G_n[S_n]$, the subgraph of G_n induced by S_n, is secure in G_n. Some of these components are paths on four vertices contained in a terminal G_2; these have at most 4 possible attackers, three inside and possibly one outside the G_2. A defense is described in the proof of Lemma 2.2.

The other components of $G_n[S_n]$ consist of paths running through levels to level $n-3$ where the last vertex of the path is the parent of two children in two terminal G_2's. See Figure 4.

There is only one attack on such a component to worry about, and here is the defense: u_j defends u_{j-1}, $2 \leq j \leq t$, and u_1 defends itself. The parts of the component in the terminal G_2's can take care of themselves.

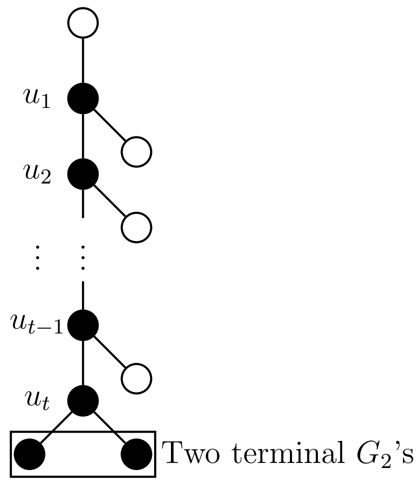

Figure 4: A component of $G_n[S_n]$, with attackers

Theorem 3.1. $\lim\limits_{n \to \infty} \frac{|S_n|}{|V(G_n)|} = \frac{11}{20}$.

Proof. Suppose that $n \geq 4$. For $0 \leq k \leq n-3$, let a_k be the number of elements of S_n on level k of G_n, let b_k be the number of non-members of S_n on level k which have their parent in S_n, and let $c_k = 2^k - (a_k + b_k)$, the number of non-members of S_n on level k not parented in S_n. Thus, for instance, $\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. If $0 \leq k \leq n-4$, then, by the instructions for forming S_n, $a_{k+1} = a_k + c_k$, $b_{k+1} = a_k$, and $c_{k+1} = 2b_k + c_k$. That is,

$$\begin{pmatrix} a_{k+1} \\ b_{k+1} \\ c_{k+1} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} = A \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} = A^{k+1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} (A^{k+1})_{1,3} \\ (A^{k+1})_{2,3} \\ (A^{k+1})_{3,3} \end{pmatrix}$$

By the standard trick of diagonalizing $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$, we have obtained closed forms for the powers of A. For instance, if $k \equiv 0 \bmod 4$,

$$A^k = \begin{bmatrix} \frac{2^{k+1}+3}{5} & \frac{2^{k+1}-2}{5} & \frac{2^{k+1}-2}{5} \\ \frac{2^k-1}{5} & \frac{2^k+4}{5} & \frac{2^k-1}{5} \\ \frac{2^{k+1}-2}{5} & \frac{2^{k+1}-2}{5} & \frac{2^{k+1}+3}{5} \end{bmatrix}.$$

A^k for $k \equiv 1$, 2, or 3 mod 4 can be obtained by multiplying A to a power $\equiv 0$ mod 4 by A, A^2, or A^3; one more multiplication by A can prove the validity of these formulae, by induction on k. You needn't go through the hassle of diagonalization to verify our claims!

We have $a_k = (A^k)_{1,3} = \frac{2^{k+1}+\delta_k}{5}$ where

$$\delta_k = \begin{cases} -2 \text{ if } k \equiv 0 \bmod 4 \\ 1 \text{ if } k \equiv 1 \bmod 4 \\ 2 \text{ if } k \equiv 2 \bmod 4 \\ -1 \text{ if } k \equiv 3 \bmod 4 \end{cases}.$$

Therefore, the number u_n of elements of S_n on levels $0, \ldots, n-3$ is

$$u_n = \sum_{k=1}^{n-3} a_k = \frac{1}{5}\left[\sum_{k=1}^{n-3} 2^{k+1} + \sum_{k=1}^{n-3} \delta_k\right] = \frac{2^{n-1}}{5} + d_n,\ |d_n| \le \frac{4}{5}.$$

Meanwhile, the number of elements of S_n on levels $n-2$, $n-1$, and n is $4 \cdot 2^{n-2} = 2^n$. Thus

$$\begin{aligned} \frac{|S_n|}{|V(G_n)|} &= \frac{2^n + \frac{2^{n-1}}{5} + d_n}{2^{n+1}-1} \\ &= \frac{2^n(1+\frac{1}{10})}{2^{n+1}-1} + \frac{d_n}{2^{n+1}-1} \\ &\to \frac{1}{2} \cdot \frac{11}{10} + 0 = \frac{11}{20} \text{ as } n \to \infty. \blacksquare \end{aligned}$$

Corollary 3.2. $\limsup_{n\to\infty} q_s(G_n) \le \frac{11}{20}$.

With the aid of Lemma 2.1 it is not hard to prove that $\gamma_s(G_n) = |S_n|$ for $n \in \{3, 4, 5\}$. To our surprise, $\gamma_s(G_6) \leq 69 = |S_6|-1$ and $\gamma_s(G_7) \leq 138 = |S_7|-2$, as shown in Figure 5, and in Figure 6 we see that $\gamma_s(G_8) \leq 278 = |S_8| - 3$.

It seems clear that $\gamma_s(G_n) < |S_n|$ for all $n > 5$, although we are not prepared to prove this. But if, as seems likely, $|S_n| - \gamma_s(G_n) = O(n)$, or if, even more likely, $|S_n| - \gamma_s(G_n) = o(2^n)$, then it would follow that $\lim_{n\to\infty} q_s(G_n) = \frac{11}{20}$.

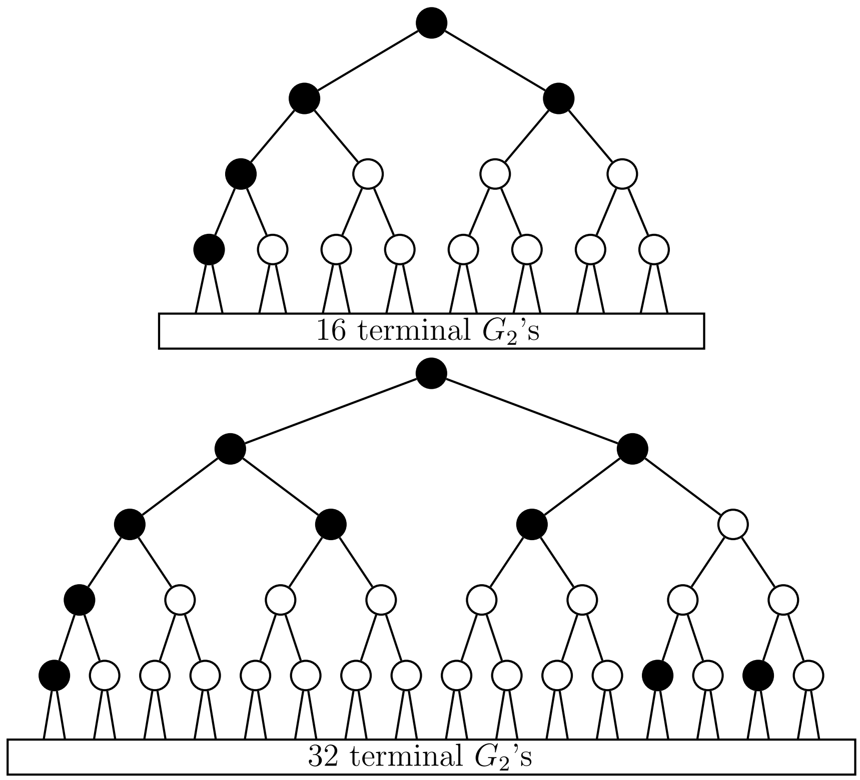

Figure 5: Secure-dominating sets showing that $\gamma_s(G_6) \leq 69$ and $\gamma_s(G_7) \leq 138$

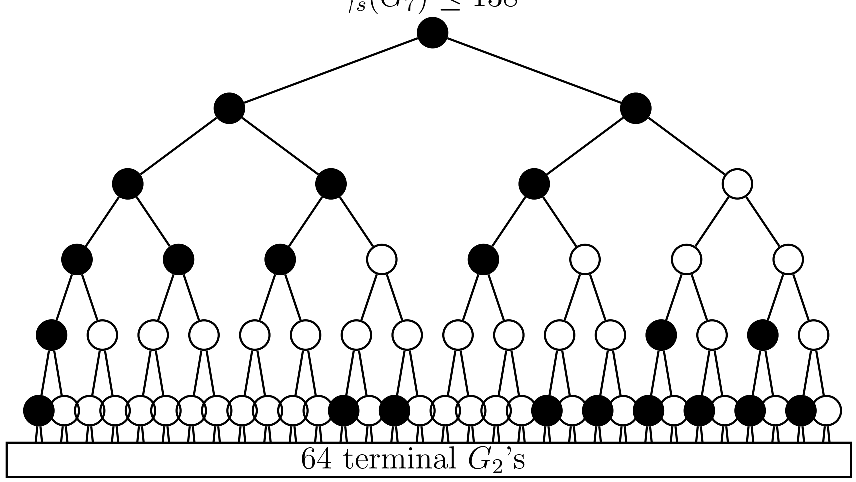

Figure 6: A secure-dominating set that shows that $\gamma_s(G_8) \leq 278$

Let G_∞ denote the infinite binary tree with no leafs and a root. This graph contains infinitely many copies of G_n for each $n = 1, 2, \ldots$; let G_n stand for the unique copy that shares its root with G_∞.

Delete instruction 1 from the instructions for constructing S_n, and replace n in instructions 2 and 3 by ∞. Ignoring some now-absurd phrases, like "with $k \leq \infty - 4$", we now have instructions for constructing an infinite secure-dominating set S_∞ in G_∞. (The connected components of $G_\infty[S_\infty]$ are one-way infinite paths, and they are secure in G_∞.)

By a calculation similar to one in the proof of Theorem 3.1,

$$\lim_{n\to\infty} \frac{|S_\infty \cap V(G_n)|}{|V(G_n)|} = \frac{2}{5},$$

which is intriguing, because $\frac{2}{5} < \frac{1}{2}$. We wonder: what is the infimum of the values $\limsup_{n\to\infty} \frac{|S\cap V(G_n)|}{|V(G_n)|}$ as S ranges over secure-dominating sets in G_∞? By using Lemma 2.4 as in the proof of Theorem 2.5 it can be seen that for any secure-dominating set S in G_∞, $\liminf_{n\to\infty} \frac{|S\cap V(G_n)|}{|V(G_n)|} \geq \frac{4}{15}$.

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