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A recursive relation and some statistical properties for the Möbius function

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Abstract

An elementary recursive relation for the Möbius function $\mu(n)$ is obtained by two ways. With this recursive relation, $\mu(n)$ can be calculated without directly knowing the factorization of the n. $\mu(1) \sim \mu(2 \times 10^7)$ are calculated recursively one by one. Based on these 2×10^7 samples, the empirical probabilities of $\mu(n)$ of taking $-1, 0,$ and 1 in classic statistics are calculated and compared with the theoretical probabilities in number theory. The numerical consistency between these two kinds of probability show that $\mu(n)$ could be seen as an independent random sequence when n is large. The expectation and variance of the $\mu(n)$ are 0 and $6n/\pi^2$, respectively. Furthermore, we show that any conjecture of the Mertens type is false in probability sense, and present an upper bound for cumulative sums of $\mu(n)$ with a certain probability.

1 Introduction

The Möbius function is defined for a positive integer n by

$$\mu(n) = \begin{cases} 1 & n = 1 \\ 0 & \text{if } n \text{ is divisible by a prime square} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \end{cases} \tag{1}$$

It is shown that Möbius function and its associated Möbius transform are important for solving different mathematical and/or scientific problems (eg., Schroeder, 2008). In

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physics, the Möbius function and its associated Möbius transform are used in inverse black body radiation problem (eg., Chen, 1987; 1990), inversion of specific heat data for phonon densities of states (eg., Chen et al., 1990), solution of integral equations regarding Fermi and Bose systems (eg., Chen, 2010), inverse transmissivity problem (Ji et al., 2006), and so on. All of these studies are related to how to calculate $\mu(n)$ if special methods are not used.

To calculate the Möbius function, many algorithms are presented and most of them are based on the factorization of its argument. A famous one is vectorized sieving (eg., Lioen and Lune, 1994; Kuznetsov, 2011). On the other hand, in their book, Hardy and Wright (2008) showed that the Möbius function is the sum of the primitive n-th roots of unity; i.e.,

$$\mu(n) = \sum_{\substack{1 \le k \le n \\ \gcd(k,n)=1}} \exp(\frac{2\pi i k}{n}) \tag{2}$$

Formula (2) can be used to calculate the Möbius function without knowing the factorization of n. However, the computational complexity is not low.

Here we present a recursive relation to calculate the Möbius function without directly knowing the factorization of n as Formula (2) does, but with less computational complexity. We calculate Möbius function from $\mu(1)$ to $\mu(2 \times 10^7)$ with this recursive relation and discuss some statistical properties of the Möbius function.

2 A recursive relation for Möbius function

The recursive relation for Möbius function can be obtained by two ways. One is from Möbius transform, and the other is from the Redheffer Matrix related to Mertens function, which is the cumulative sum of the Möbius function.

2.1 The recursive relation from Möbius transfrom

According to pair potential model for cohesive energy (Chen, 1994), the cohesive energy E for each atom in an infinite linear chain can be expressed as a sum of pairwise potentials,

$$E(x) = \sum_{n=1}^{\infty} \Phi(nx) \tag{3}$$

Using the Chen-Möbius formula (e.g., Chen, 2010; Wang, 2013),

$$\Phi(x) = \sum_{n=1}^{\infty} \mu(n) E(nx) \tag{4}$$

We can write the following Matrix equality according to expression (3) and (4),

$$\begin{pmatrix} E(x) \\ E(2x) \\ E(3x) \\ E(4x) \\ E(5x) \\ E(6x) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \Phi(x) \\ \Phi(2x) \\ \Phi(3x) \\ \Phi(4x) \\ \Phi(5x) \\ \Phi(6x) \\ \vdots \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} \Phi(x) \\ \Phi(2x) \\ \Phi(3x) \\ \Phi(4x) \\ \Phi(5x) \\ \Phi(6x) \\ \vdots \end{pmatrix} = \begin{pmatrix} \mu(1) & \mu(2) & \mu(3) & \mu(4) & \mu(5) & \mu(6) & \cdots \\ 0 & \mu(1) & 0 & \mu(2) & 0 & \mu(3) & \cdots \\ 0 & 0 & \mu(1) & 0 & 0 & \mu(2) & \cdots \\ 0 & 0 & 0 & \mu(1) & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \mu(1) & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \mu(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} E(x) \\ E(2x) \\ E(3x) \\ E(4x) \\ E(5x) \\ E(6x) \\ \vdots \end{pmatrix} \tag{6}$$

Let

$$\Phi = [\Phi(x) \quad \Phi(2x) \quad \Phi(3x) \quad \Phi(4x) \quad \Phi(5x) \quad \Phi(6x) \quad \ldots]^{\mathrm{T}}$$

and

$$E = [E(x) \quad E(2x) \quad E(3x) \quad E(4x) \quad E(5x) \quad E(6x) \quad \ldots]^{\mathrm{T}}$$

Matrix equality (5) and (6) can be rewritten as the following,

$$E = U\Phi \tag{7}$$

$$\Phi = VE \tag{8}$$

Obviously,

$$UV = VU = V^T U^T = I \tag{9}$$

Hence the values of the Möbius function, which are the elements of the first row of the matrix V, can be obtained from the inverse matrix of U. Because the matrix U is a triangular one, in which $U = \{u_{ij}\}$ with $u_{ij} = 1$ if and only if $i|j$, one can get,

$$v_{1i} = -\sum_{k=1}^{i-1} v_{1k} u_{ki}, i = 2, 3, \cdots \tag{10}$$

Based on recursive relation (10), we can obtain the recursive relation for Möbius function as the following, which is a special case for (poset) Möbius function in the incidence algebra[1]

$$\mu(n) = -\sum_{k=1}^{n-1} l_{nk}\mu(k), n = 2, 3, \cdots ; l_{nk} = \begin{cases} 1 & k|n \\ 0 & \text{else} \end{cases} \tag{11}$$

2.2 The recursive relation from Redheffer Matrix

It is well known that the Mertens function, which is the cumulative sum of the Möbius function, is the determinant of the Redheffer matrix. The Redheffer matrix $R = \{r_{ij}\}$ is defined by $r_{ij} = 1$ if $j = 1$ or $i|j$, and $r_{ij} = 0$ otherwise, ie.,

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 1 & \cdots \\ 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{12}$$

R can be decomposed as follows:

[1]Russ Woodroofe. Private Communication, August 22, 2016

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = S + U \tag{13}$$

where $S = \{s_{ij}\} = 1$ if and only if $j = 1$ and $i \neq 1$; U is the same to matrix equality (7) with $u_{ij} = 1$ if and only if $i|j$.

It can be shown that the inverse of U is V which is in matrix equality (8), that is,

$$V = \{v_{ij}\} = \begin{cases} \mu(\frac{j}{i}) & i|j \\ 0 & \text{else} \end{cases} \tag{14}$$

In fact, the ij-th entry of the product of $U \times V$, p_{ij}, is,

$$p_{ij} = \sum_{k=1}^{n} u_{ik} v_{kj} \tag{15}$$

According to the definition of U and V in (14), $u_{ik}v_{kj}$ is 0 unless $i|k$ and $k|j$, which means that $p_{ij} = 0$ if $i \nmid j$. If $i|j$, by using the well known

$$\sum_{i|n} \mu(i) = \begin{cases} 1 & n = 1 \\ 0 & \text{else} \end{cases}$$

one can get,

$$p_{ij} = \sum_{k(i|k \text{ and } k|j)} \mu(\frac{j}{k}) = \sum_{k'|(j/i)} \mu(\frac{j/i}{k'}) = \begin{cases} 1 & j = i \\ 0 & \text{else} \end{cases}$$

Therefore, $U \times V = \{p_{ij}\} = I$ and equality (14) holds. With the same procedures in the subsection above, we can obtain the recursive relation (11) for the Möbius function.

3 Some statistical properties for Möbius function

With the recursive relation (11), we calculated the Möbius function from $\mu(1)$ to $\mu(2\times 10^7)$. These values are used for the numerical test on some statistical properties of Möbius sequence $\mu(n)$, if $\mu(n)$ is seen as an independent random sequence although it has a deterministic recursive rule. In fact, as n is large enough, the random assumption above is reasonable.

3.1 The expectation and variance of the $\mu(n)$

First we calculated the probabilities of $\mu(n)$ of taking the values -1, 0 and 1. In this respect, there are two useful results from Hardy and Wright (2008) as follows,

1. $\mu(n) = \pm 1$ or $|\mu(n)| = 1$ if a number n is squarefree, and the probability (p_t) that a number should be squarefree is $\frac{6}{\pi^2}$, more precisely,

$$\sum_{n=1}^{x} |\mu(n)| = \frac{6}{\pi^2}x + O\left(\sqrt{x}\right) \tag{16}$$

2. Among the squarefree numbers, those for $\mu(n) = 1$ and those for $\mu(n) = -1$ occur with about the same frequency.

Therefore, if $\mu(k)(k = 1, 2, \ldots n)$ denotes the k-th value of $\mu(n)$ and p_{t_k} the probability, the corresponding distribution rule is shown in Table 1.

Table 1: The distribution rule for $\mu(k)$

$\mu(k)$	-1	0	1
p_{t_k}	$3/\pi^2$	$1-6/\pi^2$	$3/\pi^2$

The p_t is different from that in the classic statistics (See details in Hardy and Wright (2008), P. 354). To use the methods in classic statistics, it is first necessary to test the consistency between them. The classic probability here is,

$$p_e = \frac{N_{\mu(n)=m}}{n}, (m = -1, 0, 1) \tag{17}$$

where p_e is the classic probability that $\mu(n) = m(m = -1, 0, 1)$, $N_{\mu(n)=m}$ is the frequency of $\mu(n) = m$. We calculate the frequencies and p_es of $\mu(n)$ of taking $-1, 0, 1$ and that of

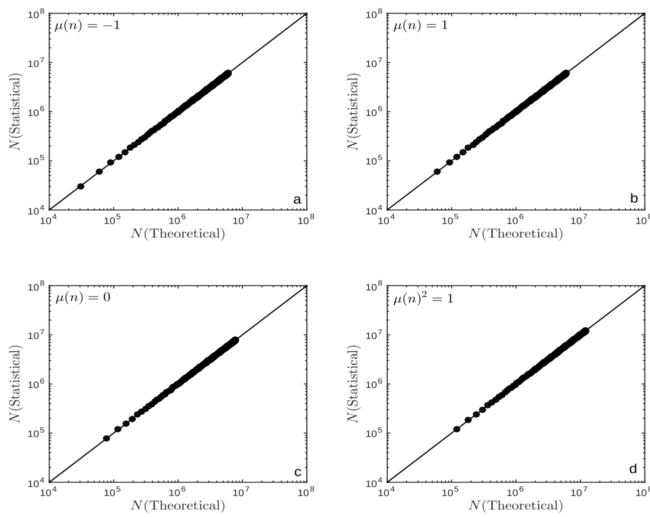

Fig. 1: The comparison of frequencies observed (N(Statistical)) with those calculated by number theory (N(Theorical)) in blocks with different length of N. The solid line in each subfigure is the reference line. The related data are shown in Table 3 and 4 in the appendix.

$|\mu(n)| = \mu^2(n) = 1$ in 200 blocks with different length by using 2×10^7 $\mu(n)$s above, respectively. Figure 1 shows the comparison of these frequencies observed with those calculated by $N \times p_t$ in different blocks of length N. It can be seen that the frequencies observed are consistent with those calculated. Figure 2 shows the numerical comparison of classic probability p_e with the p_t. It also can be observed that these two kinds of probabilities are numerically consistent. Detailed numerical results are shown in Table 3 and 4 in the appendix. The consistencies above show that the p_t is equivalent numerically to the classic probability p_e as defined in (17). Similar numerical support can be found in Good and Churchhouse (1968). Based on these numerical results, we can take $\mu(n)$ as an independent random sequence although it has a deterministic recursive rule and use classic statistical method to study it.

Accordingly, the expectation and variance of the $\mu(k)$ are $E(\mu(k)) = 0$ and $D(\mu(k)) = 6/\pi^2$ from Table 1, respectively. These results are consistent with the conjecture of Good and

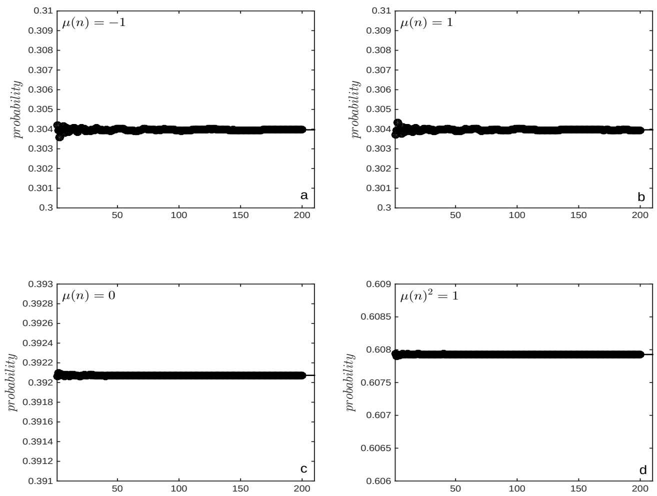

Fig. 2: The comparison of classic probability p_e (solid dotts) with the p_t (solid lines) in blocks with different length of N. In each subfigure, the x-axis is ordinal number of each block. The related data are shown in Table 3 and 4 in the appendix.

Churchhouse (1968). The expectation and variance of the $\mu(k)$ will be used in the following section.

3.2 Mertens conjecture in a statistical point of view

Mertens function of a positive integer n is defined as the cumulative sums of $\mu(n)$,

$$M(n) = \sum_{k=1}^{n} \mu(k) \tag{18}$$

An old conjecture, "Mertens conjecture" , proposed that $|M(n)| < n^{1/2}$ for all n. This was disproved by Odlyzko and te Riele (1985). In this subsection, we recheck Mertens conjecture in a statistical point of view, for $\mu(n)$ is seen as an independent random sequence although it has a deterministic recursive rule.

According to the central limit theorem, if n is large enough, for any x, we have,

$$\lim_{n\to\infty} P\left\{\frac{\sum\limits_{k=1}^{n}\mu(k)-E(\sum\limits_{k=1}^{n}\mu(k))}{\sqrt{D(\sum\limits_{k=1}^{n}\mu(k))}} \leq x\right\} \begin{aligned} &= \lim_{n\to\infty} P\left\{\frac{M(n)}{\sqrt{6n/\pi^2}} \leq x\right\} \\ &= \int\limits_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2})\mathrm{d}t \\ &= \Phi(x) \end{aligned} \tag{19}$$

where, $E\left(\sum_{k=1}^{n}\mu(k)\right) = 0$, $D\left(\sum_{k=1}^{n}\mu(k)\right) = 6n/\pi^2$ according to subsection 3.1. And (19) means,

$$\frac{M(n)}{\sqrt{6n/\pi^2}} \backsim N(0,1) \tag{20}$$

as n is large.

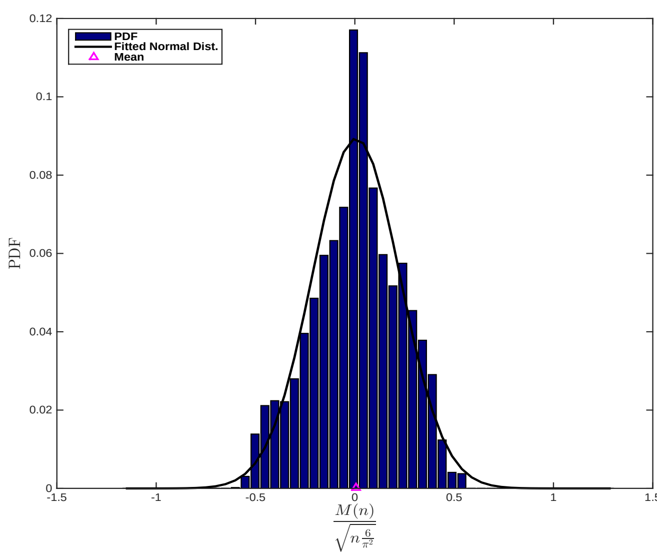

Fig. 3: The probability density function for $\frac{M(n)}{\sqrt{6n/\pi^2}}$ when $n = 500000$.

Figure 3 shows the probability density function for $\frac{M(n)}{\sqrt{6n/\pi^2}}$ when $n = 500000$. It can be observed that the distribution of (20) is reasonable. Another similar numerical support for this can be found in Good and Churchhouse (1968).

With equality (19), the probability of $M(n) > \sqrt{n}$ can be obtained. Clearly,

$$P\left\{M(n) > \sqrt{n}\right\} = 1 - \Phi(\frac{1}{\sqrt{6/\pi^2}}) \approx 0.0998 \tag{21}$$

That is, the probability of $M(n) > \sqrt{n}$ is about 0.0998 but not 0, which means that Mertens conjecture is not true. Furthermore, any conjecture of the Mertens type, viz.

$$|M(n)| < C\sqrt{n} \tag{22}$$

where C is any positive constant, is false, unless C is large enough.

3.3 Upper bound of cumulative sums of $\mu(n)$ sequence

From (19), one can get when n is large,

$$\frac{\sum_{k=1}^{n}\mu(k) - E(\sum_{k=1}^{n}\mu(k))}{\sqrt{D(\sum_{k=1}^{n}\mu(k))}} = \frac{\sum_{k=1}^{n}\mu(k) - nu}{\sqrt{n}\sigma} \sim N(0,1) \tag{23}$$

Then a confidence interval for expectation u with a known standard variance σ and a probability of $1-\alpha$ is,

$$\left[-\frac{\sigma}{\sqrt{n}}K_{\alpha/2}, \frac{\sigma}{\sqrt{n}}K_{\alpha/2}\right] \tag{24}$$

where,

$$\int_{-K_{\alpha/2}}^{K_{\alpha/2}} \frac{1}{\sqrt{2\pi}}\exp(-\frac{t^2}{2})\mathrm{d}t = 1-\alpha \tag{25}$$

From (24) and (25), one can infer further that the upper bound of $\sum_{k=1}^{n}\mu(k)$, with $u=0$ and $\sigma = \sqrt{6/\pi^2}$ for $\mu(k)$, is,

$$\sum_{k=1}^{n} \mu(k) = M(n) \leq \sigma K_{\alpha/2}\sqrt{n} = \sqrt{6/\pi^2}K_{\alpha/2}\sqrt{n} \tag{26}$$

The inequality (26) holds with a probability of $1-\alpha$.

In fact, the inequality of Mertens type is only a special case of (26) with a fixed probability of $1-\alpha$.

4 Discussion

4.1 The calculation of $\mu(n)$ with the recursive relation

In theory, we can calculate $\mu(n)$ for any large n recursively with the recursive relation obtained in section 2. However, in order to calculate $\mu(n)$, we need to know $\mu(1), \mu(2), \mu(3), \ldots\ldots \mu(n-1)$. Usually $\mu(1), \mu(2), \mu(3), \ldots\ldots \mu(n-1)$ are stored in an array which demands a much larger amount of computer memory if n is large. In this paper, we only calculate the values of $\mu(n)$ from $\mu(1)$ to $\mu(2\times 10^7)$ because of the memory limitation of our desktop computer and computing time. To obtain more numerical results of $\mu(n)$ with large n, both the faster and/or optimization algorithm for the recursive relation here and better hardware platform are required. It is a probable way by which the calculations are divided into blocks and are computed with GPU, or quantum computer will be used in the future.

4.2 The independent randomness of $\mu(n)$

Based on the numerical consistency between empirical statistical quantities for only 2×10^7 $\mu(n)$ and those from number theory (eg., N(Statistical)) and N(Theorical), p_e and p_t), we use classic statistical method to study $\mu(n)$ regardless of the strict validity of the independent randomness of $\mu(n)$. In the respect of the independent randomness of $\mu(n)$, there are some discussions (eg., Sarnak, 2012). Although $\mu(n)$ is deterministic from the recursive relation in section 2, it is visually random and independent. Figure 4 shows that the variation of $\sum_{k=1}^{n} \mu(k)/n$ with n when $n = 500000$. It can be observed that $\mu(n)$ has some properties of the independent random variable.

The above can be viewed restrictly from another definition of $\mu(n)$,

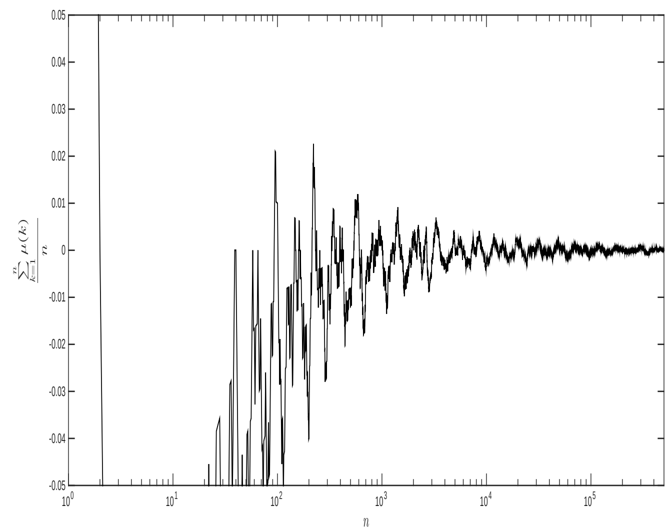

Fig. 4: Variation of $\frac{\sum_{k=1}^{n} \mu(k)}{n}$ with n when $n = 500000$.

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is non-squarefree} \\ (-1)^{\omega(n)} & \text{if } n \text{ is squarefree} \end{cases} \tag{27}$$

where $\omega(n)$ is the number of distinct prime factors.

According to the Erdös-Kac Theorem, $\omega(n)$ is independent and random when n is large, so may be $\mu(n)$ for $\mu(n) = (-1)^{\omega(n)} = \exp[iz\omega(n)]|_{z=\pi}$ when n is squarefree and if we take n in random order.

Furthermore, $\mu(n)$ is either $(-1)^{\omega(n)}$ or 0 because of its definition of (27), so we have $M(n) = M(n') = \sum_{k=1}^{n'} \mu(k) = \sum_{k=1}^{n'} (-1)^{\omega(k)}$ (n' is squarefree). The randomness of $M(n')$ should be stronger than $\mu(n)$ if we take n' in random order. Numerical results of Good and Churchhouse (1968) show $M(n)$ in blocks of length N has asymptotically a normal distribution with mean zero and variance of $6N/\pi^2$ (where N is large). These numerical results can be rechecked as follows. From the rules in mathematical statistics, we know that the observed values of a discrete random variable X ($X = x_1, x_2, \ldots, x_n$) lie in the following interval with probability $p > 1 - \alpha$, for real number α with $0 < \alpha < 1$,

$$\left[\overline{X} - \frac{\Delta X}{\sqrt{\alpha}}, \overline{X} + \frac{\Delta X}{\sqrt{\alpha}}\right] \tag{28}$$

where $\overline{X} = \sum_{k=1}^{n} x_k p_k$, $(\Delta X)^2 = \sum_{k=1}^{n} (x_k - \overline{X})^2 p_k$, $p_k := P(x = x_k)$.

Obviously, for $M(n) = M(n') = \sum_{k=1}^{n'} (-1)^{\omega(k)}$, $p_k = 6/\pi^2$ according to subsection (3.1), and further one have,

$$\overline{X} = \frac{6}{\pi^2} \sum_{k=1}^{n'_{\text{even}}} 1 + \frac{6}{\pi^2} \sum_{l=1}^{n'_{\text{odd}}} -1 = 0$$

$$(\Delta X)^2 = \frac{6}{\pi^2} \sum_{k=1}^{n'_{\text{even}}} 1 + \frac{6}{\pi^2} \sum_{l=1}^{n'_{\text{odd}}} 1 = \frac{6n'}{\pi^2}$$

where n'_{even} is squarefree with even number of distinct prime factors, n'_{odd} with odd number of distinct prime factors.

Therefore, from (28), we can obtain an upper bound for $M(n)$ similar to (26) as follows, with probability $p > 1 - \alpha$,

$$\sum_{k=1}^{n} \mu(k) = M(n) = \sum_{k=1}^{n'} (-1)^{\omega(k)} \leq \frac{\sqrt{6/\pi^2}}{\sqrt{\alpha}} \sqrt{n'} \leq \frac{\sqrt{6/\pi^2}}{\sqrt{\alpha}} \sqrt{n} \tag{29}$$

If α takes $\frac{6}{\pi^2}$, then $M(n) \leq \sqrt{n}$ with a probability $p > 1 - 6/\pi^2 \approx 0.3920$, which means that the Mertens conjecture is not true.

On the other hand, we can check whether $\mu(n)$ is periodic or not by estimating its power spectral density (PSD). We calculate the PSD for $\mu(n)$ series from $\mu(1)$ to $\mu(2 \times 10^7)$ by taking n as time. The results are shown in Figure 5. It can be found that $\mu(n)$ has no apparent periodicity because the PSD of $\mu(n)$ have no distinguished peak(s).

Although the independent randomness of $\mu(n)$ is an unsolved problem so far, we can analyze $\mu(n)$ by way of statistics because $\mu(n)$ has a complicated and non-periodic distribution, as those statistical approaches applied to chaos.

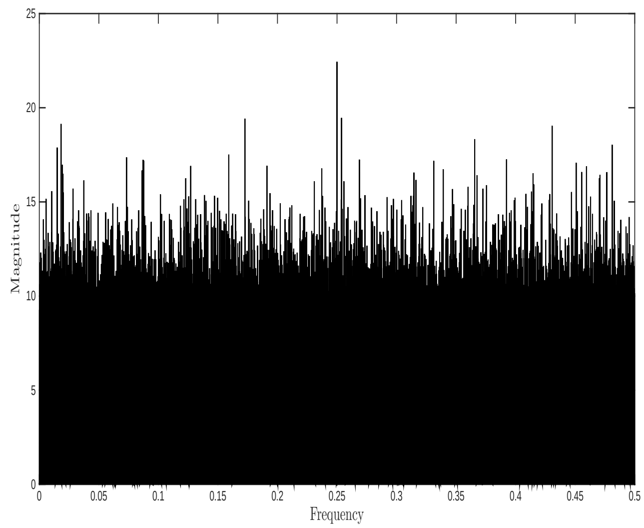

Fig. 5: The power spectral density (PSD) for $\mu(n)$ series from $\mu(1)$ to $\mu(2 \times 10^7)$.

4.3 The probability of $\sum_{k=1}^{n} |\mu(k)| > Cn$

Similarly, we can calculate the probability of $\sum_{k=1}^{n} |\mu(k)| > Cn$. The distribution law for $|\mu(n)|$ can be obtained as shown in Table 2.

Table 2: The distribution rule for $|\mu(k)|$

$\|\mu(k)\|$	0	1
p_{t_k}	$1 - 6/\pi^2$	$6/\pi^2$

And $E(|\mu(k)|) = 6/\pi^2$ and $D(|\mu(k)|) = 6/\pi^2(1 - 6/\pi^2)$.

According to central limit theorem, for any x, we have,

$$\lim_{n\to\infty} P\left\{\frac{\sum\limits_{k=1}^{n}|\mu(k)|-E(\sum\limits_{k=1}^{n}|\mu(k)|)}{\sqrt{D(\sum\limits_{k=1}^{n}|\mu(k)|)}}\le x\right\} \begin{aligned} &= \lim_{n\to\infty} P\left\{\frac{\sum\limits_{k=1}^{n}|\mu(k)|-6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}}\le x\right\} \\ &= \int\limits_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp(-\frac{t^2}{2})\mathrm{d}t \\ &= \Phi(x) \end{aligned} \tag{30}$$

And (30) means,

$$\frac{\sum\limits_{k=1}^{n}|\mu(k)|-6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}}\sim N(0,1) \tag{31}$$

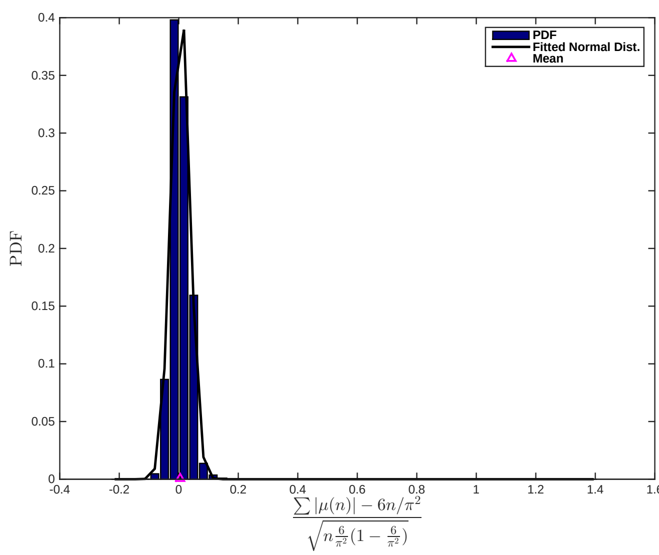

Fig. 6: The probability density function for $\frac{\sum_{k=1}^{n}|\mu(k)|-6n/\pi^2}{\sqrt{6n(1-6/\pi^2)/\pi^2}}$ when $n = 500000$.

Figure 6 shows the probability density function for $\frac{\sum_{k=1}^{n}|\mu(k)|-6n/\pi^2}{\sqrt{6n(1-6/\pi^2)/\pi^2}}$ when $n = 500000$. It can be seen that the distribution of (31) is reasonable.

With equality (30), the probability of $\sum\limits_{k=1}^{n} |\mu(k)| > Cn$ (where C is a constant) can be obtained. Clearly,

$$
\begin{aligned}
P\left\{\sum_{k=1}^{n} |\mu(k)| > Cn\right\} &= P\left\{\frac{\sum\limits_{k=1}^{n}|\mu(k)|-6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}} > \frac{Cn-6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}}\right\} \\
&= 1-P\left\{\frac{\sum\limits_{k=1}^{n}|\mu(k)|-6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}} \leq \frac{Cn-6n/\pi^2}{\sqrt{n6/\pi^2(1-6/\pi^2)}}\right\} \\
&= 1-\Phi\left\{\frac{(C-6/\pi^2)n}{\sqrt{n6/\pi^2(1-6/\pi^2)}}\right\}
\end{aligned}
\tag{32}
$$

With (32), when n is large

$$
P\left\{\sum_{k=1}^{n} |\mu(k)| > Cn\right\} = \begin{cases} 0 & C > 6/\pi^2 \\ 1/2 & C = 6/\pi^2 \\ 1 & C < 6/\pi^2 \end{cases}
\tag{33}
$$

5 Conclusions

Based on the results and discussion above, some conclusions can be drawn as follows,

(1) An elementary recursive relation for Möbius function is obtained by two ways. One is from Möbius transform, and the other is from Redheffer Matrix. With this recursive relation, $\mu(n)$ can be calculated without directly knowing the factorization of n, in which the most complex operation is only the Mod.

(2) With this relation, $\mu(1) \sim \mu(2\times 10^7)$ are calculated recursively. Based on these 2×10^7 samples, we calculate the frequencies and empirical probabilities for $\mu(n)$ of taking $-1, 0, 1$, so does for $|\mu| = 1$. Then, we compare them with those in number theory. It can be found these two kinds of frequencies and probabilities are numerically consistent.

(3) Based on these numerical results, we take $\mu(n)$ as an independent random sequence although it has a deterministic recursive rule. The expectation and variance of the $\mu(k)$ are $E(\mu(k)) = 0$ and $D(\mu(k)) = 6/\pi^2$, respectively.

(4) We show that the Mertens conjecture, even any conjecture of the Mertens type, is false in a probability sense, and present an upper bound for cumulative sums of $\mu(n)$

as $\sum_{k=1}^{n}\mu(k) \leqslant \sqrt{6/\pi^2}K_{\alpha/2}\sqrt{n}$ with a probability of $1-\alpha$.

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Appendix

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$

N	N_{-1}	N_{-1}^{T}	p_e (0.3039635509)	N_1	N_1^{T}	p_e(0.3039635509)
100000	30421	30396.4	0.3042100000	30373	30396.4 0	0.3037300000
200000	60791	60792.7	0.3039550000	60790	60792.7 0	0.3039500000
300000	91079	91189.1	0.3035966667	91299	91189.1 0	0.3043300000
400000	121577	121585.4	0.3039425000	121588	121585.4	0.3039700000
500000	151982	151981.8	0.3039640000	151976	151981.8	0.3039520000
600000	182492	182378.1	0.3041533333	182262	182378.1	0.3037700000
700000	212666	212774.5	0.3038085714	212892	212774.5	0.3041314286
800000	243181	243170.8	0.3039762500	243161	243170.8	0.3039512500
900000	273678	273567.2	0.3040866667	273453	273567.2	0.3038366667
1000000	303857	303963.6	0.3038570000	304069	303963.6	0.3040690000
1100000	334263	334359.9	0.3038754545	334464	334359.9	0.3040581818
1200000	364832	364756.3	0.3040266667	364677	364756.3	0.3038975000
1300000	395192	395152.6	0.3039938462	395111	395152.6	0.3039315385
1400000	425669	425549.0	0.3040492857	425422	425549.0	0.3038728571
1500000	456092	455945.3	0.3040613333	455799	455945.3	0.3038660000
1600000	486262	486341.7	0.3039137500	486430	486341.7	0.3040187500
1700000	516688	516738.0	0.3039341176	516792	516738.0	0.3039952941
1800000	546936	547134.4	0.3038533333	547340	547134.4	0.3040777778
1900000	577533	577530.7	0.3039647368	577544	577530.7	0.3039705263
2000000	608062	607927.1	0.3040310000	607815	607927.1	0.3039075000
2100000	638508	638323.5	0.3040514286	638142	638323.5	0.3038771429
2200000	668846	668719.8	0.3040209091	668600	668719.8	0.3039090909
2300000	699203	699116.2	0.3040013043	699016	699116.2	0.3039200000
2400000	729384	729512.5	0.3039100000	729638	729512.5	0.3040158333
2500000	759726	759908.9	0.3038904000	760088	759908.9	0.3040352000
2600000	790230	790305.2	0.3039346154	790377	790305.2	0.3039911538
2700000	820674	820701.6	0.3039533333	820722	820701.6	0.3039711111
2800000	850937	851097.9	0.3039060714	851251	851097.9	0.3040182143

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$ (continued)

N	N_{-1}	N_{-1}^{T}	p_e (0.3039635509)	N_1	N_1^{T}	p_e(0.3039635509)
2900000	881525	881494.3	0.3039741379	881457	881494.3	0.3039506897
3000000	911833	911890.7	0.3039443333	911940	911890.7	0.3039800000
3100000	942195	942287.0	0.3039338710	942372	942287.0	0.3039909677
3200000	972925	972683.4	0.3040390625	972438	972683.4	0.3038868750
3300000	1003355	1003079.7	0.3040469697	1002803	1003079.7	0.3038796970
3400000	1033623	1033476.1	0.3040067647	1033331	1033476.1	0.3039208824
3500000	1063947	1063872.4	0.3039848571	1063809	1063872.4	0.3039454286
3600000	1094166	1094268.8	0.3039350000	1094378	1094268.8	0.3039938889
3700000	1124662	1124665.1	0.3039627027	1124661	1124665.1	0.3039624324
3800000	1154989	1155061.5	0.3039444737	1155144	1155061.5	0.3039852632
3900000	1185390	1185457.8	0.3039461538	1185540	1185457.8	0.3039846154
4000000	1215772	1215854.2	0.3039430000	1215964	1215854.2	0.3039910000
4100000	1246259	1246250.6	0.3039656098	1246258	1246250.6	0.3039653659
4200000	1276499	1276646.9	0.3039283333	1276799	1276646.9	0.3039997619
4300000	1306851	1307043.3	0.3039188372	1307252	1307043.3	0.3040120930
4400000	1337169	1337439.6	0.3039020455	1337720	1337439.6	0.3040272727
4500000	1367757	1367836.0	0.3039460000	1367930	1367836.0	0.3039844444
4600000	1398114	1398232.3	0.3039378261	1398354	1398232.3	0.3039900000
4700000	1428655	1428628.7	0.3039691489	1428604	1428628.7	0.3039582979
4800000	1459035	1459025.0	0.3039656250	1459025	1459025.0	0.3039635417
4900000	1489629	1489421.4	0.3040059184	1489223	1489421.4	0.3039230612
5000000	1520171	1519817.8	0.3040342000	1519462	1519817.8	0.3038924000
5100000	1550492	1550214.1	0.3040180392	1549940	1550214.1	0.3039098039
5200000	1580813	1580610.5	0.3040025000	1580407	1580610.5	0.3039244231
5300000	1611343	1611006.8	0.3040269811	1610658	1611006.8	0.3038977358
5400000	1641700	1641403.2	0.3040185185	1641101	1641403.2	0.3039075926
5500000	1672026	1671799.5	0.3040047273	1671570	1671799.5	0.3039218182
5600000	1702290	1702195.9	0.3039803571	1702102	1702195.9	0.3039467857
5700000	1732409	1732592.2	0.3039314035	1732772	1732592.2	0.3039950877

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$ (continued)

N	N_{-1}	N_{-1}^{T}	p_e (0.3039635509)	N_1	N_1^{T}	p_e(0.3039635509)
5800000	1762745	1762988.6	0.3039215517	1763227	1762988.6	0.3040046552
5900000	1793191	1793385.0	0.3039306780	1793578	1793385.0	0.3039962712
6000000	1823629	1823781.3	0.3039381667	1823934	1823781.3	0.3039890000
6100000	1854020	1854177.7	0.3039377049	1854339	1854177.7	0.3039900000
6200000	1884442	1884574.0	0.3039422581	1884702	1884574.0	0.3039841935
6300000	1914862	1914970.4	0.3039463492	1915069	1914970.4	0.3039792063
6400000	1945017	1945366.7	0.3039089062	1945722	1945366.7	0.3040190625
6500000	1975304	1975763.1	0.3038929231	1976224	1975763.1	0.3040344615
6600000	2005774	2006159.4	0.3039051515	2006549	2006159.4	0.3040225758
6700000	2036272	2036555.8	0.3039211940	2036848	2036555.8	0.3040071642
6800000	2066700	2066952.1	0.3039264706	2067230	2066952.1	0.3040044118
6900000	2097201	2097348.5	0.3039421739	2097508	2097348.5	0.3039866667
7000000	2127845	2127744.9	0.3039778571	2127660	2127744.9	0.3039514286
7100000	2158603	2158141.2	0.3040285915	2157693	2158141.2	0.3039004225
7200000	2188842	2188537.6	0.3040058333	2188246	2188537.6	0.3039230556
7300000	2219259	2218933.9	0.3040080822	2218614	2218933.9	0.3039197260
7400000	2249571	2249330.3	0.3039960811	2249086	2249330.3	0.3039305405
7500000	2279799	2279726.6	0.3039732000	2279642	2279726.6	0.3039522667
7600000	2310208	2310123.0	0.3039747368	2310025	2310123.0	0.3039506579
7700000	2340640	2340519.3	0.3039792208	2340390	2340519.3	0.3039467532
7800000	2370882	2370915.7	0.3039592308	2370941	2370915.7	0.3039667949
7900000	2401399	2401312.1	0.3039745570	2401226	2401312.1	0.3039526582
8000000	2431794	2431708.4	0.3039742500	2431614	2431708.4	0.3039517500
8100000	2462023	2462104.8	0.3039534568	2462176	2462104.8	0.3039723457
8200000	2492372	2492501.1	0.3039478049	2492617	2492501.1	0.3039776829
8300000	2522888	2522897.5	0.3039624096	2522895	2522897.5	0.3039632530
8400000	2553324	2553293.8	0.3039671429	2553258	2553293.8	0.3039592857
8500000	2583606	2583690.2	0.3039536471	2583767	2583690.2	0.3039725882
8600000	2614145	2614086.5	0.3039703488	2614013	2614086.5	0.3039550000

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$ (continued)

N	N_{-1}	N_{-1}^{T}	p_e (0.3039635509)	N_1	N_1^{T}	p_e(0.3039635509)
8700000	2644780	2644482.9	0.3039977011	2644179	2644482.9	0.3039286207
8800000	2675302	2674879.2	0.3040115909	2674436	2674879.2	0.3039131818
8900000	2705498	2705275.6	0.3039885393	2705055	2705275.6	0.3039387640
9000000	2735835	2735672.0	0.3039816667	2735510	2735672.0	0.3039455556
9100000	2766254	2766068.3	0.3039839560	2765878	2766068.3	0.3039426374
9200000	2796643	2796464.7	0.3039829348	2796283	2796464.7	0.3039438043
9300000	2826988	2826861.0	0.3039772043	2826739	2826861.0	0.3039504301
9400000	2857344	2857257.4	0.3039727660	2857182	2857257.4	0.3039555319
9500000	2887597	2887653.7	0.3039575789	2887738	2887653.7	0.3039724211
9600000	2918022	2918050.1	0.3039606250	2918090	2918050.1	0.3039677083
9700000	2948303	2948446.4	0.3039487629	2948619	2948446.4	0.3039813402
9800000	2978545	2978842.8	0.3039331633	2979162	2978842.8	0.3039961224
9900000	3008869	3009239.2	0.3039261616	3009632	3009239.2	0.3040032323
10000000	3039114	3039635.5	0.3039114000	3040179	3039635.5	0.3040179000
10100000	3069664	3070031.9	0.3039271287	3070433	3070031.9	0.3040032673
10200000	3099868	3100428.2	0.3039086275	3101022	3100428.2	0.3040217647
10300000	3130347	3130824.6	0.3039171845	3131324	3130824.6	0.3040120388
10400000	3160726	3161220.9	0.3039159615	3161732	3161220.9	0.3040126923
10500000	3191184	3191617.3	0.3039222857	3192071	3191617.3	0.3040067619
10600000	3221657	3222013.6	0.3039299057	3222379	3222013.6	0.3039980189
10700000	3252192	3252410.0	0.3039431776	3252641	3252410.0	0.3039851402
10800000	3282557	3282806.4	0.3039404630	3283075	3282806.4	0.3039884259
10900000	3312997	3313202.7	0.3039446789	3313433	3313202.7	0.3039846789
11000000	3343506	3343599.1	0.3039550909	3343717	3343599.1	0.3039742727
11100000	3374021	3373995.4	0.3039658559	3373983	3373995.4	0.3039624324
11200000	3404404	3404391.8	0.3039646429	3404391	3404391.8	0.3039634821
11300000	3434866	3434788.1	0.3039704425	3434730	3434788.1	0.3039584071
11400000	3465178	3465184.5	0.3039629825	3465217	3465184.5	0.3039664035
11500000	3495624	3495580.8	0.3039673043	3495548	3495580.8	0.3039606957

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$ (continued)

N	N_{-1}	N_{-1}^{T}	p_e (0.3039635509)	N_1	N_1^{T}	p_e(0.3039635509)
11600000	3526196	3525977.2	0.3039824138	3525776	3525977.2	0.3039462069
11700000	3556658	3556373.5	0.3039878632	3556101	3556373.5	0.3039402564
11800000	3587053	3586769.9	0.3039875424	3586510	3586769.9	0.3039415254
11900000	3617356	3617166.3	0.3039794958	3617005	3617166.3	0.3039500000
12000000	3647916	3647562.6	0.3039930000	3647225	3647562.6	0.3039354167
12100000	3678241	3677959.0	0.3039868595	3677700	3677959.0	0.3039421488
12200000	3708699	3708355.3	0.3039917213	3708029	3708355.3	0.3039368033
12300000	3738934	3738751.7	0.3039783740	3738582	3738751.7	0.3039497561
12400000	3769650	3769148.0	0.3040040323	3768656	3769148.0	0.3039238710
12500000	3800046	3799544.4	0.3040036800	3799053	3799544.4	0.3039242400
12600000	3830268	3829940.7	0.3039895238	3829614	3829940.7	0.3039376190
12700000	3860688	3860337.1	0.3039911811	3859996	3860337.1	0.3039366929
12800000	3891324	3890733.5	0.3040096875	3890154	3890733.5	0.3039182812
12900000	3921804	3921129.8	0.3040158140	3920470	3921129.8	0.3039124031
13000000	3951997	3951526.2	0.3039997692	3951078	3951526.2	0.3039290769
13100000	3982307	3981922.5	0.3039929008	3981550	3981922.5	0.3039351145
13200000	4012612	4012318.9	0.3039857576	4012029	4012318.9	0.3039415909
13300000	4042839	4042715.2	0.3039728571	4042586	4042715.2	0.3039538346
13400000	4073252	4073111.6	0.3039740299	4072973	4073111.6	0.3039532090
13500000	4103418	4103507.9	0.3039568889	4103604	4103507.9	0.3039706667
13600000	4133875	4133904.3	0.3039613971	4133949	4133904.3	0.3039668382
13700000	4164371	4164300.6	0.3039686861	4164242	4164300.6	0.3039592701
13800000	4194712	4194697.0	0.3039646377	4194686	4194697.0	0.3039627536
13900000	4225082	4225093.4	0.3039627338	4225116	4225093.4	0.3039651799
14000000	4255460	4255489.7	0.3039614286	4255515	4255489.7	0.3039653571
14100000	4285537	4285886.1	0.3039387943	4286245	4285886.1	0.3039890071
14200000	4315864	4316282.4	0.3039340845	4316717	4316282.4	0.3039941549
14300000	4346397	4346678.8	0.3039438462	4346976	4346678.8	0.3039843357
14400000	4377007	4377075.1	0.3039588194	4377145	4377075.1	0.3039684028

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$ (continued)

N	N_{-1}	N_{-1}^{T}	p_e (0.3039635509)	N_1	N_1^{T}	p_e(0.3039635509)
14500000	4407260	4407471.5	0.3039489655	4407689	4407471.5	0.3039785517
14600000	4437718	4437867.8	0.3039532877	4438026	4437867.8	0.3039743836
14700000	4467953	4468264.2	0.3039423810	4468578	4468264.2	0.3039848980
14800000	4498439	4498660.6	0.3039485811	4498896	4498660.6	0.3039794595
14900000	4528734	4529056.9	0.3039418792	4529375	4529056.9	0.3039848993
15000000	4559090	4559453.3	0.3039393333	4559806	4559453.3	0.3039870667
15100000	4589534	4589849.6	0.3039426490	4590159	4589849.6	0.3039840397
15200000	4619931	4620246.0	0.3039428289	4620559	4620246.0	0.3039841447
15300000	4650361	4650642.3	0.3039451634	4650909	4650642.3	0.3039809804
15400000	4680591	4681038.7	0.3039344805	4681480	4681038.7	0.3039922078
15500000	4711009	4711435.0	0.3039360645	4711840	4711435.0	0.3039896774
15600000	4741664	4741831.4	0.3039528205	4741970	4741831.4	0.3039724359
15700000	4771947	4772227.7	0.3039456688	4772486	4772227.7	0.3039800000
15800000	4802171	4802624.1	0.3039348734	4803057	4802624.1	0.3039909494
15900000	4832617	4833020.5	0.3039381761	4833397	4833020.5	0.3039872327
16000000	4863075	4863416.8	0.3039421875	4863726	4863416.8	0.3039828750
16100000	4893399	4893813.2	0.3039378261	4894201	4893813.2	0.3039876398
16200000	4923789	4924209.5	0.3039375926	4924609	4924209.5	0.3039882099
16300000	4954368	4954605.9	0.3039489571	4954833	4954605.9	0.3039774847
16400000	4984493	4985002.2	0.3039325000	4985486	4985002.2	0.3039930488
16500000	5014966	5015398.6	0.3039373333	5015812	5015398.6	0.3039886061
16600000	5045514	5045794.9	0.3039466265	5046060	5045794.9	0.3039795181
16700000	5075937	5076191.3	0.3039483234	5076424	5076191.3	0.3039774850
16800000	5106464	5106587.7	0.3039561905	5106696	5106587.7	0.3039700000
16900000	5136984	5136984.0	0.3039635503	5136960	5136984.0	0.3039621302
17000000	5167548	5167380.4	0.3039734118	5167198	5167380.4	0.3039528235
17100000	5197950	5197776.7	0.3039736842	5197569	5197776.7	0.3039514035
17200000	5228073	5228173.1	0.3039577326	5228230	5228173.1	0.3039668605
17300000	5258509	5258569.4	0.3039600578	5258594	5258569.4	0.3039649711

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$ (continued)

N	N_{-1}	N^{T}_{-1}	p_e (0.3039635509)	N_1	N^{T}_1	p_e(0.3039635509)
17400000	5289011	5288965.8	0.3039661494	5288905	5288965.8	0.3039600575
17500000	5319622	5319362.1	0.3039784000	5319080	5319362.1	0.3039474286
17600000	5350004	5349758.5	0.3039775000	5349492	5349758.5	0.3039484091
17700000	5380404	5380154.9	0.3039776271	5379884	5380154.9	0.3039482486
17800000	5410798	5410551.2	0.3039774157	5410288	5410551.2	0.3039487640
17900000	5441131	5440947.6	0.3039737989	5440745	5440947.6	0.3039522346
18000000	5471651	5471343.9	0.3039806111	5471007	5471343.9	0.3039448333
18100000	5501933	5501740.3	0.3039741989	5501523	5501740.3	0.3039515470
18200000	5532239	5532136.6	0.3039691758	5532002	5532136.6	0.3039561538
18300000	5562800	5562533.0	0.3039781421	5562238	5562533.0	0.3039474317
18400000	5593117	5592929.3	0.3039737500	5592716	5592929.3	0.3039519565
18500000	5623395	5623325.7	0.3039672973	5623233	5623325.7	0.3039585405
18600000	5653805	5653722.0	0.3039680108	5653622	5653722.0	0.3039581720
18700000	5684167	5684118.4	0.3039661497	5684061	5684118.4	0.3039604813
18800000	5714614	5714514.8	0.3039688298	5714406	5714514.8	0.3039577660
18900000	5745032	5744911.1	0.3039699471	5744799	5744911.1	0.3039576190
19000000	5775455	5775307.5	0.3039713158	5775166	5775307.5	0.3039561053
19100000	5805894	5805703.8	0.3039735079	5805527	5805703.8	0.3039542932
19200000	5836329	5836100.2	0.3039754687	5835874	5836100.2	0.3039517708
19300000	5866712	5866496.5	0.3039747150	5866291	5866496.5	0.3039529016
19400000	5897278	5896892.9	0.3039834021	5896534	5896892.9	0.3039450515
19500000	5927580	5927289.2	0.3039784615	5927013	5927289.2	0.3039493846
19600000	5957970	5957685.6	0.3039780612	5957416	5957685.6	0.3039497959
19700000	5988317	5988082.0	0.3039754822	5987859	5988082.0	0.3039522335
19800000	6018758	6018478.3	0.3039776768	6018226	6018478.3	0.3039508081
19900000	6049156	6048874.7	0.3039776884	6048626	6048874.7	0.3039510553
20000000	6079759	6079271.0	0.3039879500	6078820	6079271.0	0.3039410000

Table 3: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = -1/1$ (continued)

N	N_{-1}	N_{-1}^{T}	p_e (0.3039635509)	N_1	N_1^{T}	p_e(0.3039635509)

Note: N: Length of the block from $\mu(1)$ to $\mu(N)$; N_{-1}: Frequency of $\mu(n) = -1$; $N_{-1}^{\mathrm{T}}(= N \times p_t)$: Frequency of $\mu(n) = -1$ from number theory; $p_e(= \frac{N_{-1}}{N})$: Empirical probability; The number in bracket is theoretical probability from number theory p_t. N_1 and N_1^{T} are those for $\mu(n) = 1$.

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$

N	N_0	N_0^{T}	p_e (0.3920728981)	$N_{\|\mu(n)\|=1}$	$N^{\mathrm{T}}_{\|\mu(n)\|=1}$	p_e(0.6079271019)
100000	39206	39207.3	0.3920600000	60794	60792.7	0.6079400000
200000	78419	78414.6	0.3920950000	121581	121585.4	0.6079050000
300000	117622	117621.9	0.3920733333	182378	182378.1	0.6079266667
400000	156835	156829.2	0.3920875000	243165	243170.8	0.6079125000
500000	196042	196036.4	0.3920840000	303958	303963.6	0.6079160000
600000	235246	235243.7	0.3920766667	364754	364756.3	0.6079233333
700000	274442	274451.0	0.3920600000	425558	425549.0	0.6079400000
800000	313658	313658.3	0.3920725000	486342	486341.7	0.6079275000
900000	352869	352865.6	0.3920766667	547131	547134.4	0.6079233333
1000000	392074	392072.9	0.3920740000	607926	607927.1	0.6079260000
1100000	431273	431280.2	0.3920663636	668727	668719.8	0.6079336364
1200000	470491	470487.5	0.3920758333	729509	729512.5	0.6079241667
1300000	509697	509694.8	0.3920746154	790303	790305.2	0.6079253846
1400000	548909	548902.1	0.3920778571	851091	851097.9	0.6079221429
1500000	588109	588109.3	0.3920726667	911891	911890.7	0.6079273333
1600000	627308	627316.6	0.3920675000	972692	972683.4	0.6079325000
1700000	666520	666523.9	0.3920705882	1033480	1033476.1	0.6079294118
1800000	705724	705731.2	0.3920688889	1094276	1094268.8	0.6079311111
1900000	744923	744938.5	0.3920647368	1155077	1155061.5	0.6079352632
2000000	784123	784145.8	0.3920615000	1215877	1215854.2	0.6079385000
2100000	823350	823353.1	0.3920714286	1276650	1276646.9	0.6079285714
2200000	862554	862560.4	0.3920700000	1337446	1337439.6	0.6079300000
2300000	901781	901767.7	0.3920786957	1398219	1398232.3	0.6079213043
2400000	940978	940975.0	0.3920741667	1459022	1459025.0	0.6079258333
2500000	980186	980182.2	0.3920744000	1519814	1519817.8	0.6079256000
2600000	1019393	1019389.5	0.3920742308	1580607	1580610.5	0.6079257692
2700000	1058604	1058596.8	0.3920755556	1641396	1641403.2	0.6079244444
2800000	1097812	1097804.1	0.3920757143	1702188	1702195.9	0.6079242857
2900000	1137018	1137011.4	0.3920751724	1762982	1762988.6	0.6079248276

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$ (continued)

N	N_0	N_0^{T}	p_e (0.3920728981)	$N_{\|\mu(n)\|=1}$	$N_{\|\mu(n)\|=1}^{\mathrm{T}}$	p_e(0.6079271019)
3000000	1176227	1176218.7	0.3920756667	1823773	1823781.3	0.6079243333
3100000	1215433	1215426.0	0.3920751613	1884567	1884574.0	0.6079248387
3200000	1254637	1254633.3	0.3920740625	1945363	1945366.7	0.6079259375
3300000	1293842	1293840.6	0.3920733333	2006158	2006159.4	0.6079266667
3400000	1333046	1333047.9	0.3920723529	2066954	2066952.1	0.6079276471
3500000	1372244	1372255.1	0.3920697143	2127756	2127744.9	0.6079302857
3600000	1411456	1411462.4	0.3920711111	2188544	2188537.6	0.6079288889
3700000	1450677	1450669.7	0.3920748649	2249323	2249330.3	0.6079251351
3800000	1489867	1489877.0	0.3920702632	2310133	2310123.0	0.6079297368
3900000	1529070	1529084.3	0.3920692308	2370930	2370915.7	0.6079307692
4000000	1568264	1568291.6	0.3920660000	2431736	2431708.4	0.6079340000
4100000	1607483	1607498.9	0.3920690244	2492517	2492501.1	0.6079309756
4200000	1646702	1646706.2	0.3920719048	2553298	2553293.8	0.6079280952
4300000	1685897	1685913.5	0.3920690698	2614103	2614086.5	0.6079309302
4400000	1725111	1725120.8	0.3920706818	2674889	2674879.2	0.6079293182
4500000	1764313	1764328.0	0.3920695556	2735687	2735672.0	0.6079304444
4600000	1803532	1803535.3	0.3920721739	2796468	2796464.7	0.6079278261
4700000	1842741	1842742.6	0.3920725532	2857259	2857257.4	0.6079274468
4800000	1881940	1881949.9	0.3920708333	2918060	2918050.1	0.6079291667
4900000	1921148	1921157.2	0.3920710204	2978852	2978842.8	0.6079289796
5000000	1960367	1960364.5	0.3920734000	3039633	3039635.5	0.6079266000
5100000	1999568	1999571.8	0.3920721569	3100432	3100428.2	0.6079278431
5200000	2038780	2038779.1	0.3920730769	3161220	3161220.9	0.6079269231
5300000	2077999	2077986.4	0.3920752830	3222001	3222013.6	0.6079247170
5400000	2117199	2117193.6	0.3920738889	3282801	3282806.4	0.6079261111
5500000	2156404	2156400.9	0.3920734545	3343596	3343599.1	0.6079265455
5600000	2195608	2195608.2	0.3920728571	3404392	3404391.8	0.6079271429
5700000	2234819	2234815.5	0.3920735088	3465181	3465184.5	0.6079264912
5800000	2274028	2274022.8	0.3920737931	3525972	3525977.2	0.6079262069

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$ (continued)

N	N_0	N_0^{T}	p_e (0.3920728981)	$N_{\|\mu(n)\|=1}$	$N^{\mathrm{T}}_{\|\mu(n)\|=1}$	p_e(0.6079271019)
5900000	2313231	2313230.1	0.3920730508	3586769	3586769.9	0.6079269492
6000000	2352437	2352437.4	0.3920728333	3647563	3647562.6	0.6079271667
6100000	2391641	2391644.7	0.3920722951	3708359	3708355.3	0.6079277049
6200000	2430856	2430852.0	0.3920735484	3769144	3769148.0	0.6079264516
6300000	2470069	2470059.3	0.3920744444	3829931	3829940.7	0.6079255556
6400000	2509261	2509266.5	0.3920720313	3890739	3890733.5	0.6079279687
6500000	2548472	2548473.8	0.3920726154	3951528	3951526.2	0.6079273846
6600000	2587677	2587681.1	0.3920722727	4012323	4012318.9	0.6079277273
6700000	2626880	2626888.4	0.3920716418	4073120	4073111.6	0.6079283582
6800000	2666070	2666095.7	0.3920691176	4133930	4133904.3	0.6079308824
6900000	2705291	2705303.0	0.3920711594	4194709	4194697.0	0.6079288406
7000000	2744495	2744510.3	0.3920707143	4255505	4255489.7	0.6079292857
7100000	2783704	2783717.6	0.3920709859	4316296	4316282.4	0.6079290141
7200000	2822912	2822924.9	0.3920711111	4377088	4377075.1	0.6079288889
7300000	2862127	2862132.2	0.3920721918	4437873	4437867.8	0.6079278082
7400000	2901343	2901339.4	0.3920733784	4498657	4498660.6	0.6079266216
7500000	2940559	2940546.7	0.3920745333	4559441	4559453.3	0.6079254667
7600000	2979767	2979754.0	0.3920746053	4620233	4620246.0	0.6079253947
7700000	3018970	3018961.3	0.3920740260	4681030	4681038.7	0.6079259740
7800000	3058177	3058168.6	0.3920739744	4741823	4741831.4	0.6079260256
7900000	3097375	3097375.9	0.3920727848	4802625	4802624.1	0.6079272152
8000000	3136592	3136583.2	0.3920740000	4863408	4863416.8	0.6079260000
8100000	3175801	3175790.5	0.3920741975	4924199	4924209.5	0.6079258025
8200000	3215011	3214997.8	0.3920745122	4984989	4985002.2	0.6079254878
8300000	3254217	3254205.1	0.3920743373	5045783	5045794.9	0.6079256627
8400000	3293418	3293412.3	0.3920735714	5106582	5106587.7	0.6079264286
8500000	3332627	3332619.6	0.3920737647	5167373	5167380.4	0.6079262353
8600000	3371842	3371826.9	0.3920746512	5228158	5228173.1	0.6079253488
8700000	3411041	3411034.2	0.3920736782	5288959	5288965.8	0.6079263218

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$ (continued)

N	N_0	N_0^{T}	p_e (0.3920728981)	$N_{\lvert\mu(n)\rvert=1}$	$N^{\mathrm{T}}_{\lvert\mu(n)\rvert=1}$	p_e(0.6079271019)
8800000	3450262	3450241.5	0.3920752273	5349738	5349758.5	0.6079247727
8900000	3489447	3489448.8	0.3920726966	5410553	5410551.2	0.6079273034
9000000	3528655	3528656.1	0.3920727778	5471345	5471343.9	0.6079272222
9100000	3567868	3567863.4	0.3920734066	5532132	5532136.6	0.6079265934
9200000	3607074	3607070.7	0.3920732609	5592926	5592929.3	0.6079267391
9300000	3646273	3646278.0	0.3920723656	5653727	5653722.0	0.6079276344
9400000	3685474	3685485.2	0.3920717021	5714526	5714514.8	0.6079282979
9500000	3724665	3724692.5	0.3920700000	5775335	5775307.5	0.6079300000
9600000	3763888	3763899.8	0.3920716667	5836112	5836100.2	0.6079283333
9700000	3803078	3803107.1	0.3920698969	5896922	5896892.9	0.6079301031
9800000	3842293	3842314.4	0.3920707143	5957707	5957685.6	0.6079292857
9900000	3881499	3881521.7	0.3920706061	6018501	6018478.3	0.6079293939
10000000	3920707	3920729.0	0.3920707000	6079293	6079271.0	0.6079293000
10100000	3959903	3959936.3	0.3920696040	6140097	6140063.7	0.6079303960
10200000	3999110	3999143.6	0.3920696078	6200890	6200856.4	0.6079303922
10300000	4038329	4038350.9	0.3920707767	6261671	6261649.1	0.6079292233
10400000	4077542	4077558.1	0.3920713462	6322458	6322441.9	0.6079286538
10500000	4116745	4116765.4	0.3920709524	6383255	6383234.6	0.6079290476
10600000	4155964	4155972.7	0.3920720755	6444036	6444027.3	0.6079279245
10700000	4195167	4195180.0	0.3920716822	6504833	6504820.0	0.6079283178
10800000	4234368	4234387.3	0.3920711111	6565632	6565612.7	0.6079288889
10900000	4273570	4273594.6	0.3920706422	6626430	6626405.4	0.6079293578
11000000	4312777	4312801.9	0.3920706364	6687223	6687198.1	0.6079293636
11100000	4351996	4352009.2	0.3920717117	6748004	6747990.8	0.6079282883
11200000	4391205	4391216.5	0.3920718750	6808795	6808783.5	0.6079281250
11300000	4430404	4430423.7	0.3920711504	6869596	6869576.3	0.6079288496
11400000	4469605	4469631.0	0.3920706140	6930395	6930369.0	0.6079293860
11500000	4508828	4508838.3	0.3920720000	6991172	6991161.7	0.6079280000
11600000	4548028	4548045.6	0.3920713793	7051972	7051954.4	0.6079286207

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$ (continued)

N	N_0	N_0^{T}	p_e (0.3920728981)	$N_{\|\mu(n)\|=1}$	$N^{\mathrm{T}}_{\|\mu(n)\|=1}$	p_e(0.6079271019)
11700000	4587241	4587252.9	0.3920718803	7112759	7112747.1	0.6079281197
11800000	4626437	4626460.2	0.3920709322	7173563	7173539.8	0.6079290678
11900000	4665639	4665667.5	0.3920705042	7234361	7234332.5	0.6079294958
12000000	4704859	4704874.8	0.3920715833	7295141	7295125.2	0.6079284167
12100000	4744059	4744082.1	0.3920709917	7355941	7355917.9	0.6079290083
12200000	4783272	4783289.4	0.3920714754	7416728	7416710.6	0.6079285246
12300000	4822484	4822496.6	0.3920718699	7477516	7477503.4	0.6079281301
12400000	4861694	4861703.9	0.3920720968	7538306	7538296.1	0.6079279032
12500000	4900901	4900911.2	0.3920720800	7599099	7599088.8	0.6079279200
12600000	4940118	4940118.5	0.3920728571	7659882	7659881.5	0.6079271429
12700000	4979316	4979325.8	0.3920721260	7720684	7720674.2	0.6079278740
12800000	5018522	5018533.1	0.3920720313	7781478	7781466.9	0.6079279687
12900000	5057726	5057740.4	0.3920717829	7842274	7842259.6	0.6079282171
13000000	5096925	5096947.7	0.3920711538	7903075	7903052.3	0.6079288462
13100000	5136143	5136155.0	0.3920719847	7963857	7963845.0	0.6079280153
13200000	5175359	5175362.3	0.3920726515	8024641	8024637.7	0.6079273485
13300000	5214575	5214569.5	0.3920733083	8085425	8085430.5	0.6079266917
13400000	5253775	5253776.8	0.3920727612	8146225	8146223.2	0.6079272388
13500000	5292978	5292984.1	0.3920724444	8207022	8207015.9	0.6079275556
13600000	5332176	5332191.4	0.3920717647	8267824	8267808.6	0.6079282353
13700000	5371387	5371398.7	0.3920720438	8328613	8328601.3	0.6079279562
13800000	5410602	5410606.0	0.3920726087	8389398	8389394.0	0.6079273913
13900000	5449802	5449813.3	0.3920720863	8450198	8450186.7	0.6079279137
14000000	5489025	5489020.6	0.3920732143	8510975	8510979.4	0.6079267857
14100000	5528218	5528227.9	0.3920721986	8571782	8571772.1	0.6079278014
14200000	5567419	5567435.2	0.3920717606	8632581	8632564.8	0.6079282394
14300000	5606627	5606642.4	0.3920718182	8693373	8693357.6	0.6079281818
14400000	5645848	5645849.7	0.3920727778	8754152	8754150.3	0.6079272222
14500000	5685051	5685057.0	0.3920724828	8814949	8814943.0	0.6079275172

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$ (continued)

N	N_0	N_0^{T}	p_e (0.3920728981)	$N_{\|\mu(n)\|=1}$	$N_{\|\mu(n)\|=1}^{\mathrm{T}}$	p_e(0.6079271019)
14600000	5724256	5724264.3	0.3920723288	8875744	8875735.7	0.6079276712
14700000	5763469	5763471.6	0.3920727211	8936531	8936528.4	0.6079272789
14800000	5802665	5802678.9	0.3920719595	8997335	8997321.1	0.6079280405
14900000	5841891	5841886.2	0.3920732215	9058109	9058113.8	0.6079267785
15000000	5881104	5881093.5	0.3920736000	9118896	9118906.5	0.6079264000
15100000	5920307	5920300.8	0.3920733113	9179693	9179699.2	0.6079266887
15200000	5959510	5959508.1	0.3920730263	9240490	9240491.9	0.6079269737
15300000	5998730	5998715.3	0.3920738562	9301270	9301284.7	0.6079261438
15400000	6037929	6037922.6	0.3920733117	9362071	9362077.4	0.6079266883
15500000	6077151	6077129.9	0.3920742581	9422849	9422870.1	0.6079257419
15600000	6116366	6116337.2	0.3920747436	9483634	9483662.8	0.6079252564
15700000	6155567	6155544.5	0.3920743312	9544433	9544455.5	0.6079256688
15800000	6194772	6194751.8	0.3920741772	9605228	9605248.2	0.6079258228
15900000	6233986	6233959.1	0.3920745912	9666014	9666040.9	0.6079254088
16000000	6273199	6273166.4	0.3920749375	9726801	9726833.6	0.6079250625
16100000	6312400	6312373.7	0.3920745342	9787600	9787626.3	0.6079254658
16200000	6351602	6351580.9	0.3920741975	9848398	9848419.1	0.6079258025
16300000	6390799	6390788.2	0.3920735583	9909201	9909211.8	0.6079264417
16400000	6430021	6429995.5	0.3920744512	9969979	9970004.5	0.6079255488
16500000	6469222	6469202.8	0.3920740606	10030778	10030797.2	0.6079259394
16600000	6508426	6508410.1	0.3920738554	10091574	10091589.9	0.6079261446
16700000	6547639	6547617.4	0.3920741916	10152361	10152382.6	0.6079258084
16800000	6586840	6586824.7	0.3920738095	10213160	10213175.3	0.6079261905
16900000	6626056	6626032.0	0.3920743195	10273944	10273968.0	0.6079256805
17000000	6665254	6665239.3	0.3920737647	10334746	10334760.7	0.6079262353
17100000	6704481	6704446.6	0.3920749123	10395519	10395553.4	0.6079250877
17200000	6743697	6743653.8	0.3920754070	10456303	10456346.2	0.6079245930
17300000	6782897	6782861.1	0.3920749711	10517103	10517138.9	0.6079250289
17400000	6822084	6822068.4	0.3920737931	10577916	10577931.6	0.6079262069

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$ (continued)

N	N_0	N_0^{T}	p_e (0.3920728981)	$N_{\lvert\mu(n)\rvert=1}$	$N_{\lvert\mu(n)\rvert=1}^{\mathrm{T}}$	p_e(0.6079271019)
17500000	6861298	6861275.7	0.3920741714	10638702	10638724.3	0.6079258286
17600000	6900504	6900483.0	0.3920740909	10699496	10699517.0	0.6079259091
17700000	6939712	6939690.3	0.3920741243	10760288	10760309.7	0.6079258757
17800000	6978914	6978897.6	0.3920738202	10821086	10821102.4	0.6079261798
17900000	7018124	7018104.9	0.3920739665	10881876	10881895.1	0.6079260335
18000000	7057342	7057312.2	0.3920745556	10942658	10942687.8	0.6079254444
18100000	7096544	7096519.5	0.3920742541	11003456	11003480.5	0.6079257459
18200000	7135759	7135726.7	0.3920746703	11064241	11064273.3	0.6079253297
18300000	7174962	7174934.0	0.3920744262	11125038	11125066.0	0.6079255738
18400000	7214167	7214141.3	0.3920742935	11185833	11185858.7	0.6079257065
18500000	7253372	7253348.6	0.3920741622	11246628	11246651.4	0.6079258378
18600000	7292573	7292555.9	0.3920738172	11307427	11307444.1	0.6079261828
18700000	7331772	7331763.2	0.3920733690	11368228	11368236.8	0.6079266310
18800000	7370980	7370970.5	0.3920734043	11429020	11429029.5	0.6079265957
18900000	7410169	7410177.8	0.3920724339	11489831	11489822.2	0.6079275661
19000000	7449379	7449385.1	0.3920725789	11550621	11550614.9	0.6079274211
19100000	7488579	7488592.4	0.3920721990	11611421	11611407.6	0.6079278010
19200000	7527797	7527799.6	0.3920727604	11672203	11672200.4	0.6079272396
19300000	7566997	7567006.9	0.3920723834	11733003	11732993.1	0.6079276166
19400000	7606188	7606214.2	0.3920715464	11793812	11793785.8	0.6079284536
19500000	7645407	7645421.5	0.3920721538	11854593	11854578.5	0.6079278462
19600000	7684614	7684628.8	0.3920721429	11915386	11915371.2	0.6079278571
19700000	7723824	7723836.1	0.3920722843	11976176	11976163.9	0.6079277157
19800000	7763016	7763043.4	0.3920715152	12036984	12036956.6	0.6079284848
19900000	7802218	7802250.7	0.3920712563	12097782	12097749.3	0.6079287437
20000000	7841421	7841458.0	0.3920710500	12158579	12158542.0	0.6079289500

Table 4: The comparison of empirical probability and frequency with those from number theory when $\mu(n) = 0/|\mu(n)| = 1$ (continued)

N	N_0	N_0^{T}	p_e (0.3920728981)	$N_{\|\mu(n)\|=1}$	$N_{\|\mu(n)\|=1}^{\mathrm{T}}$	p_e(0.6079271019)

Note: N: Length of the block from $\mu(1)$ to $\mu(N)$; N_0: Frequency of $\mu(n) = 0$; $N_0^{\mathrm{T}}(= N \times p_t)$: Frequency of $\mu(n) = 0$ from number theory; $p_e(= \frac{N_0}{N})$: Empirical probability; The number in bracket is theoretical probability from number theory p_t. $N_{|\mu(n)|=1}$ and $N_{|\mu(n)|=1}^{\mathrm{T}}$ are those for $|\mu(n)| = 1$.