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Numerical method for a neutral delay differential problem

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Abstract

In this paper, the initial-value problem for a linear first order neutral delay differential equation is considered. To solve this problem numerically, we construct a fitted difference scheme on a uniform mesh which is succeeded by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with weight and remainder term in integral form. Also, the method is first-order convergent in the discrete maximum norm. Furthermore, numerical illustrations provide support of the theoretical results.

1 Introduction

Delay differential equations (DDEs) often appear in various phenomena in science and engineering such as dynamical diseases, population ecology, economics, control theory, biosciences [3, 8, 10, 12, 14, 20, 23]. Besides, neutral delay differential equations (NDDEs) are connected to real world life problems. In fact, the NDDEs arise in modelling of neural networks [16], population dynamics [13], time lag cell growth [2], lossless transmission lines and partial element equivalent circuits [5]. Furthermore, the existence and uniqueness of solution to NDDEs are discussed in [4, 6, 7, 9, 15, 17, 18, 19, 26].

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Up to now, there are many researchers who have investigated oscillation, Hopf bifurcation and numerical aspects [11, 21, 22, 24, 25]. Moreover, the authors in [7] studied the numerical solutions of DDEs and NDDEs by using standard methods, such as Euler, Heun, Runge-Kutta methods.
As far as we know, NDDEs cannot be solved explicitly; therefore, it is important to develop effective numerical methods to solve them.
Motivated by the previous work, we consider the following neutral delay differential problem in the interval $\overline{I} = [0, T]$:

$$u'(t) + a(t)u'(t-r) + b(t)u(t) + c(t)u(t-r) = f(t), \quad t \in I, \tag{1}$$

$$u(t) = \varphi(t), \quad t \in I_0, \tag{2}$$

where $I = (0, T] = \cup_{p=1}^{m} I_p$, $I_p = \{t : r_{p-1} < t \leqslant r_p\}$, $1 \leqslant p \leqslant m$ and $r_s = sr$, $0 \leqslant s \leqslant m$ and, $I_0 = [-r, 0]$ (for simplicity we suppose that T/r is integer; i.e., $T = mr$). $a(t)$, $b(t) \geqslant \beta > 0$, $c(t)$, $f(t)$ and $\varphi(t)$ are given sufficiently smooth functions satisfying certain regularity conditions in $\overline{I}$ and I_0, to be specified r is a constant delay. The numerical method presented here comprises of a finite-difference scheme on a uniform mesh. We have derived this approach on the basis of the method of integral identities with the use of interpolating quadrature rules with the weight and remainder terms in integral form. These result in a local truncation error containing only the first derivative of an exact solution and hence facilitates examination of convergence. The aim of this paper is discretizing (1)-(2) using a numerical method, which is composed of an implicit fitted finite difference scheme on a uniform mesh. The structure of this paper is as follows. In section 2, we state some important properties of the exact solution. The finite difference discretization is introduced in Section 3. In section 4, we present the error analysis for the approximate solution. Convergence is proved in the discrete maximum norm. In section 5, we formulate the iterative algorithm for solving the discrete problem and present numerical results which validate the theoretical analysis computationally. The paper concludes with a summary. Throughout the paper, C denotes a generic positive constant. Some specific, fixed constants of this kind are indicated by subscripting C and D.

2 The continuous problem

Here we give a priori estimates for the solution of (1)-(2), which are needed in later sections for the analysis of the appropriate numerical solution. For any continuous function $g(t)$, we use $\|g\|_\infty$ for the continuous maximum norm on the corresponding interval.

Lemma 2.1. *Let a, b, c, $f \in C(\overline{I})$, $\varphi \in C^1(I_0)$. Then for the solution u of the problem (1)-(2) the following estimates hold:*

$$\|u\|_{\infty,I_p} \leqslant C_p,\ 1 \leqslant p \leqslant m \tag{3}$$

$$\|u'\|_{\infty,I_p} \leqslant D_p,\ 1 \leqslant p \leqslant m \tag{4}$$

where

$$C_1 = |\varphi(0)| + \beta^{-1}[\|f\|_{\infty,I_1} + \|a\|_{\infty,I_1}\|\varphi'\|_{\infty,I_0} + \|c\|_{\infty,I_1}\|\varphi\|_{\infty,I_0}],$$

$$D_1 = \|f\|_{\infty,I_1} + \|a\|_{\infty,I_1}\|\varphi'\|_{\infty,I_0} + \|b\|_{\infty,I_1} C_1 + \|c\|_{\infty,I_1}\|\varphi\|_{\infty,I_0},$$

$$C_p = |\varphi(0)| + \beta^{-1}[\|f\|_{\infty,I_p} + \|a\|_{\infty,I_p} D_{p-1} + \|c\|_{\infty,I_p} C_{p-1}],\ p = 2, 3, ...m,$$

$$D_p = \|f\|_{\infty,I_p} + \|a\|_{\infty,I_p} D_{p-1} + \|b\|_{\infty,I_p} C_p + \|c\|_{\infty,I_p} C_{p-1},\ p = 2, 3, ...m.$$

Proof. The proof is by induction on p. From (1) we have

$$u(t) = \varphi(0)\, e^{-\int_0^t b(\eta)d\eta} + \int_0^t F(\xi) e^{-\int_\xi^t b(\eta)d\eta} d\xi$$

with

$$F(t) = f(t) - a(t)\, u'(t-r) - c(t)\, u(t-r).$$

So,

$$|u(t)| \leqslant |\varphi(0)|\, e^{-\int_0^t b(\eta)d\eta} + \int_0^t [|f(\xi)| + |a(\xi)|\,|u'(\xi - r)| + |c(\xi)|\,|u(\xi - r)|] e^{-\int_\xi^t b(\eta)d\eta} d\xi \tag{5}$$

$$\leqslant |\varphi(0)|\, e^{-\beta t} + \int_0^t [|f(\xi)| + |a(\xi)|\, |\varphi'(\xi - r)| + |c(\xi)|\, |\varphi(\xi - r)|\,] e^{-\beta(t-\xi)} d\xi,$$

for $p = 1$ $(t \in I_1)$

$$|u(t)| \leqslant |\varphi(0)| + \beta^{-1}[\|f\|_{\infty,I_1} + \|a\|_{\infty,I_1} \|\varphi'\|_{\infty,I_0} + \|c\|_{\infty,I_1} \|\varphi\|_{\infty,I_0}](1 - e^{-\beta t}) \equiv C_1.$$

Next, from (1)

$$|u'(t)| \leqslant |f(t)| + |a(t)|\, |\varphi'(t - r)| + |b(t)|\, |u(t)| + |c(t)|\, |\varphi(t - r)|$$

$$\leqslant \|f\|_{\infty,I_1} + \|a\|_{\infty,I_1} \|\varphi'\|_{\infty,I_0} + \|b\|_{\infty,I_1} C_1 + \|c\|_{\infty,I_1} \|\varphi\|_{\infty,I_0} \equiv D_1$$

therefore the inequalities (3) and (4) hold for $p = 1$. Let the inequalities (3) and (4) be true for $p = k$. That is

$$C_k = |\varphi(0)| + \beta^{-1}[\|f\|_{\infty,I_k} + \|a\|_{\infty,I_k} D_{k-1} + \|c\|_{\infty,I_k} C_{k-1}],$$

$$D_k = \|f\|_{\infty,I_k} + \|a\|_{\infty,I_k} D_{k-1} + \|b\|_{\infty,I_k} C_k + \|c\|_{\infty,I_k} C_{k-1}.$$

For $t \in I_{k+1}$ because of (5) we get

$$|u(t)| \leqslant |\varphi(0)|\, e^{-\beta t} + \int_0^t [|f(\xi)| + |a(\xi)|\, |u'(\xi - r)| + |c(\xi)|\, |u(\xi - r)|\,] e^{-\beta(t-\xi)} d\xi$$

$$\leqslant |\varphi(0)| + \beta^{-1}[\|f\|_{\infty,I_{k+1}} + \|a\|_{\infty,I_{k+1}} \|u'\|_{\infty,I_k} + \|c\|_{\infty,I_{k+1}} \|u\|_{\infty,I_k}](1 - e^{-\beta t})$$

and also from (1)

$$|u'(t)| \leqslant |f(t)| + |a(t)|\, |u'(t - r)| + |b(t)|\, |u(t)| + |c(t)|\, |u(t - r)|$$

$$\leqslant \|f\|_{\infty,I_{k+1}} + \|a\|_{\infty,I_{k+1}} \|u'\|_{\infty,I_k} + \|b\|_{\infty,I_{k+1}} C_k + \|c\|_{\infty,I_{k+1}} C_{k-1}$$

and hence the inequalities (3) and (4) hold for $p = k + 1$. □

3 The difference scheme and mesh

Let ω_{N_0} be a uniform mesh on $\overline{I}$:

$$\omega_{N_0} = \{t_i = i\tau,\ i = 1, 2, ...N_0,\ \tau = T/N_0 = r/N\}$$

which contains N mesh point at each subinterval I_p $(1 \leqslant p \leqslant m)$:

$$\omega_{N_p} = \{t_i : (p-1)N + 1 \leqslant i \leqslant pN\},\ 1 \leqslant p \leqslant m,$$

and consequently

$$\omega_{N_0} = \cup_{p=1}^{m} \omega_{N_p}.$$

For any mesh function $g(t)$, we use $g_i = g(t_i)$ and y_i to denote an approximation of $u(t)$ at t_i and

$$g_{\bar{t},i} = (g_i - g_{i-1})/\tau, \quad \|g\|_{\infty,N,p} = \|g\|_{\infty,\omega_{N,p}} := \max_{1 \leqslant i \leqslant N} |g_i|\,.$$

For the difference approximation the problem (1), we use the following identity

$$\tau^{-1}\int_{t_{i-1}}^{t_i} Lu(t)\psi_i(t)dt = \tau^{-1}\int_{t_{i-1}}^{t_i} f(t)\psi_i(t)dt, \quad 1 \leqslant i \leqslant N_0, \tag{6}$$

with basis function

$$\psi_i(t) = e^{-\int\limits_t^{t_i} b(\eta)d\eta},\ t_{i-1} \leqslant t \leqslant t_i.$$

We note that the function $\psi_i(t)$ is a solution of the problem

$$-\psi_i(t) + b(t)\psi_i(t) = 0,\ t_{i-1} < t \leqslant t_i,\ \psi_i(t_i) = 1. \tag{7}$$

The relation (6) is rewritten as

$$\tau^{-1}\int_{t_{i-1}}^{t_i} u'(t)\psi_i(t)dt + \tau^{-1}\int_{t_{i-1}}^{t_i} a(t)u'(t-r)\psi_i(t)dt + \tau^{-1}\int_{t_{i-1}}^{t_i} b(t)u(t)\psi_i(t)dt$$

$$+\tau^{-1}\int_{t_{i-1}}^{t_i} c(t)u(t-r)\psi_i(t)dt = \tau^{-1}\int_{t_{i-1}}^{t_i} f(t)\psi_i(t)dt.$$

Using the formulas (2.1) and (2.2) from [1] on each interval (t_{i-1}, t_i) taking into account (7) we have following precise relation

$$\ell u_i \equiv A_i u_{\bar{t},i} + B_i u_{\bar{t},i-N} + C_i u_i + D_i u_{i-N} = F_i + R_i,\ i = 1, 2, ..., N_0, \quad (8)$$

where

$$A_i = \tau^{-1} \int_{t_{i-1}}^{t_i} \psi_i(t)dt + \tau^{-1} \int_{t_{i-1}}^{t_i} (t - t_i) b(t)\psi_i(t)dt,$$

$$B_i = \tau^{-1} \int_{t_{i-1}}^{t_i} a(t)\psi_i(t)dt + \tau^{-1} \int_{t_{i-1}}^{t_i} (t - t_i) c(t)\psi_i(t)dt,$$

$$C_i = \tau^{-1} \int_{t_{i-1}}^{t_i} b(t)\psi_i(t)dt,\ D_i = \tau^{-1} \int_{t_{i-1}}^{t_i} c(t)\psi_i(t)dt,$$

$$F_i = \tau^{-1} \int_{t_{i-1}}^{t_i} f(t)\psi_i(t)dt,$$

$$R_i = \tau^{-1} \int_{t_{i-1}}^{t_i} dt[c(t) - a'(t) - a(t)b(t)]\psi_i(t) \int_{t_{i-1}}^{t_i} u'(\xi - r)K_0(t, \xi)d\xi, \quad (9)$$

$K_0(t, \xi) = T_0(t - \xi) - \tau^{-1}(t - t_{i-1}),\ T_s(\lambda) = \lambda^s/s!,\ \lambda \geqslant 0;\ T_s(\lambda) = 0,\ \lambda < 0.$

As a consequence of (8), we propose the following difference scheme for approximating (1)-(2):

$$\ell y_i \equiv A_i y_{\bar{t},i} + B_i y_{\bar{t},i-N} + C_i y_i + D_i y_{i-N} = F_i,\ 1 \leqslant i \leqslant N_0, \quad (10)$$

$$y_i = \varphi_i,\ \ -N \leqslant i \leqslant 0. \quad (11)$$

4 Error analysis

In order to investigate the convergence of this method, note that the error function $z_i = y_i - u_i$, $0 \leqslant i \leqslant N_0$ is the solution of the following discrete problem

$$\ell z_i = R_i, \quad 1 \leqslant i \leqslant N_0, \quad (12)$$

$$z_i = 0, \quad -N_0 \leqslant i \leqslant 0, \quad (13)$$

where the truncation error R_i is given by (9).

Lemma 4.1. *If* $a \in C^1(\overline{I})$, b, c, $f \in C(\overline{I})$ *and* $\varphi \in C^1(I_0)$, *then for the truncation error* R_i *we have*

$$\|R\|_{\infty,N,p} \leqslant CN^{-1}. \tag{14}$$

Proof. From (9), we present

$$|R_i| \leqslant \tau^{-1} \int_{t_{i-1}}^{t_i} dt[|a'(t)| + |a(t)|\,|b(t)| + |c(t)|]\psi_i(t) \int_{t_{i-1}}^{t_i} |u'(\xi - r)|\, d\xi$$
$$\leqslant C\tau^{-1} \int_{t_{i-1}}^{t_i} dt\psi_i(t) \int_{t_{i-1}}^{t_i} |u'(\xi - r)|\, d\xi$$

and by virtue of Lemma 2.1 and $0 < \psi_i(t) \leqslant 1$

$$|R_i| \leqslant C\tau.$$

□

Lemma 4.2. *Let* z_i *be the solution (12)-(13) holds true. Then*

$$\|z\|_{\infty,N,p} \leqslant C \sum_{k=1}^{p} \|R\|_{\infty,\omega_{N,k}}. \tag{15}$$

Proof. From the solution (12)-(13)

$$\|z\|_{\infty,N,p} \leqslant \beta^{-1} \sum_{k=1}^{p} \|R\|_{\infty,\omega_{N,k}} Q_{p-k} \quad 1 \leqslant p \leqslant m,$$

where

$$Q_{p-k} = \begin{cases} 1, & \text{for } k = p \\ \prod_{s=k+1}^{p}(1 + \beta^{-1}(\|a\|_{\infty,I_s} + \|c\|_{\infty,I_s})), & \text{for } 0 \leqslant k \leqslant p-1. \end{cases}$$

□

Now we give the main convergence result.

Theorem 4.3. *Let* u *be the solution of (1)-(2) and* y *the solution (6)-(7). Then*

$$\|y - u\|_{\infty,\omega_{N_0}} \leqslant CN^{-1}.$$

Proof. This follows immediately by combining the previous lemmas. □

5 Numerical results

In the section, we present numerical results obtained by applying the numerical method (10)-(11) to the particular problem. We consider the test problem:

$$u'(t) + u'(t-1) + 2u(t) - u(t-1) = 0,\ t > 0$$

subject to the interval condition,

$$u(t) = e^t,\ \ -1 \leqslant t \leqslant 0.$$

The exact solution for $0 \leqslant t \leqslant 2$ is given by

$$u\,(t) = \begin{cases} e^{-2t}, & t \in [0,1]\,, \\ e^{-2t} + 3\,(t-1)e^{-2(t-1)}, & t \in [1,2]\,. \end{cases}$$

The computational results for the test problem are presented in Table 1.

Table 1 The numerical results on $(0,2]$.

Nodes t_i	Exact solution	Numerical solution $\tau = 0.015625$	Pointwise error $\lvert y-u \rvert$	Numerical solution $\tau = 0.0078125$	Pointwise error $\lvert y-u \rvert$
0.125	0.7788008	0.7788034	$2.652E-6$	0.7788014	$6.630E-7$
0.250	0.6065307	0.6065357	$5.070E-6$	0.6065319	$1.268E-6$
0.375	0.4723666	0.4723739	$7.354E-6$	0.4723684	$1.839E-6$
0.500	0.3678890	0.3678890	$9.586E-6$	0.3678818	$2.397E-6$
0.625	0.2865048	0.2865166	$1.184E-5$	0.2865078	$2.960E-6$
0.750	0.2231302	0.2231443	$1.417E-5$	0.2231337	$3.544E-6$
0.875	0.1737906	0.1737906	$1.665E-5$	0.1737781	$4.163E-6$
1.000	0.1353353	0.1353546	$1.933E-5$	0.1353401	$4.833E-6$
1.125	0.3974495	0.3974862	$3.664E-5$	0.3974587	$9.159E-6$
1.250	0.5370283	0.5370283	$4.534E-5$	0.5369943	$1.134E-5$
1.375	0.5953886	0.5953886	$4.840E-5$	0.5953523	$1.210E-5$
1.500	0.6016541	0.6016541	$4.789E-5$	0.6016182	$1.197E-5$
1.625	0.5759707	0.5760159	$4.524E-5$	0.5759820	$1.131E-5$
1.750	0.5322402	0.5322817	$4.141E-5$	0.5322506	$1.035E-5$
1.875	0.4796743	0.4797114	$3.707E-5$	0.4796836	$9.267E-6$
2.000	0.4243215	0.4243541	$3.262E-5$	0.4243296	$8.155E-6$

6 Conclusion

In this paper, we have developed a finite difference method for solving the initial-value problem for a linear first order delay differential equation with neutral type. This method was based on an exponentially fitted difference scheme on an equidistant mesh on each time subinterval. From the method, first order convergence in the discrete maximum norm resulted. A numerical example was considered using the presented method and the computational results for $N = 64, 128$ were displayed in Table 1. Numerical results were carried out to show the efficiency and accuracy of the method. The theoretical results represented undergoing research, such as nonlinear delay and neutral delay problem.

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