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The Edge Grundy Numbers of Some Graphs

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Abstract

The edge Grundy numbers of graphs in a number of different classes are determined, notably for the complete and the complete bipartite graphs, as well as for the Petersen graph, the grids, and the cubes. Except for small-graph exceptions, these edge Grundy numbers turn out to equal a natural upper bound of the edge Grundy number, or to be one less than that bound.

1 Grundy colorings and Grundy numbers

A *Grundy coloring* of a (finite simple) graph G is a proper coloring of the vertices of G with positive integers with the property that if $v \in V(G)$ is colored with $c > 1$, then all the colors $1, \ldots, c-1$ appear on neighbors of v. (*Proper* means that adjacent vertices of G bear different colors.) The *Grundy number* of G, $\Gamma(G)$, is the greatest number of colors appearing in a Grundy coloring of G.

It is easy to obtain Grundy colorings by the following method. Let $v_1, \ldots, v_n$ be an ordering of the vertices of G. The *greedy coloring* of $V(G)$ with respect to this ordering is obtained through the following instructions: color v_1 with 1; for $1 \leq i < n$, having colored $v_1, \ldots, v_i$, color v_{i+1} with the smallest positive integer not appearing on neighbors of v_{i+1} among $v_1, \ldots, v_i$. It is easy to see that the coloring that results is Grundy. There is a converse.

Lemma 1.1. *A coloring of the vertices of G is Grundy if and only if it is obtainable as a greedy coloring with respect to some ordering of $V(G)$.*

To see the "only if" assertion: given a Grundy coloring of G, order the vertices so that those colored 1 come first, then those colored 2, etc. Clearly the greedy coloring obtained from this ordering will be the coloring you started with.

If $S \subseteq V(G)$ and $\varphi : S \to \mathbb{P} = \{1, 2, \ldots\}$ we say that $v \in S$ in *Grundy-satisfied* by the coloring φ if either $\varphi(v) = 1$ or $\varphi(v) > 1$ and for every $i \in \{1, \ldots, \varphi(v) - 1\}\}$, there is a neighbour $u \in S$ of v such that $\varphi(u) = i$. We will say that φ is a *full partial Grundy coloring* of G if φ is proper ($u, v \in S$ and $uv \in E(G)$ implies $\varphi(u) \neq \varphi(v)$) and every $v \in S$ is Grundy-satisfied by φ.

Lemma 1.2. *Every full partial Grundy coloring of a graph G can be extended to a Grundy coloring of G.*

Proof. Given a full partial Grundy coloring $\varphi : S \to \mathbb{P}$, $S \subseteq V(G)$, order S by putting vertices colored 1 by φ first, then vertices colored 2, etc. Follow that ordering by an ordering of $V(G) \setminus S$. Clearly the greedy coloring with respect to this ordering of $V(G)$ will be an extension of φ. □

Corollary 1.3. *If H is an induced subgraph of G, then $\Gamma(H) \leq \Gamma(G)$.*

Proof. Every Grundy coloring of H is a full partial Grundy coloring of G, and so can be extended to a Grundy coloring of G. □

As usual, $\chi(G)$ will denote the chromatic number of G, d_G, or just d, the degree function on $V(G)$, and $\Delta(G)$ the maximum degree in G.

Lemma 1.4. *For any finite simple graph G,*

$$\chi(G) \leq \Gamma(G) \leq \Delta(G) + 1.$$

Proof. The left hand inequality arises from the requirement that every Grundy coloring be proper. The right hand inequality follows from Lemma 1.1. □

The results in these lemmas have been known from the beginning of the study of Grundy colorings [1, 2] We mention them for the purpose of transferring them to the object of our study, edge Grundy colorings. An edge Grundy coloring of G (finite and simple, as always, in this paper) is a Grundy coloring of $L(G)$, the line graph of G. The edge Grundy number of G, denoted $\Gamma'(G)$, is the Grundy number of $L(G)$: $\Gamma'(G) = \Gamma(L(G))$. Letting, as usual $\chi'(G) = \chi(L(G))$ denote the edge chromatic number (aka chromatic index) of G, we have from Lemmas 1.2 and 1.4 the following.

Lemma 1.5. *Suppose that G is a finite, simple graph.*

(a) *Any proper coloring of some of the edges of G with positive integers so that each colored edge is Grundy-satisfied can be extended to an edge Grundy coloring of G.*

(b) $\chi'(G) \leq \Gamma'(G) \leq \max_{uv \in E(G)}[d(u) + d(v) - 1]$.

Our main results will be stated in the next section. Proofs and intermediate results will be given in section 3. In section 4 we will pose some questions.

2 Results

P_n is the path on n vertices, C_n the cycle on n vertices, K_n the complete graph on n vertices, $K_{m,n}$ the complete bipartite graph with parts of sizes m and n, Q_n the n-cube, and $\mathcal{P}$ the Petersen graph. Let the Cartesian product be denoted by $\Box$; $P_m \Box P_n$ is the m by n *grid*, $C_m \Box C_n$ is the m by n *toroidal grid*, and $P_m \Box C_n$ is the m by n *cylindrical grid*.

Theorem 2.1. (Paths, cycles, grids, cylindirical grids, and toroidal grids)

(a) $\Gamma'(P_2) = 1$, $\Gamma'(P_3) = \Gamma'(P_4) = 2$, *and* $\Gamma'(P_n) = 3$ *if* $n \geq 5$.

(b) $\Gamma'(C_4) = 2$; *for all other* $n \geq 3$, $\Gamma'(C_n) = 3$.

(c) (i) $\Gamma'(P_2 \Box P_3) = \Gamma'(P_2 \Box P_4) = 4$; *for* $n \geq 5$, $\Gamma'(P_2 \Box P_n) = 5$.

(ii) $\Gamma'(P_3 \Box P_3) = \Gamma'(P_3 \Box P_4) = 6$; $6 \leq \Gamma'(P_3 \Box P_n) \leq 7$ *for* $5 \leq n \leq 7$, *and for* $n \geq 8$, $\Gamma'(P_3 \Box P_n) = 7$.

(iii) *For* $m, n \geq 4$, $\Gamma'(P_m \Box P_n) = 7$.

(d) (i) *For all* $n \geq 3$, $\Gamma'(P_2 \Box C_n) = 5$.

(ii) $\Gamma'(P_3 \Box C_3) = 6$; *for* $n \in \{4, 5, 6\}$, $6 \leq \Gamma'(P_3 \Box C_n) \leq 7$; *and for* $n \geq 7$, $\Gamma'(P_3 \Box C_n) = 7$.

(iii) *For* $m \geq 4$, $\Gamma'(P_m \Box C_3) = 7$.

(iv) *For* $m, n \geq 4$, $\Gamma'(P_m \Box C_n) = 7$.

(e) *For all* $m, n \geq 3$, $\Gamma'(C_m \Box C_n) = 7$.

Theorem 2.2. (a) $\Gamma'(\mathcal{P}) = 5$;

(b) $\Gamma'(Q_n) = 2n - 1$ *if* $n > 2$.

Theorem 2.3. (a) *If* $1 \leq m < n$, *then* $\Gamma'(K_{m,n}) = m + n - 1$.

(b) *If* $n > 1$, $\Gamma'(K_{n,n}) = 2n - 2$.

Theorem 2.4. *If* $n \geq 3$ *is odd, then* $\Gamma'(K_n) = 2n - 3$. *For* $n \geq 6$, *even,* $\Gamma'(K_n) = 2n - 4$. *Finally,* $\Gamma'(K_4) = 3$.

3 Proofs and intermediate results

Let $D(G) = \Delta(L(G)) + 1 = \max_{uv \in E(G)}[d(u) + d(v) - 1]$, the upper bound for $\Gamma'(G)$ given in Lemma 1.5.

Lemma 3.1. *If $\Gamma'(G) = D(G) > 1$ then there exist distinct edges $uv, vw \in E(G)$ such that*

$$d(u) + d(v) = d(v) + d(w) = D(G) + 1.$$

Proof. Suppose that $\varphi : E(G) \to \{1, 2, \ldots, D(G)\}$ is a proper edge Grundy coloring, with $D(G) = \varphi(uv)$ for some $uv \in E(G)$. Because uv is Grundy-satisfied by the coloring, and $D(G) > 1$, $\varphi(vw) = D(G) - 1$ for some edge vw adjacent to uv.

The number of edges of G adjacent to uv must be at least $D(G) - 1$, for uv to be Grundy-satisfied by φ, whence

$$D(G) - 1 \leq d(u) + d(v) - 2 \leq D(G) - 1,$$

so $d(u) + d(v) = D(G) + 1$. For vw to be Grundy-satisfied by φ, the number of edges of G, *other than* uv, adjacent to vw, must be at least $D(G) - 2$, whence

$$D(G) - 2 \leq d(v) + d(w) - 3 \leq D(G) - 2,$$

so $d(v) + d(w) = D(G) + 1$. □

Lemma 3.2. *Suppose that $\Gamma'(G) = D(G)$ and that $u, v, w \in V(G)$ and φ are as in the proof of Lemma 3.1. Then φ is injective on the set of edges adjacent to uv, and also on the set of edges adjacent to vw.*

The proof of Lemma 3.2 is straightforward from the proof of Lemma 3.1, and the fact that $D(G) - 1 = \Delta(L(G))$.

Proof of Theorem 2.1. (a) Since $L(P_n) \simeq P_{n-1}$ for $n > 1$, the results here follow from well known results about $\Gamma(P_m)$: clearly $\Gamma(P_1) = 1$; using Lemma 1.1 it is easy to see that $\Gamma(P_2) = \Gamma(P_3) = 2$, and $\Gamma(P_4) = 3$; by Corollary 1.3 and Lemma 1.4, for $m \geq 4$,

$$3 = \Gamma(P_4) \leq \Gamma(P_m) \leq \Delta(P_m) + 1 = 3.$$

(b) Since $L(C_n) = C_n$, the results here follow from well-known results about $\Gamma(C_n)$. For small n, these are easy to check. For $n \geq 5$, note that

$$3 \leq \Gamma(P_4) \leq \Gamma(C_n) \leq \Delta(C_n) + 1 = 3.$$

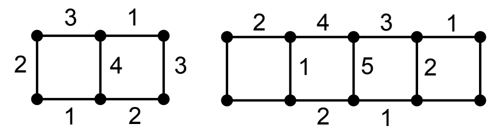

Figure 1: An edge Grundy coloring of $P_2 \square P_3$ with 4 colors, and a full partial edge Grundy coloring of $P_2 \square P_5$ with 5 colors.

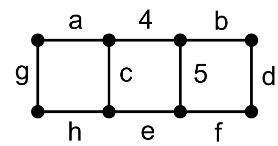

Figure 2: Case 1 in the proof that $\Gamma'(P_2 \square P_4) < 5$.

(c)(i) For $n \geq 3$, we have $\Gamma'(P_2 \square P_n) \leq 5$ by Lemma 1.5(b). By Lemma 3.1, $\Gamma'(P_2 \square P_3) < 5$. In Figure 1 we see proof that $\Gamma'(P_2 \square P_3) = 4$ and that $\Gamma'(P_2 \square P_5) = 5$. Since $P_2 \square P_5$ is a subgraph of $P_2 \square P_n$ for $n \geq 5$, this proves that $\Gamma'(P_2 \square P_n) = 5$ for all $n \geq 5$. Regarding $P_2 \square P_4$, we have

$$4 = \Gamma'(P_2 \square P_3) \leq \Gamma'(P_2 \square P_4) \leq 5.$$

To finish the proof of the claims about $\Gamma'(P_2 \square P_n)$, it suffices to show that $P_2 \square P_4$ cannot be edge Grundy colored with 5 colors. Suppose that it can. By the proof of Lemma 3.1, there are essentially two possibilities for starting such a coloring with colors 5 and 4 on adjacent edges.

Case 1: We are in the situation of Figure 2. Let the letters on the edges of this figure serve as names for the edges and also as stand-ins for the colors on those edges.

First, $b \in \{2, 1\}$, because if $b = 3$ then edge b could not possibly be Grundy-satisfied. Therefore, $3 \in \{a, c\} \cap \{e, f\}$.

If $c = 3$ then, by Lemma 3.2, $\{a, b\} = \{1, 2\}$, $f = 3$, and $\{b, e\} = \{1, 2\}$. It follows that edge f cannot be Grundy-satisfied, since, if it were, then $\{d, e\} = \{b, e\} = \{1, 2\}$ which implies that $b = d$, which means that the coloring is not proper.

Therefore $a = 3$, and $\{b, c\} = \{1, 2\}$, for the edge colored 4 to be Grundy-satisfied. Further, $\{e, f\} = \{c, 3\}$. But $e \neq c$, so $e = 3$ and $f = c$, so $\{b, f\} = \{1, 2\}$.

If $b = 2$ then $f = 1$ and the edge b cannot be Grundy-satisfied. If $b = 1$ then $f = 2$ and the edge f cannot be Grundy-satisfied.

Case 2: We are in the situation of Figure 3. Let the letters on the edges

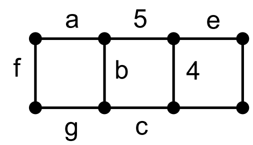

Figure 3: Case 2 in the proof that $\Gamma'(P_2 \Box P_4) < 5$.

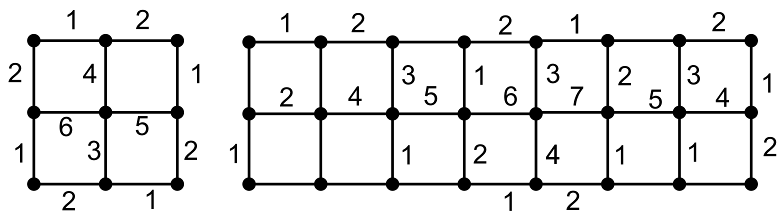

Figure 4: An edge Grundy coloring of $P_3 \Box P_3$ with 6 colors and a full partial edge Grundy coloring of $P_3 \Box P_8$ with 7 colors.

of this figure serve as names for the edges and also as stand-ins for the colors on those edges.

As in case 1, $e \neq 3$, so $e \in \{1, 2\}$, and $3 \in \{a, b\} \cap \{c, d\}$, while $c \neq b$. If $b = 3$ then $d = 3$, and $\{c, e\} = \{1, 2\}$; but then d cannot be Grundy-satisfied. Therefore, $a = 3$. By Lemma 3.2, $c \neq e$, so if $d = 3$, d cannot be Grundy-satisfied. Therefore, $c = 3$.

Then $\{e, d\} = \{1, 2\}$, which implies that either e or d cannot be Grundy-satisfied.

(c)(ii) $P_3 \Box P_n$, $n \geq 3$: By Lemmas 3.1 and 1.5, $\Gamma'(P_3 \Box P_n) \leq 6$ for $n = 3, 4$, and $\Gamma'(P_3 \Box P_n) \leq 7$ for $n \geq 5$. The colorings in Figure 4 prove that $\Gamma'(P_3 \Box P_3) = \Gamma'(P_3 \Box P_4) = 6$, $6 \leq \Gamma'(P_3 \Box P_n) \leq 7$ for $5 \leq n \leq 7$, and $\Gamma'(P_3 \Box P_n) = 7$ for $n \geq 8$.

(c)(iii) $P_m \Box P_n$, $m, n \geq 4$: By Lemma 1.5, $\Gamma'(P_m \Box P_n) \leq 7$ for all $m, n \geq 4$. To show equality, the coloring in Figure 5 suffices.

(d)(i) $P_2 \Box C_n$: By Lemma 1.5, $\Gamma'(P_2 \Box C_n) \leq 5$ for all $n \geq 3$. For $n \geq 5$, applying (c)(i), above,

$$5 = \Gamma'(P_2 \Box P_n) \leq \Gamma'(P_2 \Box C_n) \leq 5.$$

To finish the proof here, it suffices to show that

$$\Gamma'(P_2 \Box C_3) = \Gamma'(P_2 \Box C_4) = 5;$$

this is accomplished by the colorings shown in Figure 6.

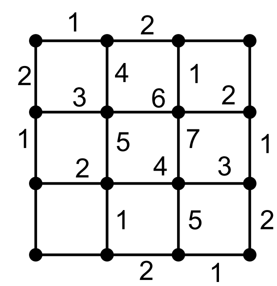

Figure 5: A full partial edge Grundy coloring of $P_4 \square P_4$ with 7 colors.

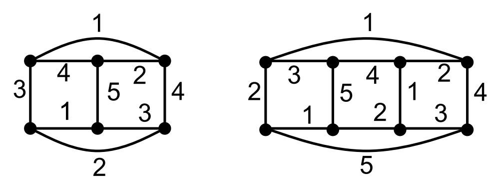

Figure 6: Edge Grundy colorings of $P_2 \square C_3$ and $P_2 \square C_4$ with 5 colors.

(d)(ii) $P_3 \square C_n, n \geq 3$: By Lemma 1.5, $\Gamma'(P_3 \square C_n) \leq 7$ for all $n \geq 3$. Because $P_3 \square P_n$ is a subgraph of $P_3 \square C_n$, from (c)(ii), above, it follows that $6 \leq \Gamma'(P_3 \square C_n) \leq 7$, $3 \leq n \leq 7$, and $\Gamma'(P_3 \square C_n) = 7$ for all $n \geq 8$. Therefore, to complete the proofs of the claims in this part of Theorem 2.1, it suffices to show that $P_3 \square C_3$ cannot be edge Grundy colored with 7 colors, and that $P_3 \square C_7$ can be so colored. The latter chore is taken care of by Figure 7

Suppose $P_3 \square C_3$ has an edge Grundy coloring with 7 colors. By Lemma 3.1, there is essentially only one possibility for the placement of the colors 6 and 7. As in the proof that $\Gamma'(P_2 \square P_4) < 5$, consider the leftmost copy of $P_3 \square C_3$ in Figure 8 and let the letters on the the edges of stand both as names of the edges and as the colors from $\{1, \ldots, 7\}$ that might be assigned to these edges.

Neither c no d can be colored 5; otherwise that edge is not Grundy-

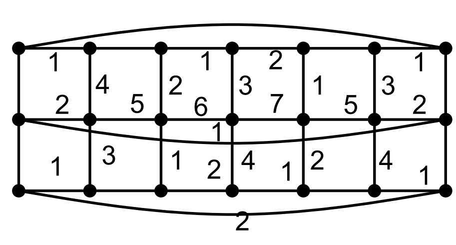

Figure 7: A full partial edge Grundy coloring of $P_3 \square C_7$ with 7 colors.

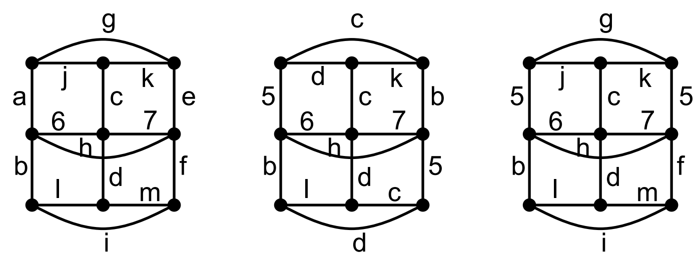

Figure 8: Three pictures of $P_3 \square C_3$ as in the proof of Theorem 2.1 (d)(ii).

satisfied by the coloring.

Case 1: $h = 5$. Then $\{a, b\} = \{e, f\} = \{1, 2, 3, 4\} \setminus \{c, d\}$. But then the edge h cannot possibly be Grundy-satisfied.

Case 2: $5 \in \{a, b\} \cap \{e, f\}$. Without loss of generality, $a = 5$.

Subcase 2.1: $f = 5$. From the assumption that the edges colored 6 and 7, and also a and f, are Grundy-satisfied, we have that

$$\{1, 2, 3, 4\} = \{c, d, e, h\} = \{c, d, b, h\} = \{g, j, b, h\} = \{e, h, i, m\},$$

which imply $b = e$ and $\{g, j\} = \{c, d\} = \{i, m\}$. Since j and c are adjacent edges, it follows that $g = c$ and $j = d$. Similarly, $m = c$ and $i = d$. The coloring now looks like the center drawing in Figure 8 (but edge names will be those in the leftmost drawing). Then $\{1, 2, 3, 4\} = \{b, c, d, h\}$, in order for the two edges colored 5 to be Grundy-satisfied. Clearly 4 cannot be the color of any of edges c, d, or h, because, if it were, that edge would not be Grundy-satisfied. Therefore, $b = 4$. Consequently, $h = 1$ and $\{l, d\} = \{c, d\} = \{2, 3\}$, because the edges colored 5 and 4 must be Grundy-satisfied. Then $l = c$; but then edges l and m, which are adjacent, bear the same color. Thus there is no such coloring.

Subcase 2.2: $e = 5$. We pursue a contradiction starting with an assumed edge Grundy coloring of $P_3 \square C_3$ that looks like the rightmost picture in Figure 8. Because the edges colored 5, 6, or 7 must be Grundy-satisfied, we have

$$\{1, 2, 3, 4\} = \{c, d, f, h\} = \{c, d, b, h\} = \{b, h, g, j\} = \{f, h, g, k\}.$$

From two of these equalities it follows that $\{c, d\} = \{g, j\} = \{g, k\}$, whence $j = k$; but the edges j and k are adjacent, and so cannot bear the same color.

(d)(iii) $P_m \square C_3$, $m \geq 4$: By Lemma 1.5, $\Gamma'(P_m \square C_3) \leq 7$ for all $m \geq 4$.

To complete the proof in this part, it suffices to show that $\Gamma'(P_4 \square C_3) = 7$; this is accomplished by the coloring in Figure 9.

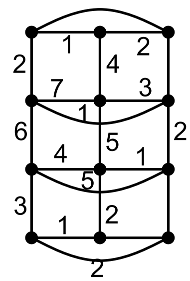

Figure 9: A full partial edge Grundy coloring of $P_4 \square C_3$ with 7 colors.

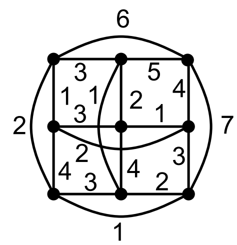

Figure 10: An edge Grundy coloring of $C_3 \square C_3$ with 7 colors.

(d)(iv) By c(iii) and Lemma 1.5, for all $m, n \geq 4$,

$$7 = \gamma'(P_m \square P_n) \leq \gamma'(P_m \square C_n) \leq 7.$$

(e) $C_m \square C_n$, $m, n \geq 3$: By Lemma 1.5 and results preceding, we have $\gamma'(C_m \square C_n) \leq 7$ for all $m, n \geq 3$, with equality when $m \geq 3, n \geq 4$. To finish the proof of the claim in this part of Theorem 2.1, it suffices to show that $\Gamma'(C_3 \square C_3) = 7$. For this, Figure 10 does the job. □

Proof of Theorem 2.2. By Lemma 1.5, $\Gamma'(\mathcal{P}) \leq D(\mathcal{P}) = 5$ and $\Gamma'(Q_n) \leq D(Q_n) = 2n - 1$. As in several parts of the proof of Theorem 2.1, we will proceed by giving full partial edge Grundy colorings with $D(G)$ colors of each graph G; we like this method because it proves a bit more than $\Gamma'(G) = D(G)$. See Figure 11 for $\mathcal{P}$.

The vertex set of Q_n is $\{0,1\}^n$, the set of binary words of length n. Assuming $n \geq 3$, let $e_0 = 0^n$, $e_k = 0^{k-1}10^{n-k}$, the binary word with 1 in position k and 0's elsewhere, $1 \leq k \leq n$. With addition defined coordinatewise, let $f_k = e_1 + e_k$, $2 \leq k \leq n$, and $g_{jk} = e_j + e_k$, $2 \leq j < k \leq n$. Color the edge $e_0 e_k$ with $2n - k$, $k = 1, \ldots, n$, and color $e_1 f_k$ with $n - k + 1$, $k = 2, \ldots, n$.

Each e_j, $2 \leq j \leq n$, is adjacent to f_j and to $n - 2$ g_{pq}'s, namely g_{ij}, $2 \leq i < j$, and g_{jk}, $j < k \leq n$. Color each edge $e_j f_j$, $2 \leq j \leq n$ with $n - 1$.

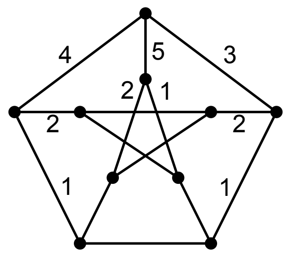

Figure 11: A full partial edge Grundy coloring of $\mathcal{P}$ with 5 colors.

Notice that e_2f_2 and e_1f_2 are both colored $n-1$ at this point. This is the only impropriety in the coloring so far; don't worry, it will be corrected soon.

The edges e_jg_{ij} and e_jg_{jk}, $2 \le j \le n$, $2 \le i < j$ and $j < k \le n$, are the edges of a bipartite graph with maximum degree $n-2$, achieved at each vertex e_j, $2 \le j \le n$. Let these edges be properly colored with $1, \dots, n-2$. Now trade the color $n-1$ on e_2f_2 for any one of the colors on $e_2g_{23}, \dots, e_2g_{2n}$; that is, let the colors of the two edges be switched. The result is a full partial edge Grundy coloring of Q_n with $2n-1 = D(Q_n)$ colors. □

A set $S \subseteq V(G)$ is *independent* (in G) if no two vertices of S are adjacent, and *dominating* if every vertex of $V(G) - S$ is adjacent to at least one vertex of S. A set $F \subseteq E(G)$ is independent in G if and only if it is independent in $L(G)$, and is edge-dominating in G if and only if it is dominating in $L(G)$. An independent set of edges is also called a *matching*.

Lemma 3.3. *If $\varphi : V(G) \to \{1, 2, \dots, t\}$ is a Grundy coloring of G, then $\{v \in V(G) \mid \varphi(v) = 1\}$ is an independent dominating set in G.*

Proof. $\{v \in V(G) \mid \varphi(v) = 1\}$ is independent because φ is a proper coloring, and dominating by the rest of the definition of Grundy colorings. □

Corollary 3.4. *If $\varphi : E(G) \to \{1, 2, \dots, t\}$ is an edge Grundy coloring of G, then $\{e \in E(G) \mid \varphi(e) = 1\}$ is an edge-dominating matching in G.*

Proof of Theorem 2.3. Suppose $1 \le m < n$. Let the vertices on one "side" of $K_{m,n}$ be $u_1, \dots, u_m$, and, on the other side, $v_1, \dots, v_n$. Color u_1v_1 with $m + n - 1 = D(K_{m,n})$. If $m = 1$, finish by coloring u_1v_j with $n - j + 1$, $j = 2, \dots, n$. Otherwise, if $2 \le m < n$, color u_jv_1 with $m+n-j$, $j = 2, \dots, m$. At this point the m edges incident to v_1 are colored with $n, n+1, \dots, m+n-1$, and no other edges are colored.

$K_{m,n} - v_1 \simeq K_{m,n-1}$ is a bipartite graph with maximum degree $n-1$, and can therefore be properly edge-colored with colors $1, \ldots, n-1$. Let it be so colored; now every edge of $K_{m,n}$ is colored, with colors $1, \ldots, m+n-1$ appearing, and, because every color $1, \ldots, n-1$ appears on the edges incident to each u_j, $j = 1, \ldots, m$, every edge is Grundy satisfied; the coloring is Grundy. Thus $\Gamma'(K_{m,n}) = D(K_{m,n}) = m+n-1$.

Now suppose that $n > 1$. Since $K_{n,n}$ contains $K_{n-1,n}$ as a subgraph,

$$2n-2 = \Gamma'(K_{n-1,n}) \leq \Gamma'(K_{n,n}) \leq D(K_{n,n}) = 2n-1.$$

To prove the claim in Theorem 2.3 about $K_{n,n}$, it suffices to show that $K_{n,n}$ has no edge Grundy coloring with $2n-1$ colors. Suppose the contrary. Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be the vertices on the two sides, and let u_1v_1 bear the color $2n-1$ in an edge Grundy coloring of $K_{n,n}$.

Clearly the only edge-dominating matchings in $K_{n,n}$ are perfect matchings, matchings which "saturate", or "cover" every vertex. For the supposed edge Grundy coloring to satisfy u_1v_1, which bears the color $2n-1$, the colors $1, \ldots, 2n-2$ must appear on the $2n-2$ edges adjacent to u_1v_1, so 1 appears on only one of those edges, say on u_1v_j, for some $j \in \{2, \ldots, n\}$. But then no edge incident to v_1 is colored 1. Therefore the edges colored 1 are not an edge-dominating matching, which contradicts Corollary 3.4. Therefore no such coloring exists. □

Proof of Theorem 2.4. $D(K_n) = 2n-3$ for $n = 2, 3, \ldots$. First suppose that $n \geq 5$ is odd. (Clearly $\Gamma'(K_3) = 3 = 2{\cdot}3-3$.) Let u and v be two distinct vertices of K_n. $K_n - \{u, v\} \simeq K_{n-2}$; since $n-2$ is odd, $\chi'(K_{n-2}) = n-2$, as is well known. A little less well known, but easy to see by elementary arguments, is that for any proper edge coloring of K_{n-2} with $n-2$ colors, there is one color missing from the edges incident to any particular vertex, and for each color there is one vertex from the edges incident to which that color is missing.

Color the edges of $K_n - \{u, v\}$ properly with colors $1, \ldots, n-2$; color uv with the color $2n-3$; color the other edges of K_n incident to v with the $n-2$ colors $2n-4, \ldots, n-1$. Now we color the $n-2$ edges other than uv which are incident to u with the colors $1, \ldots, n-2$, but with some care: the color $j \in \{1, \ldots, n-2\}$ is assigned to the edge uw such that w is the vertex of $K_n - \{u, v\}$ from the edges of which the color j is missing, in the proper edge coloring of $K_n - \{u, v\}$. This way of coloring those edges incident to u not only makes the coloring of the edges of K_n proper – it also makes the coloring Grundy, because at each vertex $w \in V(K_n) \setminus \{u, v\}$, all the colors

$1, \ldots, n-2$ now appear on the edges incident to w.

Now suppose that $n \geq 4$ is even. We will first show that $\Gamma'(K_n) < D(K_n) = 2n-3$. Suppose the contrary: suppose that K_n is edge Grundy colored with colors $1, \ldots, 2n-3$. Let uv be an edge colored $2n-3$.

Then colors $1, \ldots, 2n-4$ must appear on the other $2n-4$ edges incident to either u or to v. Therefore, one of u, v is not incident to an edge colored 1. Suppose it is u.

But, since the number of vertices covered by a matching is even, and the edges colored 1 form a matching (Corollary 3.4), there must be another vertex $w \in V(G) \setminus \{u, v\}$ which is not incident to an edge colored 1. Then uw is an edge not colored 1, nor adjacent to any edge colored 1, so the set of edges colored 1 is not edge-dominating, contradicting Corollary 3.4.

We leave the fingers-and-toes proof that $\Gamma'(K_4) < 4$ (and, therefore, $\Gamma'(K_4) = \chi'(K_4) = 3$) to the reader. Suppose that n is even, $n \geq 6$. We want to show that K_n has an edge Grundy coloring with $2n-4$ colors.

Let the vertices of K_n be $v_1, \ldots, v_n$. Color v_1v_2 with 1; color the edges of the K_{n-2} induced by $v_3, \ldots, v_n$ properly with $1, \ldots, n-3$. (Since $n-2 \geq 4$ is even, $\chi'(K_{n-2}) = n-3$, as is well known.) For each $j \in \{3, \ldots, n\}$, all colors $1, \ldots, n-3$ now occur on edges incident to v_j.

Color v_1v_n with $2n-4$ and v_2v_n with $2n-5$. For $j = 3, \ldots, n-1$, color v_1v_j with $j+n-5$. For $j = 4, \ldots, n-1$ color v_2v_j with $j+n-6$. Finally, color v_2v_3 with $2n-6$. The colors on edges incident to v_1 or v_2, except for v_1v_2, are indicated in the following table.

	v_3	v_4	$\cdots$	v_{n-2}	v_{n-1}	v_n
v_1	$n-2$	$n-1$	$\cdots$	$2n-7$	$2n-6$	$2n-4$
v_2	$2n-6$	$n-2$	$\cdots$	$2n-8$	$2n-7$	$2n-5$

Bearing in mind the remark above about the $(n-3)$-coloring of the edges among $v_3, \ldots, v_n$, it is straightforward to verify that every edge is Grundy-satisfied by this coloring. $\square$

4 Unfinished business

Inspection of Theorem 2.1 shows that we do not know the edge Grundy numbers of $P_3 \square P_n$ for $n = 5, 6, 7$, nor of $P_3 \square C_n$ for $n = 4, 5, 6$. No excuses. However, the first author reports that a computer search for an edge Grundy 7-coloring of $P_3 \square P_5$ definitely establishes that $\Gamma'(P_3 \square P_5) = 6$. We're still waiting for the verdict on $P_3 \square P_6$ and $P_3 \square P_7$.

If $1 \leq m < n$, is every edge Grundy coloring of $K_{m,n}$ with colors $1, \ldots, m+n-1$ necessarily just like the coloring described in the proof of Theorem 2.3? Equivalently, are all the "big" colors, $n, \ldots, m+n-1$, necessarily incident to one vertex on the side of the bipartition with n vertices? It is easy to see that this is the case if $m \in \{1, 2\}$, but for $m \geq 3$ the question seems not so easy.

The question of which edge Grundy colorings are possible of $K_{m,n}$, $1 \leq m < n$ with $m + n - 1 = \Gamma'(K_{m,n})$ colors may bear on the corresponding questions for $K_{n,n}$. Is every edge Grundy coloring of $K_{n,n}$ ($n \geq 3$) with $\Gamma'(K_{n,n}) = 2n-2$ colors obtainable as a greedy extension of an edge Grundy coloring of one of the $K_{n-1,n}$ subgraphs with $2n-2$ colors? Equivalently, is it possible to edge Grundy color $K_{n,n}$ with $2n-2$ colors so that, for each $v \in V(K_{n,n})$, the coloring restricted to the edges of $K_{n,n} - v$ fails to be an edge Grundy coloring with $2n-2$ colors?

About possible edge Grundy colorings of K_n with $\Gamma'(K_n)$ colors, one question, in view of the proof of Theorem 2.4, seems a good place to start: Supposing $n \geq 5$, is there, necessarily, for any such coloring, a K_{n-2} subgraph which is colored with $\chi'(K_{n-2})$ colors?

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