

Sums of squares of Krawtchouk polynomials, Catalan numbers, and some algebras over the Boolean lattice

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Abstract

Writing the values of Krawtchouk polynomials as matrices, we consider weighted partial sums along columns. For the general case, we find an identity that, in the symmetric case yields a formula for such partial sums. Complete sums of squares along columns involve “Super Catalan” numbers. We look as well for particular values (matrix entries) involving the Catalan numbers. Properties considered and developed in this work are applied to calculations of various dimensions that describe the structure of some *-algebras over the Boolean lattice based on inclusion/superset relations expressed algebraically using zeons [zero-square elements].

1 Introduction

Our approach to Krawtchouk polynomials is to consider the values as entries in corresponding matrices. This makes it convenient to refer to their values, indexing, and associated properties. We begin with Krawtchouk polynomials for general parameter p and derive an identity for partial sums of squares along a column. For the case $p = 1/2$ this leads to evaluations of these sums.

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We also review some basic properties of Krawtchouk matrices that we will find useful.

Two related features are considered in detail. First is the “Catalan connection” which appears when looking at complete sums of squares along columns. As well, Catalan numbers appear as particular entries in Krawtchouk matrices. The second main feature is the use of properties of Krawtchouk matrices in calculating dimensions of some algebras over the Boolean lattice. These arise when considering the lattice of subsets of $\{1, 2, \dots, n\}$ as an algebra generated by “zeons” — commuting elements which square to zero. These algebras are generated by the basic matrices corresponding to inclusion and superset, specifically, the regular representation of the zeon algebra.

References to Catalan numbers are readily available and abundant, so these have been skipped, but see [7] for a relevant discussion involving Super Catalan numbers. For sums of squares, we found the results of [1] important, though we stress that the difference in our approaches is substantial. Our approach to Krawtchouk polynomials follows [5, 6]. For the Boolean connection, full details are presented in [4]. And we found the discussion in [8] especially valuable for background.

2 Krawtchouk polynomials

We modify the generating function given in [2, 18.23], formula 18.23.3, for convenience in computations as well as in point of view. Throughout, we will use the parameter

$$r = \frac{1-p}{p}$$

with $r = 1$ corresponding to the symmetric case $p = 1/2$. Note the relation

$$\frac{1}{p} = 1 + r$$

that we will find useful.

The generating function takes the form

$$(1+z)^{N-j}(1-rz)^j = \sum_{n=0}^N z^n \phi_{nj}^N \quad (1)$$

with slight changes in notation. In particular, we prefer the matrix form ϕ_{nj}^N for the values of the Krawtchouk polynomials at integer points, $0 \leq j \leq N$.

2.1 Relations of Pascal type

First we note some identities similar to Pascal's triangle for binomial coefficients. See [6] for the case $r = 1$.

Proposition 2.1. *The following identities of Pascal type hold for $N \geq 0$, $0 \leq j, n \leq N$:*

$$\phi_{nj}^N + \phi_{n-1j}^N = \phi_{nj}^{N+1} \quad (\text{i})$$

$$\phi_{nj}^N - r \phi_{n-1j}^N = \phi_{nj+1}^{N+1} \quad (\text{ii})$$

with the boundary conditions $\phi_{-1j}^N = 0$, $0 \leq j \leq N$, for $N \geq 0$.

Proof. The first relation follows upon multiplication of the generating function (1) by $1 + z$. The second follows similarly using the factor $1 - rz$. \square

2.2 Recurrence formula

The second ingredient needed is a recurrence formula for Krawtchouk polynomials. From [2, 18.22], formula 18.22.12, Difference Equations in x , we have, replacing x by j and rearranging:

$$p(N - j)\phi_{nj+1}^N + qj\phi_{nj-1}^N = (Np + (q - p)j - n)\phi_{nj}^N$$

where we introduce the notation $q = 1 - p$ for convenience, with $r = q/p$. Dividing through by p and noting $r = q/p$, $1/p = 1 + r$, let us state

Lemma 2.2. *We have the recurrence in j*

$$(N + (r - 1)j - n(1 + r))\phi_{nj}^N = (N - j)\phi_{nj+1}^N + rj\phi_{nj-1}^N .$$

3 Sums of Squares

We now derive our main result.

3.1 Sums of squares for general r

Multiply equations (i) and (ii) of Proposition 2.1, for $N \rightarrow N - 1$, to get

$$\begin{aligned}\phi_{n\ j+1}^N \phi_{n\ j}^N &= (\phi_{n\ j}^{N-1})^2 - r(\phi_{n-1\ j}^{N-1})^2 + (1-r)\phi_{n\ j}^{N-1}\phi_{n-1\ j}^{N-1} \\ &= (\phi_{n\ j}^{N-1})^2 - r(\phi_{n-1\ j}^{N-1})^2 + \frac{1-r}{2} [(\phi_{n\ j}^N)^2 - (\phi_{n\ j}^{N-1})^2 - (\phi_{n-1\ j}^{N-1})^2] \\ &= \frac{1-r}{2}(\phi_{n\ j}^N)^2 + \frac{1+r}{2} [(\phi_{n\ j}^{N-1})^2 - (\phi_{n-1\ j}^{N-1})^2]\end{aligned}$$

where in the second line we use Proposition 2.1, (i), in the elementary identity $ab = ((a+b)^2 - a^2 - b^2)/2$. A similar formula holds replacing $j \rightarrow j - 1$. Now multiply through the relation in Lemma 2.2 by $\phi_{n\ j}^N$ to get

$$\begin{aligned}(N + (r-1)j - n(1+r))(\phi_{n\ j}^N)^2 &= (N-j)\phi_{n\ j+1}^N \phi_{n\ j}^N + rj\phi_{n\ j-1}^N \phi_{n\ j}^N \\ &= (N-j) \left(\frac{1-r}{2}(\phi_{n\ j}^N)^2 + \frac{1+r}{2} [(\phi_{n\ j}^{N-1})^2 - (\phi_{n-1\ j}^{N-1})^2] \right) \\ &\quad + rj \left(\frac{1-r}{2}(\phi_{n\ j-1}^N)^2 + \frac{1+r}{2} [(\phi_{n\ j-1}^{N-1})^2 - (\phi_{n-1\ j-1}^{N-1})^2] \right)\end{aligned}$$

With the telescoping parts collapsing, we sum n from 0 to m to get

$$\begin{aligned}\sum_{n=0}^m (N + (r-1)j - n(1+r))(\phi_{n\ j}^N)^2 &= \frac{1-r}{2}(N-j) \sum_{n=0}^m (\phi_{n\ j}^N)^2 + \frac{1+r}{2}(N-j)(\phi_{m\ j}^{N-1})^2 \\ &\quad + rj \frac{1-r}{2} \sum_{n=0}^m (\phi_{n\ j-1}^N)^2 + rj \frac{1+r}{2} (\phi_{m\ j-1}^{N-1})^2\end{aligned}$$

Multiplying through by $2/(1+r)$ we arrive at our main formula.

Theorem 3.1. *We have the sum of squares identity for Krawtchouk polynomials*

$$\begin{aligned}\sum_{n=0}^m (N - 2n)(\phi_{n\ j}^N)^2 &= (N-j)(\phi_{m\ j}^{N-1})^2 + rj(\phi_{m\ j-1}^{N-1})^2 \\ &\quad + \frac{1-r}{1+r} j \sum_{n=0}^m (r(\phi_{n\ j-1}^N)^2 + (\phi_{n\ j}^N)^2)\end{aligned}$$

Dette [1] has similar formulas, his formula (d) for Krawtchouk polynomials is most similar to ours.

4 Symmetric case

Letting $r = 1$ gives the symmetric case with generating function

$$(1+z)^{N-j}(1-z)^j = \sum_{n=0}^N z^n \Phi_{nj}^N \quad (2)$$

with the capital Φ denoting the values for this special case.

The recurrence is now

$$(N-2n)\Phi_{nj}^N = (N-j)\Phi_{nj+1}^N + j\Phi_{nj-1}^N \quad (3)$$

4.1 Basic properties

Here we recall some basic properties of the Kravchuk matrices for $r = 1$.

Proposition 4.1.

1. Row and column sign symmetries.

$$\begin{aligned} \Phi_{iN-j}^N &= (-1)^i \Phi_{ij}^N \\ \Phi_{N-ij}^N &= (-1)^j \Phi_{ij}^N \\ \Phi_{N-iN-i}^N &= (-1)^N \Phi_{ii}^N \end{aligned}$$

The first two follow readily from the generating function, the third follows from those.

For reference, here are the matrices for $N = 3$ and $N = 4$:

$$\Phi^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} \quad \Phi^4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

2. First rows. First columns. The entries in row $i = 1$ follow as the coefficient of z in the expansion of the generating function eq. (2) yielding

$$\Phi_{1j}^N = N - 2j .$$

The entries in the first column are the binomial coefficients

$$\Phi_{n0}^N = \binom{N}{n} .$$

For the second column, we set $j = 0$ in Proposition 2.1, (ii) to get

$$\Phi_{n1}^{N+1} = \Phi_{n0}^N - \Phi_{n-10}^N = \binom{N}{n} - \binom{N}{n-1} \quad (4)$$

$$= \binom{N}{n} \frac{N+1-2n}{N+1-n} \quad (5)$$

3. Binomial conjugation. The diagonal matrix B is defined by

$$B_{ii} = \binom{N}{i} .$$

Using the fact that $\Phi^N B$ is symmetric (ref. [5]), i.e., $\Phi^N B = B(\Phi^N)^*$, it follows

$$\Phi_{ji}^N = \binom{N}{i}^{-1} \binom{N}{j} \Phi_{ij}^N . \quad (6)$$

4. Sum of squares along a column, see [5] proof of Theorem 3.1.3

$$\sum_{i=0}^N (\Phi_{ij}^N)^2 = \binom{2N-2j}{N-j} \binom{2j}{j} / \binom{N}{j} \quad (7)$$

We will see that

The sum of squares of column $j/2$ of Φ^m is $(-1)^{j/2} \Phi_{mj}^{2m}$.

5. Sum of squares along a row, see [5] proof of Lemma 3.3.9

$$\sum_{j=0}^N (\Phi_{ij}^N)^2 = \sum_{k=0}^i \binom{N+1}{2k+1} \binom{2k}{k} \binom{N-2k}{i-k} \quad (8)$$

We have a result for partial sums along a column without the squares:

Theorem 4.2. *For $j \geq 2$, we have the partial sums*

$$\sum_{n=0}^m (N - 2n) \Phi_{n j}^N = (N - j) \Phi_{m j}^{N-1} + j \Phi_{m j-2}^{N-1} = (N - 1 - 2m) \Phi_{m j-1}^{N-1} + \Phi_{m j-2}^{N-1} .$$

Proof. Start with the Pascal relation Proposition 2.1, (ii), with $r = 1$ and $N \rightarrow N - 1$

$$\Phi_{n j}^{N-1} - \Phi_{n-1 j}^{N-1} = \Phi_{n j+1}^N$$

summing from 0 to m yields

$$\Phi_{m j}^{N-1} = \sum_{n=0}^m \Phi_{n j+1}^N \quad (9)$$

and similarly with $j - 2$ replacing j . Thus, summing the recurrence relation (3) over n we have

$$\sum_{n=0}^m (N - 2n) \Phi_{n j}^N = (N - j) \sum_{n=0}^m \Phi_{n j+1}^N + j \sum_{n=0}^m \Phi_{n j-1}^N$$

and using (9) with j adjusted accordingly yields the first equality. The second follows by applying the right hand side of the recurrence formula (3) with $N \rightarrow N - 1$. \square

Theorem 3.1 takes the form

Theorem 4.3. *For the symmetric Krawtchouk polynomials we have the sum of squares identity*

$$\sum_{n=0}^m (N - 2n) (\Phi_{n j}^N)^2 = (N - j) (\Phi_{m j}^{N-1})^2 + j (\Phi_{m j-1}^{N-1})^2 .$$

4.2 Special values. Catalan connection.

Proposition 4.4.

$$\Phi_{mj}^{2m} = \binom{m}{j/2} (-1)^{j/2} \frac{\binom{2m}{m}}{\binom{2m}{j}} \text{ for } j \text{ even , } \quad 0 \text{ for } j \text{ odd} \quad (10)$$

$$\Phi_{mj}^{2m+1} = \binom{m}{\lfloor j/2 \rfloor} (-1)^{\lfloor j/2 \rfloor} \frac{\binom{2m+1}{m}}{\binom{2m+1}{j}} . \quad (11)$$

Proof. Setting $N = 2m$, $j = m$, the generating function, (2) becomes

$$\begin{aligned} (1+z)^m(1-z)^m &= (1-z^2)^m \\ &= \sum z^{2k} \binom{m}{k} (-1)^k = \sum z^\ell \Phi_{\ell m}^{2m} \end{aligned}$$

hence the evaluation

$$\Phi_{\ell m}^{2m} = \binom{m}{\ell/2} (-1)^{\ell/2} \text{ for } \ell \text{ even , } \quad 0 \text{ for } \ell \text{ odd}$$

and applying the binomial conjugation, eq. (6), yields the result for $N = 2m$.

For $N = 2m + 1$, $j = m$, we have

$$(1+z)(1-z^2)^m = \sum \left(z^{2k} \binom{m}{k} (-1)^k + z^{2k+1} \binom{m}{k} (-1)^k \right)$$

which yields

$$\Phi_{\ell m}^{2m+1} = \binom{m}{\lfloor \ell/2 \rfloor} (-1)^{\lfloor \ell/2 \rfloor}$$

and binomial conjugation completes the proof. \square

Remark 4.5. Note that for N even, these can be expressed as Super Catalan numbers, in the terminology of [7], e.g., according to the relations

$$\binom{n}{k} \frac{\binom{2n}{n}}{\binom{2n}{2k}} = \frac{\binom{2n-2k}{n-k} \binom{2k}{k}}{\binom{n}{k}} = \frac{(2n-2k)! (2k)!}{(n-k)! k! n!}$$

with n replacing m and k replacing $j/2$ in (10).

Note that for $N = 2m + 1$, (11) yields for the entry in the second column middle row

$$\Phi_{m1}^{(2m+1)} = C_m = \frac{1}{m+1} \binom{2m}{m}.$$

the m^{th} Catalan number. In fact,

Proposition 4.6. Catalan Connection

We have the following evaluations involving Catalan numbers.

1. For $N = 2m$ even,

$$\begin{aligned}\Phi_{m-1,1}^{(2m)} &= C_m \\ \Phi_{m+1,1}^{(2m)} &= -C_m \\ \Phi_{m,2}^{(2m)} &= -2C_{m-1}\end{aligned}$$

2. For $N = 2m + 1$ odd,

$$\begin{aligned}\Phi_{m,1}^{(2m+1)} &= C_m \\ \Phi_{m,2}^{(2m+1)} &= \Phi_{m+1,1}^{(2m+1)} = \Phi_{m+1,2}^{(2m+1)} = -C_m\end{aligned}$$

3. Reading right-to-left along the rows yield evaluations correspondingly by sign symmetries.

Proof. The first two equations in #1 follow from eq. (4). The third follows from (10).

Similarly, for #2, the column one evaluations follow from (4) and the $\Phi_{m,2}^{2m+1}$ entry follows from (11). For $\Phi_{m+1,2}^{2m+1}$, use Pascal as follows:

$$\Phi_{m+1,1}^{(2m)} - \Phi_{m,1}^{(2m)} = \Phi_{m+1,2}^{(2m+1)} = -C_m$$

noting that $\Phi_{m,1}^{(2m)}$ vanishes. □

Remark 4.7. See [3] for worksheets on Catalan numbers, listed up to C_{20} , and on Kravchuk matrices, listed up to $N = 12$.

5 Dimensions of algebras over the Boolean lattice

Let \mathcal{B} denote the Boolean lattice of subsets of the standard n -set $\{1, 2, \dots, n\}$. The layers, each consisting of subsets of cardinality ℓ , are denoted \mathcal{B}_ℓ . We identify each element i with a variable e_i , taken together forming the generators of a commutative algebra satisfying the conditions

$$e_i^2 = 0 .$$

We call such variables *zeons*.

A subset $I \in \mathcal{B}_\ell$ is identified with the product

$$e_I = e_{i_1} \cdots e_{i_\ell}$$

for $I = \{i_1, \dots, i_\ell\}$.

We will consider some algebras generated by the zeons and determine their structure. See [4] for a full account.

Start with the linear operator \hat{e}_i of multiplication by e_i .

$$\hat{e}_i e_I = \begin{cases} e_{\{i\} \cup I}, & \text{if } i \notin I \\ 0, & \text{otherwise} \end{cases}$$

And for the dual basis $\{\delta_i\}$, the action of δ_i is given by the linear operator $\hat{\delta}_i$ defined by

$$\hat{\delta}_i e_I = \begin{cases} e_{I \setminus \{i\}}, & \text{if } i \in I \\ 0, & \text{otherwise} \end{cases}$$

For convenience we will drop the $\hat{}$ notations. With the standard inner product $\langle e_I, e_J \rangle = \delta_{IJ}$, one checks that

$$\langle e_i e_I, e_J \rangle = \langle e_I, e_i^* e_J \rangle = \langle e_I, \delta_i e_J \rangle$$

the $*$ indicating adjoint with respect to the inner product. We define the operator $T = \sum_i e_i$ and its adjoint

$$T^* = \sum_i \delta_i = \sum_i e_i^*$$

Their commutator

$$U = [T^*, T]$$

has matrix elements

$$U_{\mathbf{I}\mathbf{J}} = \begin{cases} n - 2\ell, & \text{if } \mathbf{I} = \mathbf{J} \in \mathcal{B}_\ell \\ 0, & \text{otherwise} \end{cases}$$

Recall that a finite-dimensional $*$ -algebra is a direct sum of matrix algebras. Our goal here is to find the form of some algebras generated by these operators as direct sums of matrix algebras. The description is provided by four numbers. The algebra consists of a direct sum of m_i copies of matrix algebras of degree d_i .

$$d = \text{Degree of the algebra} = \sum m_i d_i$$

$$\delta = \text{Dimension of the algebra} = \sum d_i^2$$

$$\zeta = \text{Dimension of the centralizer} = \sum m_i^2$$

$z = \text{Dimension of the center} = \text{the number of components of the decomposition}$

Remark 5.1. See, e.g., [8, Ch. 1], for an exposition in the context of group algebras.

The calculations, although accessible by elementary means, are done illustrating the connection with Kravchuk matrices.

In the discussion below, we write $N = n + 1$.

For N even, write $N = 2m$, for N odd, write $N = 2m + 1$. And

$$\lfloor \frac{n}{2} \rfloor = \begin{cases} m - 1, & \text{for } N \text{ even} \\ m, & \text{for } N \text{ odd} \end{cases}$$

5.1 Algebra generated by U

Since U is diagonal, all of the $d_i = 1$ and we only need to determine m_i . Since the m_i are exactly given by the number of sets in each layer, we see

that the m_i are precisely the binomial coefficients. See Appendix page for U .

Note that this is the "horizontal description" of $\mathcal{B}(n)$.

We have immediately

$$\begin{aligned} d &= 2^n, \text{ degree of the algebra} \\ \delta &= n + 1, \text{ dimension of the algebra} \\ z &= n + 1, \text{ dimension of the center} \end{aligned}$$

And we have the dimension of the centralizer

$$\zeta = \sum_i \binom{n}{i}^2 = \binom{2n}{n}.$$

5.2 Algebra generated by T and T^*

This is the "vertical description" of $\mathcal{B}(n)$.

The d_i may be seen directly to be $n + 1 - 2\alpha$, where $0 \leq \alpha \leq \lfloor n/2 \rfloor$. The m_i are correspondingly given by $\binom{n}{\alpha} - \binom{n}{\alpha-1}$, with $m_0 = 1$. See Appendix page for $n = 4$ as well as that for T and T^* .

For the degree, we have

$$\sum_{\alpha=0}^{\lfloor n/2 \rfloor} \left[\binom{n}{\alpha} - \binom{n}{\alpha-1} \right] (n + 1 - 2\alpha)$$

Using the relation $(\Phi^N)^2 = 2^N I$, write this as

$$\sum_{\alpha=0}^m \Phi_{1\alpha}^N \Phi_{\alpha 1}^N = \frac{1}{2} (\Phi_{11}^N)^2 = \frac{1}{2} 2^N = 2^n$$

appropriately.

And we have

$d = 2^n$, degree of the algebra

$$\delta = \sum_{\alpha=0}^{\lfloor n/2 \rfloor} (n+1-2\alpha)^2 = \binom{n+3}{3}$$

$z = 1 + \lfloor n/2 \rfloor$, dimension of the center

Proof of the formula for δ :

By sign symmetry, we have, using eq. (8),

$$\begin{aligned} \sum_{\alpha=0}^m (n+1-2\alpha)^2 &= \frac{1}{2} \sum (\Phi_{1\alpha}^N)^2 = \frac{1}{2} \left[(N+1)N + 2 \binom{N+1}{3} \right] \\ &= \binom{N+2}{3} \end{aligned}$$

as required.

The dimension of the centralizer

$$\zeta = \sum_{\alpha=0}^{\lfloor n/2 \rfloor} \left[\binom{n}{\alpha} - \binom{n}{\alpha-1} \right]^2 = \frac{1}{n+1} \binom{2n}{n} = C_n .$$

Proof of the formula for ζ :

Using eq. (7), we have

$$\sum_{\alpha=0}^m (\Phi_{\alpha 1}^N)^2 = \frac{1}{2} \frac{\binom{2N-2}{N-1} \binom{2}{1}}{\binom{N}{1}} = C_{N-1} = C_n$$

as required.

5.3 Algebra generated by TT^* and T^*T

If the $\mathcal{A}(T, T^*)$ has m_i and d_i , then here we have d_i copies of m_i all with $d = 1$.

$d = 2^n$, degree of the algebra

$$\delta = \sum_{\alpha=0}^{\lfloor n/2 \rfloor} (n+1-2\alpha) = \begin{cases} (n+2)^2/4, & \text{if } n \text{ is even} \\ (n+1)(n+3)/4 & \text{if } n \text{ is odd} \end{cases}$$

$z = 1 + \lfloor n/2 \rfloor$, dimension of the center

Proof of the formula for δ :

$$\begin{aligned} \sum_{\alpha=0}^{\lfloor \frac{n}{2} \rfloor} (n+1-2\alpha) &= \sum_0^m (N-2\alpha) \\ &= N(m+1) - m(m+1) = (m+1)(N-m) \end{aligned}$$

For N odd we get $(m+1)^2 = ((N+1)/2)^2$. Even N yields

$$(1+N/2)(N/2) = \frac{N(N+2)}{4}$$

as required.

And we have the dimension of the centralizer

$$\zeta = \sum_{\alpha=0}^{\lfloor n/2 \rfloor} (n+1-2\alpha) \left[\binom{n}{\alpha} - \binom{n}{\alpha-1} \right]^2 = \begin{cases} \binom{n}{n/2}^2, & \text{if } n \text{ is even} \\ 2 \binom{n}{\lfloor n/2 \rfloor} \binom{n-1}{\lfloor n/2 \rfloor}, & \text{if } n \text{ is odd} \end{cases}$$

Proof of the formula for ζ :

Using Theorem 3.1, we have

$$\begin{aligned} \sum_{\alpha=0}^m (N-2\alpha) (\Phi_{\alpha 1}^N)^2 &= (N-1)(\Phi_{m 1}^{N-1})^2 + (\Phi_{m 0}^{N-1})^2 \\ &= (N-1) \left(\binom{N-2}{m} - \binom{N-2}{m-1} \right)^2 + \binom{N-1}{m}^2 \\ &= (N-1) \binom{N-2}{m}^2 \left(\frac{N-1-2m}{N-1-m} \right)^2 + \binom{N-1}{m}^2. \end{aligned}$$

If N is odd, $n = N - 1 = 2m$ and we get $\binom{n}{n/2}^2$. If N is even, substituting $N = 2m$ and simplifying gives

$$2 \binom{2m-1}{m-1} \binom{2m-2}{m-1}$$

and $n = N - 1 = 2m - 1$, $m - 1 = \lfloor n/2 \rfloor$ yields the result.

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Boolean System for $n=4$

Table of TT^*

#	U	ℓ		4	2	2	2	0	0
1	4	0		0					
4	2	1		4	0	0	0		
6	0	2		6	2	2	2	0	0
4	-2	3		6	2	2	2		
1	-4	4		4					

Table of States

#	U	ℓ		4	2	2	2	0	0
1	4	0		1					
4	2	1		$e_1 + e_2 + e_3 + e_4$	$e_1 - e_2$	$e_1 + e_2 - 2e_3$	$e_1 + e_2 + e_3 - 3e_4$		
6	0	2		$e_1e_2 + e_1e_3 + e_1e_4 + e_2e_3 + e_2e_4 + e_3e_4$	$e_1e_3 + e_1e_4 - e_2e_3 - e_2e_4 + e_4$	$2e_1e_2 - e_1e_3 + e_1e_4 - e_2e_3 + e_2e_4 - 2e_3e_4$	$2e_1e_2 + 2e_1e_3 - 2e_1e_4 + 2e_2e_3 - 2e_2e_4 - 2e_3e_4$	$e_1e_3 - e_1e_4 - e_2e_3 + e_2e_4 + e_4$	$2e_1e_2 - e_1e_3 - e_1e_4 - e_2e_3 - e_2e_4 + 2e_3e_4$
4	-2	3		$e_1e_2e_3 + e_1e_2e_4 + e_1e_3e_4 + e_2e_3e_4$	$e_1e_3e_4 - e_2e_3e_4$	$2e_1e_2e_4 - e_1e_3e_4 - e_2e_3e_4$	$3e_1e_2e_3 - e_1e_2e_4 - e_1e_3e_4 - e_2e_3e_4$		
1	-4	4		$e_1e_2e_3e_4$					

Casimirs: $25^5, 9^9, 1^2$

Casimir $C = 4TT^* + (U+1)^2$ has multiplicity $\binom{n+1}{\ell} \frac{C}{n+1}$, where ℓ is the level of the vacuum for the corresponding states.

Table of Casimirs

#	U	ℓ		4	2	2	2	0	0
1	4	0		25					
4	2	1		25	9	9	9		
6	0	2		25	9	9	9	1	1
4	-2	3		25	9	9	9		
1	-4	4		25					

Algebra generated by U

Data is the same for T.

n	d	δ	ζ	z	m/d
2	4	3	6	3	1 2 1 1 1 1
3	8	4	20	4	1 3 3 1 1 1 1 1
4	16	5	70	5	1 4 6 4 1 1 1 1 1 1
5	32	6	252	6	1 5 10 10 5 1 1 1 1 1 1 1
6	64	7	924	7	1 6 15 20 15 6 1 1 1 1 1 1 1 1
7	128	8	3432	8	1 7 21 35 35 21 7 1 1 1 1 1 1 1 1 1

The dimension of the centralizer is $\binom{2n}{n}$.

Note that this is the "horizontal description" of $\mathcal{B}(n)$.

Algebra generated by T, T^*

n	d	δ	ζ	z	m/d
2	4	10	2	2	1 1 3 1
3	8	20	5	2	1 2 4 2
4	16	35	14	3	1 3 2 5 3 1
5	32	56	42	3	1 4 5 6 4 2
6	64	84	132	4	1 5 9 5 7 5 3 1
7	128	120	429	4	1 6 14 14 8 6 4 2

Note that $\delta = \binom{n+3}{3}$. ζ is the n^{th} Catalan number $\frac{1}{n+1} \binom{2n}{n}$. And $z = 1 + \lfloor n/2 \rfloor$.

This is the "vertical description" of $\mathcal{B}(n)$.

The successive rows of d_i 's are formed by adding one to each entry of a given row, padding on the right with a "1" when moving from n odd to n even.

The successive rows of m_i 's are formed as in Pascal's triangle, summing pairs of successive values. Moving from n even to n odd, pad with zero only on the left. Moving from n odd to n even, pad with zero on both right and left. So i runs from 1 to $1 + \lfloor n/2 \rfloor$.

The d_i may be found directly as $n + 1 - 2\alpha$, where $0 \leq \alpha \leq \lfloor n/2 \rfloor$. The m_i are correspondingly given by $\binom{n}{\alpha} - \binom{n}{\alpha-1}$, with $m_0 = 1$.

Algebra generated by TT^* and T^*T

n	d	δ	ζ	z	m/d
2	4	4	4	4	1 1 1 1 1 1 1 1
3	8	6	12	6	1 1 1 1 2 2 1 1 1 1 1 1
4	16	9	36	9	1 1 1 1 1 3 3 3 2 1 1 1 1 1 1 1 1 1
5	32	12	120	12	1 1 1 1 1 1 4 4 4 4 5 5 1 1 1 1 1 1 1 1 1 1 1 1
6	64	16	400	16	1 1 1 1 1 1 1 5 5 5 5 5 9 9 9 5 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
7	128	20	1400	20	1 1 1 1 1 1 1 1 6 6 6 6 6 6 14 14 14 14 14 14 1

Note that

$$\delta = \begin{cases} (n+2)^2/4, & \text{if } n \text{ is even} \\ (n+1)(n+3)/4, & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad \zeta = \begin{cases} \binom{n}{n/2}, & \text{if } n \text{ is even} \\ 2 \binom{n}{\lfloor n/2 \rfloor} \binom{n-1}{\lfloor n/2 \rfloor}, & \text{if } n \text{ is odd} \end{cases}$$

If the $\mathcal{A}(T, T^*)$ has m_i and d_i , then here we have d_i copies of m_i all with $d = 1$.