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Fuzzy almost bi-ideals in semigroups

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Abstract

The notion of fuzzy subsets was introduced by Zadeh in 1965 and the notion of almost bi-ideals was introduced by Bogdanovic in 1981. The fuzzy theory has developed in many directions and found applications in a wide variety of fields. In this paper, we define fuzzy almost bi-ideals in semigroups and give some relationship between almost bi-ideals and fuzzy almost bi-ideals of semigroups.

1 Introduction

In 1952, Good and Hughes [3] introduced the notion of bi-ideals of semigroups. The concept of a bi-ideal is not only interesting but also important in

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semigroups. In 1965, the definition of fuzzy subsets was introduced by Zadeh [9]. The fuzzy subset theory is a generalization of conventional mathematics set theory. Kuroki [7] studied various kinds of fuzzy ideals in semigroups and characterized them. Interesting research on fuzzy semigroups can be found in [8]. In 2011, Chon [2] also characterized the fuzzy bi-ideal generated by fuzzy subsets in semigroups. In 1980, Grosek and Satko [4] defined and studied the notion of left (respectively, right, two-sided) almost ideals of semigroups. Moreover, they characterized when a semigroup S contains no proper left (respectively, right, two-sided) almost ideals. In 1981, Bogdanovic [1] introduced the notion of almost bi-ideals in semigroups.

In this paper, we first give some properties of almost bi-ideals of semigroups and introduce the notion of fuzzy almost bi-ideals by using the concepts of almost bi-ideals and fuzzy ideals of semigroups. Moreover, we give some relationship between almost bi-ideals and fuzzy almost bi-ideals of semigroups.

2 Preliminary Notes

2.1 Bi-ideals in semigroups

In 1952, Good and Hughes [3] introduced the notion of bi-ideals as follows:

Definition 2.1. Let S be a semigroup and B be a subsemigroup of S. If $BSB \subseteq B$, then B is called a *bi-ideal* of S

Example 2.1. Let $\mathbb{N}$ be the set of all positive integers. Then $\mathbb{N}$ is a semigroup under the usual multiplication. Let $B = 2\mathbb{N}$. Thus B is a subsemigroup of $\mathbb{N}$ and $B\mathbb{N}B = 4\mathbb{N} \subseteq 2\mathbb{N} = B$. Hence B is a bi-ideal of $\mathbb{N}$.

2.2 Fuzzy sets

In 1965, Zadeh [9] introduced the concept of a fuzzy subset of a set. Let X be any set. A function f from X to the unit interval $[0, 1]$ is called a *fuzzy subset* of X.

Let f and g be any two fuzzy subsets of X. The fuzzy subsets $f \cap g$, $f \cup g$ of X and $f \subseteq g$ are defined as follows:

$$(f \cap g)(x) = f(x) \wedge g(x),$$
$$(f \cup g)(x) = f(x) \vee g(x),$$
$$f \subseteq g \text{ if } f(x) \leq g(x)$$

for all $x \in X$. Let f be a fuzzy subset of X. The *support* of f is defined by $suppf := \{x \in X | f(x) \neq 0\}$.

Let A be a non-empty subset of X. The *characteristic mapping* of A is a fuzzy subset of X defined by

$$C_A(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

Let s be any element in X. The *characteristic mapping* of s is a fuzzy subset of X defined by

$$C_s(x) = \begin{cases} 1 & \text{if} \quad x = s, \\ 0 & \text{if} \quad x \neq s. \end{cases}$$

Next, we recall the definitions and propositions of fuzzy ideals in semigroups from [8].

Definition 2.2. Let f be a fuzzy subset of a semigroup S.

(1) f is called a *fuzzy subsemigroup* of S if and only if $f(xy) \geq f(x) \wedge f(y)$.

(2) f is called a *fuzzy left ideal* of S if and only if $f(xy) \geq f(y)$.

(3) f is called a *fuzzy right ideal* of S if and only if $f(xy) \geq f(x)$.

(4) f is called a *fuzzy two-sided ideal* of S if and only if $f(xy) \geq f(x) \vee f(y)$.

Let f and g be two fuzzy subsets of a semigroup S. A *product* $f \circ g$ is defined by

$$(f \circ g)(x) = \begin{cases} \vee_{x=yz}\{f(y) \wedge g(z)\}, & \text{if } \exists y, z \in S, \text{ such that } x = yz, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.3. *Let f be a fuzzy subset of a semigroup S. Then the following properties hold:*

(1) f is a fuzzy subsemigroup of S if and only if $f \circ f \subseteq f$.

(2) f is a fuzzy left ideal of S if and only if $S \circ f \subseteq f$.

(3) f is a fuzzy right ideal of S if and only if $f \circ S \subseteq f$.

(4) f is a fuzzy two-sided ideal of S if and only if $S \circ f \subseteq f$ and $f \circ S \subseteq f$.

Finally, we recall the definition of fuzzy bi-ideals of semigroups.

Definition 2.4. A fuzzy subset f of a semigroup S is called a fuzzy bi-ideal of S if

$$f(xyz) \geq f(x) \wedge f(z)$$

for all $x, y, z \in S$.

Proposition 2.5. *Let f be any fuzzy subsemigroup of a semigroup S. Then f is a fuzzy bi-ideal of S if and only if $f \circ S \circ f \subseteq f$.*

2.3 Almost bi-ideals in semigroups

In 1980, Grosek and Satko [4] defined the notion of a left almost ideal of a semigroup : a non-empty subset G_L of a semigroup S is said to be a left almost ideal of S if

$$sG_L \cap G_L \neq \emptyset$$

for all $s \in S$. A right A-ideal of S is similarly defined. If G is both left and right almost ideal of S, then G is called an almost ideal of S.

In 1981, Bogdanovic [1] introduced the notion of almost bi-ideals as following:

Definition 2.6. A non-empty subset B of a semigroup S is called an *almost bi-ideal* of S if $BsB \cap B \neq \emptyset$ for all $s \in S$.

Example 2.2. Consider the semigroup $\mathbb{Z}_4$ under the usual addition. Let $B = \{\overline{1}, \overline{2}, \overline{3}\}$. Consider

$$\begin{aligned}
(B + \overline{0} + B) \cap B &= \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\} \cap \{\overline{1}, \overline{2}, \overline{3}\} = \{\overline{1}, \overline{2}, \overline{3}\} \neq \emptyset. \\
(B + \overline{1} + B) \cap B &= \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\} \cap \{\overline{1}, \overline{2}, \overline{3}\} = \{\overline{1}, \overline{2}, \overline{3}\} \neq \emptyset. \\
(B + \overline{2} + B) \cap B &= \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\} \cap \{\overline{1}, \overline{2}, \overline{3}\} = \{\overline{1}, \overline{2}, \overline{3}\} \neq \emptyset. \\
(B + \overline{3} + B) \cap B &= \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\} \cap \{\overline{1}, \overline{2}, \overline{3}\} = \{\overline{1}, \overline{2}, \overline{3}\} \neq \emptyset.
\end{aligned}$$

Then B is an almost bi-ideal of $\mathbb{Z}_4$.

Example 2.3. Consider the semigroup $S = \{a, b, c, d\}$ with the multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $B = \{a, b\}$. Then

$$\begin{aligned} BaB \cap B &= \{a\} \cap \{a, b\} = \{a\} \neq \emptyset. \\ BbB \cap B &= \{a\} \cap \{a, b\} = \{a\} \neq \emptyset. \\ BcB \cap B &= \{a\} \cap \{a, b\} = \{a\} \neq \emptyset. \\ BdB \cap B &= \{a\} \cap \{a, b\} = \{a\} \neq \emptyset. \end{aligned}$$

Therefore B is an almost bi-ideal of S.

Next, we give some properties of almost bi-ideals of semigroups.

Remark 2.7. *Every bi-ideal of a semigroup S is an almost bi-ideal of S.*

Proof. Assume that B is a bi-ideal of a semigroup S. Then $BsB \neq \emptyset$ and $BsB \subseteq BSB \subseteq B$ for all $s \in S$. Hence $BsB \cap B = BsB \neq \emptyset$.
Therefore B is an almost bi-ideal of S. □

Remark 2.8. *Let B be an almost bi-ideal of a semigroup S. Let B' be a non-empty subset of S such that $B \subseteq B' \subseteq S$. Then B' is an almost bi-ideal of S.*

Proof. Let B be an almost bi-ideal of a semigroup S such that $B \subseteq B' \subseteq S$. Then $\emptyset \neq BsB \cap B \subseteq B'sB' \cap B'$ for all $s \in S$. Thus B' is an almost bi-ideal of S. □

Remark 2.9. *The union of two almost bi-ideals of a semigroup S is an almost bi-ideal of S.*

Proof. Use Remark 2.8 □

Example 2.4. Consider a semigroup $(\mathbb{Z}_5, +)$. We have $B_1 = \{\overline{1}, \overline{3}, \overline{4}\}$ and $B_2 = \{\overline{1}, \overline{2}, \overline{4}\}$ are almost bi-ideals but $B = B_1 \cap B_2 = \{\overline{1}, \overline{4}\}$ is not an almost bi-ideal of $\mathbb{Z}_5$ because $B + \overline{0} + B = \{\overline{0}, \overline{2}, \overline{3}\}$. So, in general, the intersection of two almost bi-ideals need not be an almost bi-ideal.

3 Results

3.1 Fuzzy almost bi-ideals

In this subsection, we define fuzzy almost bi-ideals in semigroups and give some relationship between almost bi-ideals and fuzzy almost bi-ideals of semigroups.

Definition 3.1. Let f be a fuzzy subset of a semigroup S such that $f \neq 0$. f is called a *fuzzy almost bi-ideal* of S if for all $s \in S, (f \circ C_s \circ f) \cap f \neq 0$.

Theorem 3.2. *Let f be a fuzzy almost bi-ideal of a semigroup S and g be a fuzzy subset of S such that $f \subseteq g$. Then g is a fuzzy almost bi-ideal of S.*

Proof. Assume that f is a fuzzy almost bi-ideal of a semigroup S and g is a fuzzy subset of S such that $f \subseteq g$. Then for all $s \in S, (f \circ C_s \circ f) \cap f \subseteq (g \circ C_s \circ g) \cap g$ and $(f \circ C_s \circ f) \cap f \neq 0$. This implies $(g \circ C_s \circ g) \cap g \neq 0$ for all $s \in S$. Therefore g is a fuzzy almost bi-ideal. □

Corollary 3.3. *Let f and g be a fuzzy almost bi-ideal of a semigroup S. Then $f \cup g$ is a fuzzy almost bi-ideal of S.*

Proof. Since $f \subseteq f \cup g$, by Theorem 3.2, $f \cup g$ is a fuzzy almost bi-ideal of S. □

Example 3.1. Consider the semigroup $\mathbb{Z}_5$ under the usual addition. Let $f : \mathbb{Z}_5 \to [0, 1]$ be defined by $f(\overline{0}) = 0, f(\overline{1}) = 0.5, f(\overline{2}) = 0, f(\overline{3}) = 0.1, f(\overline{4}) = 0.1$ and $g : \mathbb{Z}_5 \to [0, 1]$ be defined by $g(\overline{0}) = 0, g(\overline{1}) = 0.2, g(\overline{2}) = 0.1, g(\overline{3}) = 0$ and $g(\overline{4}) = 0.2$. We have f and g are fuzzy almost bi-ideals of $\mathbb{Z}_5$ but $f \cap g$ is not a fuzzy almost bi-ideal of $\mathbb{Z}_5$.

Theorem 3.4. *Let B be a non-empty subset of a semigroup S. Then B is an almost bi-ideal of S if and only if C_B is a fuzzy almost bi-ideal of S.*

Proof. Assume that B is an almost bi-ideal of a semigroup S. Then $BsB \cap B \neq \emptyset$ for all $s \in S$. Thus there exists $x \in BsB$ and $x \in B$. So $(C_B \circ C_s \circ C_B)(x) = 1$ and $C_B(x) = 1$. Hence $(C_B \circ C_s \circ C_B) \cap C_B \neq 0$ for all $s \in S$. Therefore C_B is a fuzzy almost bi-ideal of S.

Conversely, assume that C_B is a fuzzy almost bi-ideal of S. Let $s \in S$. Then $(C_B \circ C_s \circ C_B) \cap C_B \neq 0$. Then there exists $x \in S$ such that $[(C_B \circ C_s \circ C_B) \cap C_B](x) \neq 0$. Hence $x \in BsB \cap B$. So $BsB \cap B \neq \emptyset$ for all $s \in S$. Consequently, B is an almost bi-ideal of S. □

Theorem 3.5. *Let f be a fuzzy subset of a semigroup S. Then f is a fuzzy almost bi-ideal of S if and only if $\operatorname{supp} f$ is an almost bi-ideal of S.*

Proof. Assume that f is a fuzzy almost bi-ideal of a semigroup S. Let $s \in S$. Then $(f \circ C_s \circ f) \cap f \neq 0$. Hence for each $s \in S$, there exists $x \in S$ such that $[(f \circ C_s \circ f) \cap f](x) \neq 0$. So there exist $y_1, y_2 \in S$ such that

$x = y_1sy_2, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. That is $x, y_1, y_2 \in supp\, f$. Thus $(C_{supp\, f} \circ C_s \circ C_{supp\, f})(x) \neq 0$ and $C_{supp\, f}(x) \neq 0$. Therefore $(C_{supp\, f} \circ C_s \circ C_{supp\, f}) \cap C_{supp\, f} \neq 0$. Hence $C_{supp\, f}$ is a fuzzy almost bi-ideal of S. By Theorem 3.4, $supp\, f$ is an almost bi-ideal of S.

Conversely, assume that $supp\, f$ is an almost bi-ideal of S. By Theorem 3.4, $C_{supp\, f}$ is a fuzzy almost bi-ideal of S. Then $(C_{supp\, f} \circ C_s \circ C_{supp\, f}) \cap C_{supp\, f} \neq 0$ for all $s \in S$. Then there exists $x \in S$ such that $[(C_{supp\, f} \circ C_s \circ C_{supp\, f}) \cap C_{supp\, f}](x) \neq 0$. Hence $(C_{supp\, f} \circ C_s \circ C_{supp\, f})(x) \neq 0$ and $C_{supp\, f}(x) \neq 0$. Then there exist $y_1, y_2 \in S$ such that $x = y_1sy_2, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. This means $(f \circ C_s \circ f) \cap f \neq 0$. Therefore f is a fuzzy almost bi-ideal of S. □

3.2 Minimal fuzzy almost bi-ideals

In this subsection, we define minimal fuzzy almost bi-ideals in semigroups and give some relationship between minimal almost bi-ideals and minimal fuzzy almost bi-ideals of semigroups.

Definition 3.6. A fuzzy almost bi-ideal f is called *minimal* if for each fuzzy almost bi-ideal g of S such that $g \subseteq f$, we have $supp\, g = supp\, f$.

Theorem 3.7. *Let B be a non-empty subset of a semigroup S. Then B is a minimal almost bi-ideal of S if and only if C_B is a minimal fuzzy almost bi-ideal of S.*

Proof. Assume that B is a minimal almost bi-ideal of a semigroup S. By Theorem 3.4, C_B is a fuzzy almost bi-ideal of S. Let g be a fuzzy almost bi-ideal of S such that $g \subseteq C_B$. Then $supp\ g \subseteq supp\ C_B = B$. Since $g \subseteq C_{supp\, g}$, we have $(g \circ C_B \circ g) \cap g \subseteq (C_{supp\, g} \circ C_B \circ C_{supp\, g}) \cap C_{supp\, g}$. Thus $C_{supp\, g}$ is a fuzzy almost bi-ideal of S. By Theorem 3.4, $supp\, g$ is an almost bi-ideal of S. Since B is minimal, $supp\ g = B = supp\ C_B$. Therefore C_B is minimal.

Conversely, assume that C_B is a minimal fuzzy almost bi-ideal of S. Let B' be an almost bi-ideal of S such that $B' \subseteq B$. Then $C_{B'}$ is a fuzzy almost bi-ideal of S such that $C_{B'} \subseteq C_B$. Hence $B' = supp\ C_{B'} = supp\ C_B = B$. Therefore B is minimal. □

Corollary 3.8. *A semigroup S has no proper almost bi-ideal of S if and only if for all fuzzy almost bi-ideal f of S, $supp f = S$.*

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