On certain new applications of quasi-power increasing sequences

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Abstract

In this paper, we generalize a known theorem dealing with the absolute Cesàro summability factors of infinite series. Some new and known results are also obtained.

1 Introduction

A positive sequence \(X = (X_n)\) is said to be a quasi-\(f\)-power increasing sequence if there exists a constant \(K = K(X, f) \geq 1\) such that \(K f_n X_n \geq f_m X_m\) for all \(n \geq m \geq 1\), where \(f = (f_n) = \{n^\sigma (\log n)^\eta, \eta \geq 0, 0 < \sigma < 1\}\) (see [14]). If we take \(\eta=0\), then we get a quasi-\(\sigma\)-power increasing sequence (see [13]). For any sequence \((\lambda_n)\) we write that \(\Delta \lambda_n = \lambda_n - \lambda_{n+1}\). The sequence \((\lambda_n)\) is said to be of bounded variation, denoted by \((\lambda_n) \in BV\), if \(\sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty\).

Let \(\sum a_n\) be a given infinite series. We denote by \(t_n^{\alpha, \beta}\) the \(n\)th Cesàro mean of order \((\alpha, \beta)\), with \(\alpha + \beta > -1\), of the sequence \((na_n)\), that is (see [9])

\[
t_n^{\alpha, \beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1}
\]

where

\[
A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1, \quad \text{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad \text{for} \quad n > 0. \tag{2}
\]

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Let \((u_n^{\alpha,\beta})\) be a sequence defined by (see [1])

\[
u_n^{\alpha,\beta} = \begin{cases} 
|t_n^{\alpha,\beta}|, & \alpha = 1, \beta > -1 \\
\max_{1 \leq v \leq n} |t_v^{\alpha,\beta}|, & 0 < \alpha < 1, \beta > -1. 
\end{cases}
\] (3)

A series \(\sum a_n\) is said to be summable \(|C, \alpha, \gamma, \beta; \delta|_k\), \(k \geq 1\), \(\delta \geq 0\), \(\alpha + \beta > -1\), and \(\gamma \in \mathbb{R}\), if (see [2])

\[
\sum_{n=1}^{\infty} n^{\gamma(\delta k + k - 1)} \frac{|t_n^{\alpha,\beta}|^k}{n^k} < \infty.
\] (4)

If we take \(\gamma = 1\), then the \(|C, \alpha, \beta, \gamma; \delta|_k\) summability reduces to \(|C, \alpha, \beta; \delta|_k\) summability (see [3]). If we set \(\gamma = 1\) and \(\delta = 0\), then we obtain the \(|C, \alpha, \beta|_k\) summability (see [10]). Also, if we take \(\beta = 0\), then we have \(|C, \alpha, \gamma; \delta|_k\) summability (see [16]). Furthermore, if we take \(\gamma = 1\), \(\beta = 0\), and \(\delta = 0\), then we get \(|C, \alpha|_k\) summability (see [11]). Finally, if we take \(\gamma = 1\) and \(\beta = 0\), then we get \(|C, \alpha; \delta|_k\) summability (see [12]).

2. The known results. The following theorems are known dealing with \(|C, \alpha, \gamma; \delta|_k\) summability factors of infinite series.

Theorem A ([6]). Let \((\lambda_n) \in \mathcal{B}\nu\) and let \((X_n)\) be a quasi-f-power increasing sequence for some \(\sigma\) \((0 < \sigma < 1)\) and \(\eta \geq 0\). Suppose also that there exist sequences \((\kappa_n)\) and \((\lambda_n)\) such that

\[
|\Delta \lambda_n| \leq \kappa_n
\] (5)

\[
\kappa_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\] (6)

\[
\sum_{n=1}^{\infty} n |\Delta \kappa_n| X_n < \infty
\] (7)

\[
|\lambda_n| X_n = O(1) \quad \text{as} \quad n \rightarrow \infty.
\] (8)

If the condition

\[
\sum_{n=1}^{m} n^{\gamma(\delta k + k - 1)} \frac{(u_n^{\alpha,\beta})^k}{n^k} = O(X_m) \quad \text{as} \quad m \rightarrow \infty
\] (9)

holds, then the series \(\sum a_n \lambda_n\) is summable \(|C, \alpha, \gamma; \delta|_k\), \(k \geq 1\), \(0 \leq \delta < \alpha \leq 1\), \(\gamma \in \mathbb{R}\), and \(\{k + \alpha k - \gamma(\delta k + k - 1)\} > 1\). If we set \(\eta = 0\), then we get a known result dealing with an application of quasi-\(\sigma\)-power increasing sequences (see [4]).

Theorem B ([7]). Let \((X_n)\) be a quasi-f-power increasing sequence for some
σ (0 < σ < 1) and η ≥ 0. Suppose also that there exist sequences (κ_n) and (λ_n) such that the conditions (5)-(8) are satisfied. If the condition
\[
\sum_{n=1}^{m} \eta^{(δk+k-1)} \left( \frac{u_n^α}{n^k} \right)^k = O(X_m) \quad \text{as} \quad m \to \infty
\] (10)
holds, then the series \( \sum a_nλ_n \) is summable \( | C, α, γ; δ |, k ≥ 1, 0 ≤ δ < α ≤ 1, γ ∈ R, \) and \{αk − γ(δk + k − 1)\} > 0.

**Remark.** It should be noted that condition (10) is the same as condition (9) when k=1. When k > 1, condition (10) is weaker than condition (9) but the converse is not true (see [7,15]). Also, it should be noted that the condition “(λ_n) ∈ BV” has been removed.

3. **The main result.** The aim of this paper is to generalize Theorem B for the \( | C, α, β, γ; δ | \) summability. Now, we shall prove the following theorem.

**Theorem.** Let \( (X_n) \) be a quasi-f-power increasing sequence for some \( σ \) (0 < σ < 1) and η ≥ 0. Suppose also that there exist sequences (κ_n) and (λ_n) such that the conditions (5)-(8) are satisfied. If the condition
\[
\sum_{n=1}^{m} \eta^{(δk+k-1)} \left( \frac{u_n^{α,β}}{n^k} \right)^k = O(X_m) \quad \text{as} \quad m \to \infty
\] (11)
satisfies, then the series \( \sum a_nλ_n \) is summable \( | C, α, β, γ; δ |, k ≥ 1, 0 ≤ δ < α ≤ 1, γ ∈ R, (α + β)k − γ(δk + k − 1) > 0. \)

We need the following lemmas for the proof of our theorem.

**Lemma ([1]).** If \( 0 < α ≤ 1, \ β > −1, \) and \( 1 ≤ v ≤ n, \) then
\[
| \sum_{p=0}^{v} A_{n-p}^{α-1}A_p^{β}a_p | ≤ \max_{1≤m≤v} | \sum_{p=0}^{m} A_{m-p}^{α-1}A_p^{β}a_p | . \] (12)

**Lemma 2 ([5]).** Under the conditions on \( (X_n), (κ_n) \) and \( (λ_n) \) as expressed in the statement of the theorem, then we have the following;
\[
nX_nκ_n = O(1) \quad \text{as} \quad n \to \infty, \]
\[
\sum_{n=1}^{∞} κ_nX_n < \infty. \] (14)

4. **Proof of the theorem.** Let \( (T_n^{α,β}) \) be the nth \( (C, α, β) \) mean of the sequence \( (na_nλ_n) \). Then, by (1), we have
\[
T_n^{α,β} = \frac{1}{A_n^{α+β}} \sum_{v=1}^{n} A_{n-v}^{α-1}A_v^{β}va_vλ_v. \]
First, applying Abel’s transformation and then using Lemma 1, we have that

\[
T_{n}^{\alpha,\beta} = \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta \lambda_{v} \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_{p}^{\beta} p a_{p} + \frac{\lambda_{n}}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_{v}^{\beta} v a_{v},
\]

\[
|T_{n}^{\alpha,\beta}| \leq \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n-1} |\Delta \lambda_{v}| |\sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_{p}^{\beta} p a_{p}| + \frac{|\lambda_{n}|}{A_{n}^{\alpha+\beta}} |\sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_{v}^{\beta} v a_{v}| \leq \frac{1}{A_{n}^{\alpha+\beta}} \sum_{v=1}^{n-1} A_{v}^{(\alpha+\beta)} u_{v}^{\alpha,\beta} |\Delta \lambda_{v}| + |\lambda_{n}| u_{n}^{\alpha,\beta} = T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}.
\]

To complete the proof of the theorem, by Minkowski’s inequality, it is enough to show that

\[
\sum_{n=1}^{\infty} n^{\gamma (\delta k+k-1)-k} |T_{n,r}^{\alpha,\beta}| < \infty, \quad \text{for} \quad r = 1, 2.
\]  

(15)

Whenever \( k > 1 \), we can apply Hölder’s inequality with indices \( k \) and \( k' \), where \( \frac{1}{k} + \frac{1}{k'} = 1 \),
we get that

\[
\sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-k} | T_{n,1}^{\alpha,\beta} |^k \leq \sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-k} (A_n^{\alpha+\beta} - k) \left\{ \sum_{v=1}^{n-1} (A_v^{\alpha+\beta})^k (u_v^{\alpha,\beta})^k | \Delta \lambda_v |^k \right\} \\
\times \left\{ \sum_{v=1}^{n-1} 1 \right\}^{k-1} \\
= O(1) \sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-1-(\alpha+\beta)k} \left\{ \sum_{v=1}^{n-1} v^{(\alpha+\beta)k} (u_v^{\alpha,\beta})^k k_v \right\} \\
= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (u_v^{\alpha,\beta})^k k_v \left( \sum_{n=v+1}^{m+1} \frac{1}{n^{1+(\alpha+\beta)k-\gamma(\delta k+k-1)}} \right) \\
= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (u_v^{\alpha,\beta})^k k_v \int_{v}^{\infty} \frac{dx}{x^{1+(\alpha+\beta)k-\gamma(\delta k+k-1)}} \\
= O(1) \sum_{v=1}^{m} (u_v^{\alpha,\beta})^k k_v \left( \frac{1}{vX_v} \right)^{k-1} u^{\gamma(\delta k+k-1)} \\
= O(1) \sum_{v=1}^{m} v^{\gamma(\delta k+k-1)} \left( \frac{u_v^{\alpha,\beta}}{v^k X_v^{k-1}} \right) \\
+ O(1)m\kappa_m \sum_{v=1}^{m-1} v^{\gamma(\delta k+k-1)} \left( \frac{u_v^{\alpha,\beta}}{v^k X_v^{k-1}} \right) \\
= O(1) \sum_{v=1}^{m-1} | \Delta (v\kappa_v) | \left| X_v \right| + O(1)m\kappa_m X_m \\
= O(1) \sum_{v=1}^{m-1} (v+1) \Delta \kappa_v - \kappa_v \left| X_v \right| + O(1)m\kappa_m X_m \\
= O(1) \sum_{v=1}^{m-1} v \left| \Delta \kappa_v \right| \left| X_v \right| + O(1) \sum_{v=1}^{m-1} \kappa_v X_v + O(1)m\kappa_m X_m \\
= O(1) \text{ as } m \to \infty,
\]
by virtue of the hypotheses of the theorem and Lemma 2. Finally, we have that

\[ \sum_{n=1}^{m} n^{\gamma(\delta k + k - 1) - k} | T_{n,2}^{\alpha,\beta} | k = \sum_{n=1}^{m} | \lambda_n | k - 1 | \lambda_n | n^{\gamma(\delta k + k - 1)} \frac{(u_n^{\alpha,\beta})^k}{n^k} \]

\[ = O(1) \sum_{n=1}^{m} | \lambda_n | n^{\gamma(\delta k + k - 1)} \frac{(u_n^{\alpha,\beta})^k}{n^k X_n^{k-1}} \]

\[ = O(1) \sum_{n=1}^{m} \Delta | \lambda_n | \sum_{v=1}^{n} v^{\gamma(\delta k + k - 1)} \frac{(u_v^{\alpha,\beta})^k}{v^k X_v^{k-1}} \]

\[ + O(1) | \lambda_m | \sum_{n=1}^{m-1} n^{\gamma(\delta k + k - 1)} \frac{(u_n^{\alpha,\beta})^k}{n^k X_n^{k-1}} \]

\[ = O(1) \sum_{n=1}^{m-1} \Delta \lambda_n | X_n + O(1) | \lambda_m | X_m \]

\[ = O(1) \sum_{n=1}^{m-1} \kappa_n X_n + O(1) | \lambda_m | X_m = O(1) \text{ as } m \to \infty, \]

by virtue of the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem.

5. Conclusions. If we take \( \beta = 0 \), then we obtain Theorem B. If we set \( \gamma = 1 \), then we obtain a known result under weaker conditions (see [8]). If we set \( \gamma = 1 \) and \( \delta = 0 \), then we get a new result dealing with \( | C, \alpha, \beta |_k \) summability factors. If we take \( \gamma = 1 \), then we obtain a new result concerning the \( | C, \alpha, \beta; \delta |_k \) summability factors. If we take \( \gamma = 1 \) and \( \beta = 0 \), then we have a new result dealing with \( | C, \alpha; \delta |_k \) summability factors of infinite series. Furthermore if we take \( \eta = 0 \) and \( \beta = 0 \), then we obtain Theorem A under weaker conditions. Finally, if we take \( \eta = 0 \), then we get a new result dealing with an application of quasi-\( \sigma \)-power increasing sequences.
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References


