International Journal of Mathematics and
Computer Science, **13**(2018), no. 1, 1–8

On certain new applications of quasi-power increasing sequences

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(Received September 27, 2017, Accepted October 31, 2017)

Abstract

In this paper, we generalize a known theorem dealing with the absolute Cesàro summability factors of infinite series. Some new and known results are also obtained.

1 Introduction

A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = (f_n) = \{n^\sigma (\log n)^\eta, \eta \geq 0, 0 < \sigma < 1\}$ (see [14]). If we take η=0, then we get a quasi-σ-power increasing sequence (see [13]). For any sequence (λ_n) we write that $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. The sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty$. Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α, β), with $\alpha + \beta > -1$, of the sequence (na_n), that is (see [9])

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1, \quad \texttt{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad \texttt{for} \quad n > 0. \tag{2}$$

Key words and phrases: Cesàro mean, absolute summability factor, power increasing sequence, infinite series, sequence space, Minkowski inequality, Hölder inequality.
AMS (MOS) Subject Classifications: 26D15, 40D15, 40F05, 40G99, 46A45.
ISSN 1814-0432, 2018, http://ijmcs.future-in-tech.net

Let $(u_n^{\alpha,\beta})$ be a sequence defined by (see [1])

$$u_n^{\alpha,\beta} = \begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha = 1, \beta > -1 \\ \max_{1\leq v\leq n}\left|t_v^{\alpha,\beta}\right|, & 0 < \alpha < 1, \beta > -1. \end{cases} \tag{3}$$

A series $\sum a_n$ is said to be summable $\mid C, \alpha, \gamma, \beta; \delta \mid_k$, $k \geq 1$, $\delta \geq 0$, $\alpha + \beta > -1$, and $\gamma \in R$,
if (see [2])

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)} \frac{\mid t_n^{\alpha,\beta} \mid^k}{n^k} < \infty. \tag{4}$$

If we take $\gamma = 1$, then the $\mid C, \alpha, \beta, \gamma; \delta \mid_k$ summability reduces to $\mid C, \alpha, \beta; \delta \mid_k$ summability (see [3]). If we set $\gamma = 1$ and $\delta = 0$, then we obtain the $\mid C, \alpha, \beta \mid_k$ summability (see [10]). Also, if we take $\beta = 0$, then we have $\mid C, \alpha, \gamma; \delta \mid_k$ summability (see [16]). Furthermore, if we take $\gamma = 1$, $\beta = 0$, and $\delta = 0$, then we get $\mid C, \alpha \mid_k$ summability (see [11]). Finally, if we take $\gamma = 1$ and $\beta = 0$, then we get $\mid C, \alpha; \delta \mid_k$ summability (see [12]).

2. The known results. The following theorems are known dealing with $\mid C, \alpha, \gamma; \delta \mid_k$ summability factors of infinite series.

Theorem A ([6]). Let $(\lambda_n) \in \mathcal{BV}$ and let (X_n) be a quasi-f-power increasing sequence for some σ $(0 < \sigma < 1)$ and $\eta \geq 0$. Suppose also that there exist sequences (κ_n) and (λ_n) such that

$$\mid \Delta\lambda_n \mid \leq \kappa_n \tag{5}$$

$$\kappa_n \to 0 \quad as \quad n \to \infty \tag{6}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\kappa_n \mid X_n < \infty \tag{7}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{8}$$

If the condition

$$\sum_{n=1}^{m} n^{\gamma(\delta k+k-1)} \frac{(u_n^{\alpha})^k}{n^k} = O(X_m) \quad as \quad m \to \infty \tag{9}$$

holds, then the series $\sum a_n\lambda_n$ is summable $\mid C, \alpha, \gamma; \delta \mid_k$, $k \geq 1$, $0 \leq \delta < \alpha \leq 1$,$\gamma \in R$, and $\{k + \alpha k - \gamma(\delta k + k - 1)\} > 1$.

If we set $\eta = 0$, then we get a known result dealing with an application of quasi-σ-power increasing sequences (see [4]).

Theorem B ([7]). Let (X_n) be a quasi-f-power increasing sequence for some

σ $(0 < \sigma < 1)$ and $\eta \geq 0$. Suppose also that there exist sequences (κ_n) and (λ_n) such that the conditions (5)-(8) are satisfied. If the condition

$$\sum_{n=1}^{m} n^{\gamma(\delta k+k-1)} \frac{(u_n^{\alpha})^k}{n^k \ X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty \tag{10}$$

holds, then the series $\sum a_n \lambda_n$ is summable $\mid C, \alpha, \gamma; \delta \mid_k$, $k \geq 1$, $0 \leq \delta < \alpha \leq 1$, $\gamma \in R$, and $\{\alpha k - \gamma(\delta k + k - 1)\} > 0$.

Remark. It should be noted that condition (10) is the same as condition (9) when k=1. When $k > 1$, condition (10) is weaker than condition (9) but the converse is not true (see [7,15]). Also, it should be noted that the condition "$(\lambda_n) \in \mathcal{BV}$ " has been removed.

3. The main result. The aim of this paper is to generalize Theorem B for the $\mid C, \alpha, \beta, \gamma; \delta \mid_k$ summability. Now, we shall prove the following theorem.

Theorem. Let (X_n) be a quasi-f-power increasing sequence for some σ $(0 < \sigma < 1)$ and $\eta \geq 0$. Suppose also that there exist sequences (κ_n) and (λ_n) such that the conditions (5)-(8) are satisfied. If the condition

$$\sum_{n=1}^{m} n^{\gamma(\delta k+k-1)} \frac{(u_n^{\alpha,\beta})^k}{n^k \ X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty \tag{11}$$

satisfies, then the series $\sum a_n \lambda_n$ is summable $\mid C, \alpha, \beta, \gamma; \delta \mid_k$, $k \geq 1$, $0 \leq \delta < \alpha \leq 1$, $\gamma \in R$, and $(\alpha + \beta)k - \gamma(\delta k + k - 1) > 0$.

We need the following lemmas for the proof of our theorem.

Lemma ([1]). If $0 < \alpha \leq 1$, $\beta > -1$, and $1 \leq v \leq n$, then

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \mid \leq \max_{1 \leq m \leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \mid . \tag{12}$$

Lemma 2 ([5]). Under the conditions on (X_n), (κ_n) and (λ_n) as expressed in the statement of the theorem, then we have the following;

$$n X_n \kappa_n = O(1) \quad as \quad n \to \infty, \tag{13}$$

$$\sum_{n=1}^{\infty} \kappa_n X_n < \infty. \tag{14}$$

4. Proof of the theorem. Let $(T_n^{\alpha,\beta})$ be the nth (C, α, β) mean of the sequence $(n a_n \lambda_n)$. Then, by (1), we have

$$T_n^{\alpha,\beta} \quad = \quad \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

First, applying Abel's transformation and then using Lemma 1, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p+\frac{\lambda_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v,$$

$$\begin{aligned}\mid T_n^{\alpha,\beta}\mid &\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\mid\Delta\lambda_v\mid\mid\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p\mid+\frac{\mid\lambda_n\mid}{A_n^{\alpha+\beta}}\mid\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\mid\\ &\leq \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{(\alpha+\beta)}u_v^{\alpha,\beta}\mid\Delta\lambda_v\mid+\mid\lambda_n\mid u_n^{\alpha,\beta}=T_{n,1}^{\alpha,\beta}+T_{n,2}^{\alpha,\beta}.\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is enough to show that

$$\sum_{n=1}^{\infty}n^{\gamma(\delta k+k-1)-k}\mid T_{n,r}^{\alpha,\beta}\mid^k<\infty,\quad \texttt{for}\quad r=1,2. \tag{15}$$

Whenever $k>1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-k} \mid T_{n,1}^{\alpha,\beta} \mid^k &\leq \sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-k} (A_n^{\alpha+\beta})^{-k} \{\sum_{v=1}^{n-1} (A_v^{\alpha+\beta})^k (u_v^{\alpha,\beta})^k \mid \Delta\lambda_v \mid^k\} \\
&\times \{\sum_{v=1}^{n-1} 1\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-1-(\alpha+\beta)k} \{\sum_{v=1}^{n-1} v^{(\alpha+\beta)k} (u_v^{\alpha,\beta})^k \kappa_v^k\} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (u_v^{\alpha,\beta})^k \kappa_v^k \sum_{n=v+1}^{m+1} \frac{1}{n^{1+(\alpha+\beta)k-\gamma(\delta k+k-1)}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (u_v^{\alpha,\beta})^k \kappa_v^k \int_v^\infty \frac{dx}{x^{1+(\alpha+\beta)k-\gamma(\delta k+k-1)}} \\
&= O(1) \sum_{v=1}^{m} (u_v^{\alpha,\beta})^k \kappa_v \kappa_v^{k-1} v^{\gamma(\delta k+k-1)} \\
&= O(1) \sum_{v=1}^{m} (u_v^{\alpha,\beta})^k \kappa_v \left(\frac{1}{vX_v}\right)^{k-1} v^{\gamma(\delta k+k-1)} \\
&= O(1) \sum_{v=1}^{m} v\kappa_v v^{\gamma(\delta k+k-1)} \frac{(u_v^{\alpha,\beta})^k}{v^k X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\kappa_v) \sum_{r=1}^{v} r^{\gamma(\delta k+k-1)} \frac{(u_r^{\alpha,\beta})^k}{r^k X_r^{k-1}} \\
&+ O(1) m\kappa_m \sum_{v=1}^{m} v^{\gamma(\delta k+k-1)} \frac{(u_v^{\alpha,\beta})^k}{v^k X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta(v\kappa_v) \mid X_v + O(1) m\kappa_m X_m \\
&= O(1) \sum_{v=1}^{m-1} \mid (v+1)\Delta\kappa_v - \kappa_v \mid X_v + O(1) m\kappa_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta\kappa_v \mid X_v + O(1) \sum_{v=1}^{m-1} \kappa_v X_v + O(1) m\kappa_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 2 . Finally, we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{\gamma(\delta k+k-1)-k} \mid T_{n,2}^{\alpha,\beta} \mid^k &= \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid n^{\gamma(\delta k+k-1)} \frac{(u_n^{\alpha,\beta})^k}{n^k} \\
&= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid n^{\gamma(\delta k+k-1)} \frac{(u_n^{\alpha,\beta})^k}{n^k X_n^{k-1}} \\
&= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{\gamma(\delta k+k-1)} \frac{(u_v^{\alpha,\beta})^k}{v^k X_v^{k-1}} \\
&+ O(1) \mid \lambda_m \mid \sum_{n=1}^{m} n^{\gamma(\delta k+k-1)} \frac{(u_n^{\alpha,\beta})^k}{n^k X_n^{k-1}} \\
&= O(1) \sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1) \sum_{n=1}^{m-1} \kappa_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem.

5. Conclusions. If we take $\beta = 0$, then we obtain Theorem B. If we set γ=1, then we obtain a known result under weaker conditions (see [8]). If we set $\gamma = 1$ and $\delta = 0$, then we get a new result dealing with $\mid C, \alpha, \beta \mid_k$ summability factors. If we take $\gamma = 1$, then we obtain a new result concerning the $\mid C, \alpha, \beta; \delta \mid_k$ summability factors. If we take $\gamma = 1$ and $\beta = 0$, then we have a new result dealing with $\mid C, \alpha; \delta \mid_k$ summability factors of infinite series. Furthermore if we take $\eta = 0$ and $\beta = 0$, then we obtain Theorem A under weaker conditions. Finally, if we take $\eta = 0$, then we get a new result dealing with an application of quasi-σ-power increasing sequences.

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