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The Full Likelihood Function of a Linear Stationary Gaussian Process

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Abstract

Computation of the full likelihood function of a Gaussian linear stationary process is considered. An iterative blocking method to develop the inverse of the covariance matrix is derived. This allows for the computation of the full likelihood when observations are missing or taken at irregular time intervals. we conclude by showing that the proposed method is a viable procedure both as an analytic or numerical matrix inversion techniques.

1 Introduction

Let $\{X_t\}$, where $t \in Z$, be a zero mean stationary Gaussian process. The autocovariance function $\gamma(k) = E(X_t X_{t-k})$ for $k = 0, 1, 2, \cdots$. Based on a realization of size n from the process $\{X_t\}$, $\underline{X}' = (X_1, \cdots, X_n)$, The likelihood function, $L(\underline{X}', \Sigma)$ is given by the following equation:

$$L(\underline{X}, \Sigma) = (2\pi)^{-n/2}(det(\Sigma))^{-1/2} \exp -\frac{1}{2}\underline{X}\Sigma^{-1}\underline{X} \quad (1)$$

Where $\underline{X}'$ denotes the transpose of vector $\underline{X}$, $det(\Sigma)$ is the determinant of the covariance matrix, Σ which is given by

$$\Sigma = \mathrm{T}oeplitz[\gamma(0), \cdots, \gamma(n-1)] \quad (2)$$

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Box and Jenkins (1976) popularized the class of linear stationary Autoregressive Moving Average of order (p, q), ARMA(p, q), which is defined as in the following difference equation:

$$X_t - \phi_1 X_{t-1} - \phi_p X_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} \tag{3}$$

Where a_t, if $t \in Z$, is a sequence of independent Normal random variables with zero mean and an unknown constant variance σ^2. $\{X_t\}$ is said to be causal if The following two characteristic polynomials:

$$\begin{aligned} \Phi(B) &= 1 - \phi_1 B - \cdots - \phi_p B^p \\ \Theta(B) &= 1 - \theta_1 B - \cdots - \theta_q B^q \end{aligned} \tag{4}$$

are relatively prime and have zeroes who lie outside the unit circle. Let $\Sigma_{p,q}$ be the covariance matrix of the stationary $ARMA(p, q)$ process given in Equation (2). The closed form of the inverse of the covariance matrix, $\Sigma_{p,q}^{-1}$ has been derived in Haddad (2004). However, the likelihood function given in Equation (1) maybe computed without deriving the inverse of the covariance function through state - space representation method see Brockwell and Davis (1987). A basic requirement to the computation of the full likelihood function is that the observations must be equally spaced. However, it may happen that some of the observations have gone missing or have got values that do not seem realistic and hence the corresponding covariance matrix becomes cumbersome to invert and consequently the likelihood function. Although, the likelihood function is still possible but one has to skip the step that corresponds to missing value in the State - Space method.

In this article we develop an iterative method of inversion of covariance matrix whether observations were taken equally or unequally spaced and hence the full likelihood of a Gaussian stationary process is determined. This is done in section 2. Section 3 has a generalization and concluding remarks.

2 The Inverse of the covariance matrix

Assume that the covariance matrix can be blocked into four blocks as the following:

$$\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The inverse of the blocked matrix becomes

$$\Sigma^{-1} = \begin{bmatrix} (A - CD^{-1}B)^{-1} & -A^{-1}B(D - BA^{-1})^{-1} \\ -CA^{-1}(D - CA^{-1}B)^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

A great deal of simplification occurs in the expression of Σ^{-1} if we choose submatrix D to be a scalar. This allows for the application of the Sherman Morrison (1950) on the 1×1 subblock of the inverse and hence the final form of the inverse becomes

$$\Sigma^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BCA^{-1}\nu & -A^{-1}B\nu \\ -CA^{-1}\nu & \nu \end{bmatrix}$$

where $\nu = (D - CA^{-1}B)^{-1}$.

As far as using the above to compute the inverse recursively, one may start with submatrix A to be the first diagonal entry and D is the second diagonal entry, and hence B and C follow accordingly, that is, $B = [1 \times 2 \text{ entry of } \Sigma]$ and $C = [2 \times 1 \text{ entry of } \Sigma]$. Updates of the first iteration are as the following:

1. $\nu = (D - CA^{-1}B)^{-1}$ for D
2. $-CA^{-1}\nu$ for C
3. $-A^{-1}B.\nu$ for B
4. $A^{-1} + A^{-1}BCA^{-1}\nu$ for A

This gives the inverse of the first 2×2 submatrix of Σ. moreover, the determinant of the 2×2 submatrix is given by $\Delta_2 = \Delta_1 \nu^{-1}$ with $\Delta_1 =$ the value of the first diagonal entry of Σ.

Now, the second iteration that starts with submatrix A to be the first 2×2 submatrix of Σ, where the inverse has just been computed, then D is the third diagonal entry and hence B and C follow accordingly. Updates are carried out in the sense of the sequence of the last iterative step which results in A^{-1}, the inverse of a 3×3 submatrix of Σ. Furthermore, the determinant of the 3×3 submatrix is given by $\Delta_3 = \Delta_2 \nu^{-1}$.

Continuing in this fashion till we reach the n^{th} diagonal entry and hence $(n-1)$ iterations a total number of steps to achieve the inverse . It should be noted here that at each iteration ν must be different than zero to ensure invertibility. Furthermore, the determinant, Δ_k, at the k^{th} iteration is given by $\Delta_{k-1}\nu^{-1}$. To illustrate the iterative blocking inversion method, we apply it to the covariance matrix of an AR(1) process with $n = 5$ and the second observation is missing. That is we need to invert the following correlation matrix:

$$\Omega(\phi) = \begin{bmatrix} 1 & \phi^2 & \phi^3 & \phi^4 \\ \phi^2 & 1 & \phi & \phi^2 \\ \phi^3 & \phi & 1 & \phi \\ \phi^4 & \phi^2 & \phi & 1 \end{bmatrix}$$

Iteration (1): $A = [1]$, $B = [\phi^2]$, $C = [\phi^2]$, and $D = [1]$
Updates: $\nu = (1 - \phi^4)^{-1}$,
$-CA^{-1}\nu = -\phi^2/(1 - \phi^4)$,
$-A^{-1}B\nu = -\phi^2/(1 - \phi^4)$, and
$A^{-1} + A^{-1}BCA^{-1}\nu = 1/(1 - \phi^4)$.
The result of the first iteration is:

$$A^{-1} = \begin{bmatrix} \frac{1}{1-\phi^4} & \frac{-\phi^2}{1-\phi^4} \\ \frac{-\phi^2}{1-\phi^4} & \frac{1}{1-\phi^4} \end{bmatrix}$$

and $\Delta_2 = 1 - \phi^4$
Iteration (2): $B' = C = [\phi^3, \phi]$, and $D = 1$,
Updates: $\nu = (D - CA^{-1}B)^{-1} = 1/(1 - \phi^2)$,
$-CA^{-1}\nu = [0, -\phi/(1 - \phi^2)]$,
$-A^{-1}B\nu = [0, -\phi/(1 - \phi^2)]'$, and

$$A^{-1} + A^{-1}BCA^{-1}\nu = \begin{bmatrix} \frac{1}{1-\phi^4} & \frac{-\phi^2}{1-\phi^4} \\ \frac{-\phi^2}{1-\phi^4} & \frac{1}{1-\phi^4} + \frac{\phi^2}{1-\phi^2} \end{bmatrix}$$

which results in

$$A^{-1} = \begin{bmatrix} \frac{1}{1-\phi^4} & \frac{-\phi^2}{1-\phi^4} & 0 \\ \frac{-\phi^2}{1-\phi^4} & \frac{1}{1-\phi^4} + \frac{\phi^2}{1-\phi^2} & \frac{-\phi}{1-\phi^2} \\ 0 & \frac{-\phi}{1-\phi^2} & \frac{1}{1-\phi^2} \end{bmatrix}$$

and $\Delta_3 = (1 - \phi^4)(1 - \phi^2)$
Iteration (3): $B' = C = [\phi^4, \phi^2, \phi]$, and $D = [1]$
Updates: $\nu = (D - CA^{-1}B)^{-1} = 1/(1 - \phi^2)$,
$-CA^{-1}\nu = [0, 0, -\phi/(1 - \phi^2)]$,
$-A^{-1}B\nu = [0, 0, -\phi/(1 - \phi^2)]'$, and

$$A^{-1} + A^{-1}BCA^{-1}\nu = \begin{bmatrix} \frac{1}{1-\phi^4} & \frac{-\phi^2}{1-\phi^4} & 0 \\ \frac{-\phi^2}{1-\phi^4} & \frac{1}{1-\phi^4} + \frac{\phi^2}{1-\phi^2} & \frac{-\phi}{1-\phi^2} \\ 0 & \frac{-\phi}{1-\phi^2} & \frac{1}{1-\phi^2} + \frac{\phi^2}{1-\phi^2} \end{bmatrix}$$

which results in

$$\Omega^{-1} = \begin{bmatrix} \frac{1}{1-\phi^4} & \frac{-\phi^2}{1-\phi^4} & 0 & 0 \\ \frac{-\phi^2}{1-\phi^4} & \frac{1}{1-\phi^4} + \frac{\phi^2}{1-\phi^2} & \frac{-\phi}{1-\phi^2} & 0 \\ 0 & \frac{-\phi}{1-\phi^2} & \frac{1+\phi^2}{1-\phi^2} & \frac{-\phi^2}{1-\phi^2} \\ 0 & 0 & \frac{-\phi}{1-\phi^2} & \frac{1}{1-\phi^2} \end{bmatrix}$$

and $\Delta_4 = (1-\phi^4)(1-\phi^2)^2$.

3 Discussion and Concluding remarks

The inversion method described can be used to invert any square matrix iteratively as long as A^{-1} and ν are both non zero and can serve as alternative to common methods such as Gaussian elimination method or Cofactor ratios. Moreover, one may question of the computational cost of conducting the iterative blocking method. It is $2k^2$ at the k^{th} iteration and hence the total cost for $n-1$ iterations becomes $O(n^3)$.

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