On Two Integrability Methods of Improper Integrals

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Abstract

In this paper, we introduce a new concept of $|\tilde{N}, p|_k, k \geq 1$ integrability of improper integrals. By using this definition we prove an analogous theorem due to Bor [H. Bor, On two summability methods, Math. Proc. Cambridge Philos. Soc., 97, 1985, 147–149] for improper integrals rather than infinite series.

1 Introduction

Throughout this paper we assume that $f$ is a real valued function which is continuous on $[0, \infty)$ and $s(x) = \int_0^x f(t)dt$. By $\sigma(x)$, we denote the Cesàro mean of $s(x)$. The integral $\int_0^\infty f(t)dt$ is said to be integrable $\mid C, 1 \mid_k, k \geq 1$, in the sense of Flett [4], if

$$\int_0^\infty x^{k-1} \mid \sigma'(x) \mid^k = \int_0^\infty \frac{|v(x)|^k}{x}$$

(1)

is convergent. Here, $v(x) = \frac{1}{x} \int_0^x tf(t)dt$ is called the generator of the integral $\int_0^\infty f(t)dt$.

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Let $p$ be a real valued, non-decreasing function on $[0, \infty)$ such that

$$P(x) = \int_0^x p(t) dt, p(x) \neq 0, p(0) = 0.$$  

The Riesz mean of $s(x)$ is defined by

$$\sigma_p(x) = \frac{1}{P(x)} \int_0^x p(t)s(t) dt.$$  

We say that the integral $\int_0^\infty f(t) dt$ is integrable $|\bar{N}, p |_k, k \geq 1$, if

$$\int_0^\infty \left( \frac{P(x)}{p(x)} \right)^{k-1} |\sigma'_p(x)|^k \ dx$$

is convergent. In Particular, if we take $p(x) = 1$ for all values of $x$, then $|\bar{N}, p |_k$ integrability reduces to $| C, 1 |_k$ integrability of improper integrals. Given any functions $f, g$, it is customary to write $g(x) = O(f(x))$, if there exist $\eta$ and $N$, for every $x > N$, $| \frac{g(x)}{f(x)} | \leq \eta$.

The difference between $s(x)$ and its $n$th weighted mean $\sigma_p(x)$, which is called the weighted Kronecker identity, is given by the identity

$$s(x) - \sigma_p(x) = v_p(x),$$

where

$$v_p(x) = \frac{1}{P(x)} \int_0^x P(u)f(u) du.$$  

We note that if we take $p(x) = 1$, for all values of $x$, then we have the following identity(see [3])

$$s(x) - \sigma(x) = v(x).$$

Since

$$\sigma'_p(x) = \frac{p(x)}{P(x)} v_p(x),$$

condition (3) can be rewritten as

$$s(x) = v_p(x) + \int_0^x \frac{p(u)}{P(u)} v_p(u) du.$$
In view of the identity (4), the function $v_p(x)$ is called the generator function of $s(x)$.

The absolute Nörlund summability of Fourier series and its allied series was studied by several workers. The corresponding problem of absolute summability of trigonometric integrals by the functional Nörlund methods up to the present was not been studied to the same degree. In fact, a paper of Boićun [1] is the first one which takes up the problem in a special direction. Next, in 1974, Lal [5] generalized Boićun’s result.

Lal and Ram [6], [7], Lal and Singh [8], [9] investigated several conditions for the absolute Nörlund summability of integrals associated with the Fourier integral of a function. Recently, the author ([10], [11]) established several theorems dealing with the absolute Cesàro and Riesz integrability of improper integrals, respectively. In what follows, we give the main theorem.

## 2 Main result

The aim of this paper is to prove the theorem due to Bor [2] for improper integrals. Now we can give the following theorem.

**Theorem 2.1.** Let $p$ be a real valued, non-decreasing functions on $[0, \infty)$ such that as $x \to \infty$ satisfying (2) and

$$ xp(x) = O(P(x)), \quad (5) $$

$$ P(x) = O(xp(x)). \quad (6) $$

If $\int_{0}^{\infty} f(t)dt$ is integrable $|C, 1 |_k$, then it is also integrable $|\bar{N}, p |_k, k \geq 1$.

## 3 Proof of the Theorem

Let $\sigma_p(x)$ denote the $(\bar{N}, p)$ means of the integral $\int_{0}^{\infty} f(t)dt$. Then we have

$$ \sigma_p(x) = \frac{1}{P(x)} \int_{0}^{x} p(t)s(t)dt. \quad (7) $$

Since the integral $\int_{0}^{\infty} f(t)dt$ is integrable $|C, 1 |_k$, we can write

$$ \int_{0}^{\infty} \frac{|v(x)|^k}{x} dx $$
is convergent, where \( v(x) \) is the generator function of \( s(x) \).

Differentiating the equation (7), we can write

\[
\sigma'_p(x) = \frac{p(x)}{P^2(x)} \int_0^x P(t)f(t)dt = \frac{p(x)}{P^2(x)} \int_0^x \frac{P(t)}{t}tf(t)dt.
\]

Integrating by parts of the second statement, we obtain

\[
\sigma'_p(x) = \frac{p(x)}{P^2(x)} v(x) - \frac{p(x)}{P^2(x)} \int_0^x \left( \frac{P(t)}{t} \right)' tv(t)dt
\]

\[
= \frac{p(x)}{P^2(x)} v(x) - \frac{p(x)}{P^2(x)} \int_0^x p(t)v(t)dt + \frac{p(x)}{P^2(x)} \int_0^x \frac{P(t)}{t}v(t)dt
\]

\[
= \sigma_{p,1}(x) + \sigma_{p,2}(x) + \sigma_{p,3}(x), \text{ say.}
\]

To complete the proof of the theorem, it is sufficient to show that

\[
\int_0^m \left( \frac{P(x)}{p(x)} \right)^{k-1} |\sigma_{p,r}(x)|^k dx = O(1) \text{ as } m \to \infty, \text{ for } r = 1, 2, 3.
\]

Using the condition (5), we have

\[
\int_0^m \left( \frac{P(x)}{p(x)} \right)^{k-1} |\sigma_{p,1}(x)|^k dx = \int_0^m \frac{p(x)}{P(x)} |v(x)|^k dx
\]

\[
= \int_0^m \frac{1}{x} |v(x)|^k dx
\]

\[
= O(1) \text{ as } m \to \infty
\]

by virtue of the hypotheses of the theorem.
Applying Hölder’s inequality with $k > 1$, we get

$$\int_0^m \left( \frac{P(x)}{p(x)} \right)^{k-1} |\sigma_{p,2}(x)|^k \, dx = O(1) \int_0^m \frac{p(x)dx}{P^{k+1}(x)} \left( \int_0^x p(t) \, |v(t)| \, dt \right)^k$$

$$= O(1) \int_0^m \frac{p(x)}{P(x)} \left( \int_0^x \frac{p(t) \, |v(t)|^k \, dt}{P(x)} \right)^{k-1}$$

$$= O(1) \int_0^m \frac{p(x)}{P(x)} \left( \int_0^x \frac{p(t)}{P^2(x)} \, dt \right)^k$$

$$= O(1) \int_0^m \frac{p(t)}{P(t)} \, |v(t)|^k \, dt$$

$$= O(1) \int_0^m \frac{|v(t)|^k}{t} \, dt$$

$$= O(1) \text{ as } m \to \infty$$

by virtue of the hypotheses of the theorem.

Finally, as in $\sigma_{p,2}(x)$, by (6), we have that

$$\int_0^m \left( \frac{P(x)}{p(x)} \right)^{k-1} |\sigma_{p,3}(x)|^k \, dx = O(1) \int_0^m \frac{p(x)dx}{P^{k+1}(x)} \left( \int_0^x \frac{P(t) \, |v(t)| \, dt}{t} \right)^k$$

$$= O(1) \int_0^m \frac{p(x)dx}{P^{k+1}(x)} \left( \int_0^x \frac{p(t)}{P^{k+1}(x)} \, dt \right)^k$$

$$= O(1) \text{ as } m \to \infty$$

by virtue of the hypotheses of the theorem.

So, we get

$$\int_0^\infty \left( \frac{P(x)}{p(x)} \right)^{k-1} |\sigma_p'(x)|^k \, dx$$

is convergent. This completes the proof of the theorem.

Therefore, a theorem of Bor [2] is obtained for the improper integrals.

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