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Picard Approximation of the Matrix Elements of Quantum Stochastic Differential Equations and Quantum Flows and Basic Error Estimates

Andreas Boukas

Centro Vito Volterra
Universitá di Roma TorVergata,
Via di TorVergata,
00133 Roma, Italy

email: andreasboukas@yahoo.com, boukas.andreas@ac.eap.gr

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Abstract

We describe Picard approximation schemes for the computation of the matrix elements of solutions of unitary quantum stochastic evolutions and associated quantum flows. We also provide some basic error estimates for the convergence of the approximation schemes presented. A combination of numerical and symbolic computation is required.

1 Quantum Stochastic Calculus

Let $B_t = \{B_t(\omega)/\, \omega \in \Omega\}$, $t \geq 0$, be one-dimensional Brownian motion. Integration with respect to B_t was defined by Itô in [3]. A basic result of the theory is that stochastic integral equations of the form

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s$$

can be viewed as stochastic differential equations of the form

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dB_t$$

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where differentials are handled with the use of Itô's formula

$$(dB_t)^2 = dt, \quad dB_t\, dt = dt\, dB_t = (dt)^2 = 0\ .$$

In [2], Hudson and Parthasarathy obtained a Fock space representation of Brownian motion and Poisson process.

Definition 1.1. *The Boson Fock space $\Gamma = \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$ over $L^2(\mathbb{R}_+, \mathcal{C})$ is the Hilbert space completion of the linear span of the exponential vectors $\psi(f)$ under the inner product*

$$< \psi(f), \psi(g) >= e^{<f,g>}$$

where $f, g \in L^2(\mathbb{R}_+, \mathcal{C})$ and $< f, g >= \int_0^{+\infty} \bar{f}(s)\, g(s)\, ds$ where, here and in what follows, $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

The annihilation, creation and conservation operators $A(f)$, $A^\dagger(f)$ and $\Lambda(F)$ respectively, are defined on the exponential vectors $\psi(g)$ of Γ as follows.

Definition 1.2.

$$A_t \psi(g) = \int_0^t g(s)\, ds\ \psi(g)\,;\ A_t^\dagger \psi(g) = \frac{\partial}{\partial \epsilon}|_{\epsilon=0}\, \psi(g + \epsilon \chi_{[0,t]})\,;\ \Lambda_t \psi(g) = \frac{\partial}{\partial \epsilon}|_{\epsilon=0}\, \psi(e^{\epsilon \chi_{[0,t]}} g)\ .$$

The basic quantum stochastic differentials dA_t, $dA_t^\dagger$, and $d\Lambda_t$ are defined as follows.

Definition 1.3.

$$dA_t = A_{t+dt} - A_t\,;\ dA_t^\dagger = A_{t+dt}^\dagger - A_t^\dagger]\,;\ d\Lambda_t = \Lambda_{t+dt} - \Lambda_t\ .$$

The fundamental result which connects classical with quantum stochastics is that the processes B_t and P_t defined by

$$B_t = A_t + A_t^\dagger\,;\ P_t = \Lambda_t + \sqrt{\lambda}(A_t + A_t^\dagger) + \lambda t$$

are identified through their vacuum characteristic functions

$$< \psi(0), e^{i\, s\, B_t}\, \psi(0) >= e^{-\frac{s^2}{2}\, t}\,;\ < \psi(0), e^{i\, s\, P_t}\, \psi(0) >= e^{\lambda\, (e^{i\, s} - 1)\, t}$$

with Brownian motion and Poisson process of intensity λ respectively.

Hudson and Parthasarathy defined stochastic integration with respect to the noise differentials of Definition 3 and obtained the Itô multiplication table

$\cdot$	$dA_t^\dagger$	$d\Lambda_t$	dA_t	dt
$dA_t^\dagger$	0	0	0	0
$d\Lambda_t$	$dA_t^\dagger$	$d\Lambda_t$	0	0
dA_t	dt	dA_t	0	0
dt	0	0	0	0

Within the framework of Hudson-Parthasarathy Quantum Stochastic Calculus, classical quantum mechanical evolution equations take the form

$$dU_t = -\left(\left(iH + \frac{1}{2}\,L^* L\right)\,dt + L^*\,W\,dA_t - L\,dA_t^\dagger + (I - W)\;d\Lambda_t\right)U_t\,;\,U_0 = I\;, \tag{1.1}$$

where, for each $t \geq 0$, U_t is a unitary operator defined on the tensor product $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$ of a system Hilbert space $\mathcal{H}$ and the noise (or reservoir) Fock space Γ. Here H, L, W are in $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, with W unitary and H self-adjoint. In all cases, I denotes the appropriate identity operator. Here and in what follows we identify time-independent, bounded, system space operators X with their ampliation $X \otimes I$ to $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$. As in [2], all Hilbert space inner products are linear on the right.

The quantum stochastic differential equation satisfied by the *quantum flow*

$$j_t(X) = U_t^*\,X\,U_t \tag{1.2}$$

where X is a bounded system space operator, is

$$\begin{aligned} dj_t(X) &= j_t\left(i\;[H, X] - \frac{1}{2}\,(L^* L X + X L^* L - 2 L^* X L)\right)\,dt \\ &\quad + j_t\,([L^*, X]\;W)\;dA_t + j_t\,(W^*\;[X, L])\;dA_t^\dagger + j_t\,(W^*\,X\,W - X)\;d\Lambda_t\;, \\ j_0(X) &= X, \quad t \in [0, T]\;. \end{aligned}$$

The commutation relations associated with the operator processes A_t, $A_t^\dagger$ are the Canonical (or Heisenberg) Commutation Relations (CCR), namely

$$\left[A_t, A_t^\dagger\right] = t\,I\;.$$

Applications of quantum stochastic calculus to the control of quantum evolutions and Langevin equations (quantum flows) can be found, for example, in [1] and the references within it. However, to the author's best knowledge, no work has been done in the direction of performing actual numerical computations, most likely with the use of a computer. That would require the implementation of suitable algorithms whose reliability depends on the existence of good norm estimates. It is that gap that this paper aspires to close.

2 Matrix Elements and Approximation Schemes

The fundamental theorems of the Hudson-Parthasarathy quantum stochastic calculus give formulas for expressing the matrix elements of quantum stochastic integrals in terms of ordinary Riemann-Lebesgue integrals.

Theorem 2.1. *Let*

$$M(t) = \int_0^t E(s)\, d\Lambda(s) + F(s)\, dA(s) + G(s)\, dA^\dagger(s) + H(s)\, ds$$

where E, F, G, H *are adapted processes. Let also* $u \otimes \psi(f)$ *and* $v \otimes \psi(g)$ *be in the exponential domain of* $\mathcal{H} \otimes \Gamma$. *Then*

$$< u \otimes \psi(f), M(t)\, v \otimes \psi(g) >=$$
$$\int_0^t < u \otimes \psi(f), \big(\bar{f}(s)\, g(s)\, E(s) + g(s)\, F(s) + \bar{f}(s)\, G(s) + H(s)\big)\, v \otimes \psi(g) > \, ds \ .$$

Proof. See Theorem 4.1 of [2]. □

Theorem 2.2. *Let*

$$M(t) = \int_0^t E(s)\, d\Lambda(s) + F(s)\, dA(s) + G(s)\, dA^\dagger(s) + H(s)\, ds$$

and

$$M'(t) = \int_0^t E'(s)\, d\Lambda(s) + F'(s)\, dA(s) + G'(s)\, dA^\dagger(s) + H'(s)\, ds$$

where E, F, G, H, E', F', G', H' *are adapted processes. Let also* $u \otimes \psi(f)$ *and* $v \otimes \psi(g)$ *be in the exponential domain of* $\mathcal{H} \otimes \Gamma$. *Then*

$$< M(t)\, u \otimes \psi(f), M'(t)\, v \otimes \psi(g) >=$$
$$\int_0^t \big(< M(s)\, u \otimes \psi(f), \big(\bar{f}(s)\, g(s) E'(s) + g(s)\, F'(s) + \bar{f}(s)\, G'(s) + H'(s)\big)\, v \otimes \psi(g) >$$
$$+ < (\bar{g}(s)\, f(s)\, E(s) + f(s)\, F(s) + \bar{g}(s)\, G(s) + H(s))\, u \otimes \psi(f), M'(s)\, v \otimes \psi(g) >$$
$$+ < (f(s) E(s) + G(s))\, u \otimes \psi(f), (g(s) E'(s) + G'(s))\, v \otimes \psi(g) >)\, ds \ .$$

Proof. See Theorem 4.3 of [2]. □

We are interested in defining approximation schemes which can be used to compute the matrix elements

$$< u \otimes \psi(f), U_t\, v \otimes \psi(g) >, \;\; < u \otimes \psi(f), j_t(X)\, v \otimes \psi(g) >$$

related to the quantum flow (1.2) and the Hudson-Parthasarathy stochastic differential equation

$$dU_t = \left(K\, dt + B\, dA_t + C\, dA_t^{\dagger} + D\, d\Lambda_t\right) U_t \,;\, U_0 = I \tag{2.3}$$

with initial condition where $t \in [0, T]$ for some $T > 0$, and K, B, C, D are bounded system space operators of the form appearing in (1.1), i.e

$$K = -\left(iH + \frac{1}{2}\, L^* L\right) \,;\, B = -L^*\, W \,;\, C = L \,;\, D = W - I \,.$$

2.1 Unitary Evolutions

Equation (2.3) has the integral form

$$U_t = I + \int_0^t K\, U_s\, ds + B\, U_s\, dA_s + C\, U_s\, dA_s^{\dagger} + D\, U_s\, d\Lambda_s \tag{2.4}$$

defined (see [2], Proposition 7.1) as the $[0, T]$-uniform limit of the sequence $U_n = \{U_{n,t} \,/\, t \geq 0\}$ defined recursively on the exponential domain of $\mathcal{H} \otimes \Gamma$ by

$$U_{0,t} = I \tag{2.5}$$

and, for $n \geq 1$,

$$U_{n,t} = I + \int_0^t K\, U_{n-1,s}\, ds + B\, U_{n-1,s}\, dA_s + C\, U_{n-1,s}\, dA_s^{\dagger} + D\, U_{n-1,s}\, \Lambda_s \,. \tag{2.6}$$

By Theorem 1, the matrix elements of (2.6) are given, for $n \geq 1$, by the recursion scheme

$$\begin{aligned} &< u \otimes \psi(f), U_{n,t}\, v \otimes \psi(g) > \qquad (2.7) \\ &= < u \otimes \psi(f), v \otimes \psi(g) > + \int_0^t \big(\overline{f}(s) g(s) < u \otimes \psi(f), D\, U_{n-1,s}\, v \otimes \psi(g) > \\ &+ g(s) < u \otimes \psi(f), B\, U_{n-1,s}\, v \otimes \psi(g) > + \overline{f}(s) < u \otimes \psi(f), C\, U_{n-1,s}\, v \otimes \psi(g) > \\ &+ < u \otimes \psi(f), K\, U_{n-1,s}\, v \otimes \psi(g) > \big)\; ds \,. \end{aligned}$$

Letting

$$u_{D^*} = D^* u\,;\; u_{B^*} = B^* u\,;\; u_{C^*} = C^* u\,;\; u_{K^*} = K^* u$$

we can rewrite approximation scheme (2.7) as:

Approximation Scheme 1. *(Unitary Evolution)*

$$< u \otimes \psi(f), U_{n,t}\, v \otimes \psi(g) >= \tag{2.8}$$
$$< u \otimes \psi(f), v \otimes \psi(g) > + \int_0^t \big(\overline{f}(s) g(s) < u_{D^*} \otimes \psi(f), U_{n-1,s}\, v \otimes \psi(g) >$$
$$+ g(s) < u_{B^*} \otimes \psi(f), U_{n-1,s}\, v \otimes \psi(g) > + \overline{f}(s) < u_{C^*} \otimes \psi(f), U_{n-1,s}\, v \otimes \psi(g) >$$
$$+ < u_{K^*} \otimes \psi(f), U_{n-1,s}\, v \otimes \psi(g) >\big)\, ds$$

with

$$< u \otimes \psi(f), U_{0,t}\, v \otimes \psi(g) >=< u \otimes \psi(f), v \otimes \psi(g) > \;\; .$$

The limit form of (2.8) as $n \to +\infty$ is:

$$< u \otimes \psi(f), U_t\, v \otimes \psi(g) >=$$
$$< u \otimes \psi(f), v \otimes \psi(g) > + \int_0^t \big(\overline{f}(s) g(s) < u_{D^*} \otimes \psi(f), U_s\, v \otimes \psi(g) >$$
$$+ g(s) < u_{B^*} \otimes \psi(f), U_s\, v \otimes \psi(g) > + \overline{f}(s) < u_{C^*} \otimes \psi(f), U_s\, v \otimes \psi(g) >$$
$$+ < u_{K^*} \otimes \psi(f), U_s\, v \otimes \psi(g) >\big)\, ds$$

which, by subtraction of the cases $t = t_n$ and $t = t_{n-1}$ where $t_{n-1} \leq t_n$, implies

$$< u \otimes \psi(f), U_{t_n}\, v \otimes \psi(g) > - < u \otimes \psi(f), U_{t_{n-1}}\, v \otimes \psi(g) >$$
$$= \int_{t_{n-1}}^{t_n} \big(\overline{f}(s) g(s) < u_{D^*} \otimes \psi(f), U_s\, v \otimes \psi(g) > + g(s) < u_{B^*} \otimes \psi(f), U_s\, v \otimes \psi(g) >$$
$$+ \overline{f}(s) < u_{C^*} \otimes \psi(f), U_s\, v \otimes \psi(g) > + < u_{K^*} \otimes \psi(f), U_s\, v \otimes \psi(g) >\big)\, ds \;,$$

or as:

Approximation Scheme 2. *(Time Approximation of Unitary Evolution)*

$$< u \otimes \psi(f), U_{t_n}\, v \otimes \psi(g) >=$$
$$< u \otimes \psi(f), U_{t_{n-1}}\, v \otimes \psi(g) > + \int_{t_{n-1}}^{t_n} \big(\overline{f}(s) g(s) < u_{D^*} \otimes \psi(f), U_s\, v \otimes \psi(g) >$$
$$+ g(s) < u_{B^*} \otimes \psi(f), U_s\, v \otimes \psi(g) > + \overline{f}(s) < u_{C^*} \otimes \psi(f), U_s\, v \otimes \psi(g) >$$
$$+ < u_{K^*} \otimes \psi(f), U_s\, v \otimes \psi(g) >\big)\, ds \;.$$

2.2 Quantum Flows

The integral form of the quantum flow equation (1.2) is

$$j_t(X) = X + \int_0^t j_s(\hat{K})\,ds + j_s(\hat{B})\,dA_s + j_s(\hat{C})\,dA_s^\dagger + j_s(\hat{D})\,d\Lambda_s$$

where

$$\hat{K} = i[H,X] - \frac{1}{2}(L^*LX + XL^*L - 2L^*XL)\,;\ \hat{B} = [L^*,X]\,W\,;\ \hat{C} = W^*\,[X,L]\,;\ \hat{D} = W^*\,X\,W - X\,.$$

The corresponding approximation scheme is

$$j_{0,t}(X) = X \tag{2.9}$$

and for $n \geq 1$

$$j_{n,t}(X) = X + \int_0^t j_{n-1,s}(\hat{K})\,ds + j_{n-1,s}(\hat{B})\,dA_s + j_{n-1,s}(\hat{C})\,dA_s^\dagger + j_{n-1,s}(\hat{D})\,d\Lambda_s\,. \tag{2.10}$$

The matrix element form of the approximation scheme (2.9) and (2.10) is

$$\begin{aligned} &< u \otimes \psi(f), j_{n,t}(X)\,v \otimes \psi(g) > = < u \otimes \psi(f), X\,v \otimes \psi(g) > + \qquad (2.11)\\ &< u \otimes \psi(f), \left(\int_0^t j_{n-1,s}(\hat{K})\,ds + j_{n-1,s}(\hat{B})\,dA_s + j_{n-1,s}(\hat{C})\,dA_s^\dagger + j_{n-1,s}(\hat{D})\,d\Lambda_s\right) v \otimes \psi(g) > \end{aligned}$$

which by Theorem 1 yields:

Approximation Scheme 3. *(General Quantum Flow)*

$$\begin{aligned} &< u \otimes \psi(f), j_{n,t}(X)\,v \otimes \psi(g) > = < u \otimes \psi(f), X\,v \otimes \psi(g) >\\ &+ \int_0^t \Big(\overline{f}(s)g(s) < u \otimes \psi(f), j_{n-1,s}(\hat{D})\,v \otimes \psi(g) > + g(s) < u \otimes \psi(f), j_{n-1,s}(\hat{B})\,v \otimes \psi(g) >\\ &+ \overline{f}(s) < u \otimes \psi(f), j_{n-1,s}(\hat{C})\,v \otimes \psi(g) > + < u \otimes \psi(f), j_{n-1,s}(\hat{K})\,v \otimes \psi(g) > \Big)\,ds\,. \end{aligned}$$

Notice that for $n = 1$ recursion (2.11) becomes

$$< u\otimes\psi(f), j_{1,t}(X)\, v\otimes\psi(g) >$$
$$=< u\otimes\psi(f), X\, v\otimes\psi(g) > + < u\otimes\psi(f), \left(\int_0^t \hat{K}\, ds + \hat{B}\, dA_s + \hat{C}\, dA_s^{\dagger} + \hat{D}\, d\Lambda_s\right) v\otimes\psi(g) >$$
$$=< u\otimes\psi(f), X\, v\otimes\psi(g) > + \int_0^t \Big(\overline{f}(s)g(s) < u\otimes\psi(f), \hat{D}\, v\otimes\psi(g) >$$
$$+g(s) < u\otimes\psi(f), \hat{B}\, v\otimes\psi(g) > +\overline{f}(s) < u\otimes\psi(f), \hat{C}\, v\otimes\psi(g) > + < u\otimes\psi(f), \hat{K}\, v\otimes\psi(g) >\Big)\, ds$$
$$=< u\otimes\psi(f), X\, v\otimes\psi(g) > + \int_0^t \big(\overline{f}(s)g(s) < u_{\hat{D}}\otimes\psi(f), v\otimes\psi(g) >$$
$$+g(s) < u_{\hat{B}}\otimes\psi(f), v\otimes\psi(g) > +\overline{f}(s) < u_{\hat{C}}\otimes\psi(f), v\otimes\psi(g) > + < u_{\hat{K}}\otimes\psi(f), v\otimes\psi(g) >\big)\, ds$$

and so, letting $u_{X^*} = X^*\, u$, we have

$$< u\otimes\psi(f), j_{1,t}(X)\, v\otimes\psi(g) >$$
$$=< u_{X^*}\otimes\psi(f), v\otimes\psi(g) > + \int_0^t \overline{f}(s)g(s)\, ds < u_{\hat{D}}\otimes\psi(f), v\otimes\psi(g) >$$
$$+ \int_0^t g(s)\, ds < u_{\hat{B}}\otimes\psi(f), v\otimes\psi(g) > + \int_0^t \overline{f}(s)\, ds < u_{\hat{C}}\otimes\psi(f), v\otimes\psi(g) >$$
$$+ \int_0^t ds < u_{\hat{K}}\otimes\psi(f), v\otimes\psi(g) > \quad .$$

The general theory of quantum flows, in the context of Hudson-Parthasarathy calculus, can be found in [4]. We now consider flows $\{j_t(X)\,/\,t\geq 0\}$ of the standard quantum mechanical form

$$j_t(X) = U_t^*\, X\, U_t$$

where U_t is , for each $t\geq 0$, a unitary operator.

Proposition 2.3. *Let X be a bounded system space operator, let U_t and $U_{n,t}$ be, for each $t\in[0,T]$ and $n\geq 1$, as in (2.4) and (2.6) respectively, and let U_t^* and $U_{n,t}^*$ be their adjoints. If*

$$j_t(X) = U_t^*\, X\, U_t$$

and

$$j_{n,t}(X) = U_{n,t}^*\, X\, U_{n,t} \tag{2.12}$$

then

$$\lim_{n\to\infty} < u\otimes\psi(f), j_{n,t}(X)\, v\otimes\psi(g) > = < u\otimes\psi(f), j_t(X)\, v\otimes\psi(g) >$$

for all $u\otimes\psi(f)$ *and* $v\otimes\psi(g)$ *in the exponential domain of* $\mathcal{H}\otimes\Gamma$. *Convergence is uniform on* $[0,T]$.

Proof. We have

$$\begin{aligned}
&| < u\otimes\psi(f), j_t(X)\, v\otimes\psi(g) > - < u\otimes\psi(f), j_{n,t}(X)\, v\otimes\psi(g) > | \\
&= | < u\otimes\psi(f), (j_t(X) - j_{n,t}(X))\; v\otimes\psi(g) > | \\
&= | < u\otimes\psi(f), \left(U_t^*\, X\, U_t - U_{n,t}^*\, X\, U_{n,t}\right)\; v\otimes\psi(g) > | \\
&= | < u\otimes\psi(f), \left((U_t^* - U_{n,t}^*)\, X\, U_t + U_{n,t}^*\, X\, (U_t - U_{n,t})\right)\; v\otimes\psi(g) > | \\
&\leq | < u\otimes\psi(f), (U_t^* - U_{n,t}^*)\, X\, U_t\, v\otimes\psi(g) > | + | < u\otimes\psi(f), U_{n,t}^*\, X\, (U_t - U_{n,t})\, v\otimes\psi(g) > | \\
&= | < (U_t - U_{n,t})\, u\otimes\psi(f), X\, U_t\, v\otimes\psi(g) > | + | < U_{n,t}\, u\otimes\psi(f), X\, (U_t - U_{n,t})\, v\otimes\psi(g) > | \\
&\leq \|(U_t - U_{n,t})\, u\otimes\psi(f)\|\, \|X\|\, \|U_t\|\, \|v\otimes\psi(g)\| + \|U_{n,t}\, u\otimes\psi(f)\|\, \|X\|\, \|(U_t - U_{n,t})\, v\otimes\psi(g)\| \\
&\leq \|(U_t - U_{n,t})\, u\otimes\psi(f)\|\, \|X\|\, \|v\otimes\psi(g)\| + \|U_{n,t}\, u\otimes\psi(f)\|\, \|X\|\, \|(U_t - U_{n,t})\, v\otimes\psi(g)\|
\end{aligned}$$

since $\|U_t\| = 1$. Since $U_{n,t}$ converges to U_t on the exponential domain of $\mathcal{H}\otimes\Gamma$ uniformly with respect to t and $\|U_{n,t}\, u\otimes\psi(f)\|$ is bounded, it follows that

$$| < u\otimes\psi(f), j_{n,t}(X)\, v\otimes\psi(g) > - < u\otimes\psi(f), j_t(X)\, v\otimes\psi(g) > | \to 0$$

as $n \to +\infty$. □

The approximation scheme for the matrix element associated with (2.12) is obtained, with the use of Theorems 1 and 2 as follows:

$$< u \otimes \psi(f), U_{n,t}^* \, X \, U_{k,t} \, v \otimes \psi(g) > = < U_{n,t} \, u \otimes \psi(f), X \, U_{k,t} \, v \otimes \psi(g) >$$
$$= < \left(I + \int_0^t K \, U_{n-1,s} \, ds + B \, U_{n-1,s} \, dA_s + C \, U_{n-1,s} \, dA_s^\dagger + D \, U_{n-1,s} \, d\Lambda_s \right) u \otimes \psi(f),$$
$$X \, \left(I + \int_0^t K \, U_{k-1,s} \, ds + B \, U_{k-1,s} \, dA_s + C \, U_{k-1,s} \, dA_s^\dagger + D \, U_{k-1,s} \, d\Lambda_s \right) v \otimes \psi(g) >$$
$$= < u \otimes \psi(f), (X \, v) \otimes \psi(g) > +$$
$$+ < u \otimes \psi(f), \left(\int_0^t X \, K \, U_{k-1,s} \, ds + X \, B \, U_{k-1,s} \, dA_s + X \, C \, U_{k-1,s} \, dA_s^\dagger + X \, D \, U_{k-1,s} \, d\Lambda_s \right) v \otimes \psi(g) >$$
$$+ < \left(\int_0^t K \, U_{n-1,s} \, ds + B \, U_{n-1,s} \, dA_s + C \, U_{n-1,s} \, dA_s^\dagger + D \, U_{n-1,s} \, d\Lambda_s \right) u \otimes \psi(f), (X \, v) \otimes \psi(g) >$$
$$+ < \left(\int_0^t K \, U_{n-1,s} \, ds + B \, U_{n-1,s} \, dA_s + C \, U_{n-1,s} \, dA_s^\dagger + D \, U_{n-1,s} \, d\Lambda_s \right) u \otimes \psi(f),$$

$$\left(\int_0^t X \, K \, U_{k-1,s} \, ds + X \, B \, U_{k-1,s} \, dA_s + X \, C \, U_{k-1,s} \, dA_s^\dagger + X \, D \, U_{k-1,s} \, d\Lambda_s \right) v \otimes \psi(g) >$$
$$= < u \otimes \psi(f), (X \, v) \otimes \psi(g) >$$
$$+ \int_0^t < u \otimes \psi(f), \left(\bar{f}(s) \, g(s) \, X \, D + g(s) \, X \, B + \bar{f}(s) \, X \, C + X \, K \right) \, U_{k-1,s} \, v \otimes \psi(g) > \, ds$$
$$+ \int_0^t < (\bar{g}(s) \, f(s) \, X^* \, D + f(s) \, X^* \, B + \bar{g}(s) \, X^* \, C + X^* \, K) \, U_{k-1,s} \, u \otimes \psi(f), v \otimes \psi(g) > \, ds$$
$$+ \int_0^t \big(< (U_{n,s} - 1) \, u \otimes \psi(f), \left(\bar{f}(s) \, g(s) \, X \, D + g(s) \, X \, B + \bar{f}(s) \, X \, C + X \, K \right) \, U_{k-1,s} \, v \otimes \psi(g) >$$
$$+ < (\bar{g}(s) \, f(s) \, D + f(s) \, B + \bar{g}(s) \, C + \, K) \, U_{n-1,s} \, u \otimes \psi(f), X \, (U_{k,s} - 1) \, v \otimes \psi(g) >$$
$$+ < (f(s) \, D + C) \, U_{n-1,s} \, u \otimes \psi(f), (g(s) \, X \, D + X \, C) \, U_{k-1,s} \, v \otimes \psi(g) > \big) \, ds \, .$$

Using (2.6) we obtain:

Approximation Scheme 4. *(Quantum Mechanical Flows) For* $n, k \geq 1$

$$
\begin{aligned}
& < u \otimes \psi(f), U^*_{n,t}\, X\, U_{k,t}\, v \otimes \psi(g) > = < u \otimes \psi(f), (X\, v) \otimes \psi(g) > \qquad (2.13)\\
& + \int_0^t \Big(\bar{f}(s)\, g(s) \ < u \otimes \psi(f), U^*_{n,s}\, X\, D\, U_{k-1,s}\, v \otimes \psi(g) > \\
& + g(s) \ < u \otimes \psi(f), U^*_{n,s}\, X\, B\, U_{k-1,s}\, v \otimes \psi(g) > \\
& + \bar{f}(s) \ < u \otimes \psi(f), U^*_{n,s}\, X\, C\, U_{k-1,s}\, v \otimes \psi(g) > + < u \otimes \psi(f), U^*_{n,s}\, X\, K\, U_{k-1,s}\, v \otimes \psi(g) > \\
& + \bar{f}(s)\, g(s) \ < u \otimes \psi(f), U^*_{n-1,s}\, D^*\, X\, U_{k,s}\, v \otimes \psi(g) > + \bar{f}(s) \ < u \otimes \psi(f), U^*_{n-1,s}\, B^*\, X\, U_{k,s}\, v \otimes \psi(g) > \\
& + g(s) \ < u \otimes \psi(f), U^*_{n-1,s}\, C^*\, X\, U_{k,s}\, v \otimes \psi(g) > + < u \otimes \psi(f), U^*_{n-1,s}\, K^*\, X\, U_{k,s}\, v \otimes \psi(g) > \\
& + \bar{f}(s)\, g(s) \ < u \otimes \psi(f), U^*_{n-1,s}\, D^*\, X\, D\, U_{k-1,s}\, v \otimes \psi(g) > \\
& + \bar{f}(s) \ < u \otimes \psi(f), U^*_{n-1,s}\, D^*\, X\, C\, U_{k-1,s}\, v \otimes \psi(g) > \\
& + g(s) \ < u \otimes \psi(f), U^*_{n-1,s}\, C^*\, X\, D\, U_{k-1,s}\, v \otimes \psi(g) > \\
& + < u \otimes \psi(f), U^*_{n-1,s}\, C^*\, X\, C\, U_{k-1,s}\, v \otimes \psi(g) > \Big)\, ds \ .
\end{aligned}
$$

Letting $n = k$ in (2.13) we obtain the value of the matrix element

$$< u \otimes \psi(f), j_{n,t}(X)\, v \otimes \psi(g) > \ .$$

3 Basic Error Estimates

Proposition 3.1. *Let* $\epsilon > 0$*, and let* U_t *and* $U_{n,t}$*, where* $0 \leq t \leq T < +\infty$*, be defined respectively by (2.4) and (2.6). Then for all* $u \otimes \psi(f)$ *and* $v \otimes \psi(g)$ *in the exponential domain of* $\mathcal{H} \otimes \Gamma$*, with* g *locally bounded and* $u, v \neq 0$*:*

$$| < u \otimes \psi(f), U_t\, v \otimes \psi(g) > - < u \otimes \psi(f), U_{n,t}\, v \otimes \psi(g) > | < \epsilon \qquad (3.14)$$

for all $t \in [0, T]$ *provided that*

$$\sqrt{\frac{\lambda^{n+1}\, T^{n+1}}{(n+1)!}} < \frac{\epsilon}{\|u\|\, \|v\|\, e^{\frac{\|f\|^2}{2}}\, e^{\frac{\|g\|^2}{2}}} \qquad (3.15)$$

where $\epsilon > 0$ *is the required degree of accuracy and*

$$\|f\|^2 = \int_0^{+\infty} |f(s)|^2\, ds\, ;\ \|g\|^2 = \int_0^{+\infty} |g(s)|^2\, ds\, ;\ \lambda = 6\, \alpha(T)^2\, e^T\, M$$
$$M = \max\left(\|K\|, \|B\|, \|C\|, \|D\|\right)\, ;\ \alpha(T) = \sup_{0 \leq s \leq T} \max\left(|g(s)|^2, |g(s)|, 1\right)\ .$$

Proof.

$$
\begin{aligned}
&| < u\otimes\psi(f), U_t\, v\otimes\psi(g) > - < u\otimes\psi(f), U_{n,t}\, v\otimes\psi(g) > |^2 \\
&= | < u\otimes\psi(f), (U_t - U_{n,t})\, v\otimes\psi(g) > |^2 \leq \|u\otimes\psi(f)\|^2\, \|(U_t - U_{n,t})\, v\otimes\psi(g)\|^2 \\
&= \|u\otimes\psi(f)\|^2\, \| \left(\int_0^t K\,(U_{s_1} - U_{n-1,s_1})\, ds_1 + B\,(U_{s_1} - U_{n-1,s_1})\, dA_{s_1} + C\,(U_{s_1} - U_{n-1,s_1})\, dA^{\dagger}_{s_1} \right. \\
&\left. + D\,(U_{s_1} - U_{n-1,s_1})\, d\Lambda_{s_1} \right)\, v\otimes\psi(g)\|^2
\end{aligned}
$$

which by Corollary 1 and Theorem 4.4 of [2] is

$$
\begin{aligned}
&\leq 6\,\alpha(T)^2\,\|u\otimes\psi(f)\|^2 \int_0^T e^{t-s_1}\,\{\|K\,(U_{s_1} - U_{n-1,s_1})\, v\otimes\psi(g)\|^2 \\
&+ \|B\,(U_{s_1} - U_{n-1,s_1})\, v\otimes\psi(g)\|^2 + \|C\,(U_{s_1} - U_{n-1,s_1})\, v\otimes\psi(g)\|^2 \\
&+ \|D\,(U_{s_1} - U_{n-1,s_1})\, v\otimes\psi(g)\|^2\}\, ds_1 \\
&\leq 6\,\alpha(T)^2\,(\max\{\|K\|,\|B\|,\|C\|,\|D\|\})^2\,\|u\otimes\psi(f)\|^2 \int_0^T e^{t-s_1}\,\|(U_{s_1} - U_{n-1,s_1})\, v\otimes\psi(g)\|^2\, ds_1 \\
&= \lambda\|u\otimes\psi(f)\|^2 \int_0^T \|(U_{s_1} - U_{n-1,s_1})\, v\otimes\psi(g)\|^2\, ds_1 \\
&\leq \lambda^2\|u\otimes\psi(f)\|^2 \int_0^T \int_0^{s_1} \|(U_{s_2} - U_{n-2,s_2})\, v\otimes\psi(g)\|^2\, ds_2\, ds_1 \\
&\leq \lambda^3\|u\otimes\psi(f)\|^2 \int_0^T \int_0^{s_1} \int_0^{s_2} \|(U_{s_3} - U_{n-3,s_3})\, v\otimes\psi(g)\|^2\, ds_3\, ds_2\, ds_1 \\
&\vdots \\
&\leq \lambda^n\|u\otimes\psi(f)\|^2 \int_0^T \int_0^{s_1} \int_0^{s_2} ... \int_0^{s_n - 1} \|(U_{s_n} - U_{0,s_n})\, v\otimes\psi(g)\|^2\, ds_n ... ds_3\, ds_2\, ds_1
\end{aligned}
$$

which using

$$
\begin{aligned}
U_{s_n} &= I + \int_0^{s_n} K\,U_{s_{n+1}}\, ds_{n+1} + B\,U_{s_{n+1}}\, dA_{s_{n+1}} + C\,U_{s_{n+1}}\, dA^{\dagger}_{s_{n+1}} + D\,U_{s_{n+1}}\, d\Lambda_{s_{n+1}} \\
U_{0,s_n} &= I
\end{aligned}
$$

and the unitarity of $U_{s_{n+1}}$, becomes

$$\leq \lambda^{n+1} \|u \otimes \psi(f)\|^2 \int_0^T \int_0^{s_1} \int_0^{s_2} ... \int_0^{s_n - 1} \int_0^{s_n} \|U_{s_{n+1}} v \otimes \psi(g)\|^2 \, ds_{n+1} \, ds_n ... \, ds_3 \, ds_2 \, ds_1$$
$$= \lambda^{n+1} \|u \otimes \psi(f)\|^2 \int_0^T \int_0^{s_1} \int_0^{s_2} ... \int_0^{s_n - 1} \int_0^{s_n} \|v \otimes \psi(g)\|^2 \, ds_{n+1} \, ds_n ... \, ds_3 \, ds_2 \, ds_1$$
$$= \|u \otimes \psi(f)\|^2 \, \|v \otimes \psi(g)\|^2 \, \lambda^{n+1} \int_0^T \int_0^{s_1} \int_0^{s_2} ... \int_0^{s_n - 1} \int_0^{s_n} ds_{n+1} \, ds_n ... \, ds_3 \, ds_2 \, ds_1$$
$$= \|u \otimes \psi(f)\|^2 \, \|v \otimes \psi(g)\|^2 \, \lambda^{n+1} \, \frac{T^{n+1}}{(n+1)!}$$
$$= \|u\|^2 \, \|v\|^2 \, e^{\|f\|^2} \, e^{\|g\|^2} \, \lambda^{n+1} \, \frac{T^{n+1}}{(n+1)!}$$

which is less than ϵ^2 provided that (3.15) is satisfied. □

Corollary 3.2. *In the notation of Proposition 2*

$$| \; | < u \otimes \psi(f), U_t \, v \otimes \psi(g) > | - | < u \otimes \psi(f), U_{n,t} \, v \otimes \psi(g) > | \; | < \epsilon$$

for all $t \in [0, T]$ *and* u, v, f, g *with* $u, v \neq 0$*, provided that*

$$\sqrt{\frac{\lambda^{n+1} \, T^{n+1}}{(n+1)!}} < \frac{\epsilon}{\|u\| \, \|v\| \, e^{\frac{\|f\|^2}{2}} \, e^{\frac{\|g\|^2}{2}}} \; .$$

Proof. The proof follows by applying the triangle inequality to (3.14). □

Proposition 3.3. *In the notation of Proposition 1*

$$| < u \otimes \psi(f), j_{n,t}(X) \, v \otimes \psi(g) > - < u \otimes \psi(f), j_t(X) \, v \otimes \psi(g) > | < \epsilon \quad (3.16)$$

for all $t \in [0, T]$ *and* u, v, f, g *with* $u, v \neq 0$*, provided that*

$$max \left(\sqrt{\frac{\lambda^{n+1} \, T^{n+1}}{(n+1)!}}, \frac{\lambda^{n+1} \, T^{n+1}}{(n+1)!} \right) < \frac{\epsilon}{3 \, \|u\| \, \|v\| \, e^{\frac{\|f\|^2}{2}} \, e^{\frac{\|g\|^2}{2}} \, \|X\|} \; .$$

Proof.

$$| < u \otimes \psi(f), j_{n,t}(X) \, v \otimes \psi(g) > - < u \otimes \psi(f), j_t(X) \, v \otimes \psi(g) > |$$
$$\leq \|(U_t - U_{n,t}) \, u \otimes \psi(f)\| \, \|X\| \, \|v \otimes \psi(g)\|$$
$$+ \left(\|(U_t - U_{n,t}) \, u \otimes \psi(f)\| + \|u \otimes \psi(f)\| \right) \, \|X\| \, \|(U_t - U_{n,t}) \, v \otimes \psi(g)\|$$

and using, as in the proof of Proposition 2, the inequality

$$\|(U_t - U_{n,t})\, a \otimes \psi(b)\|^2 \leq \|a \otimes \psi(b)\|^2\, \frac{\lambda^{n+1}\, T^{n+1}}{(n+1)!}$$

we obtain

$$\begin{aligned} &| < u \otimes \psi(f), j_{n,t}(X)\, v \otimes \psi(g) > - < u \otimes \psi(f), j_t(X)\, v \otimes \psi(g) > | \\ &\leq \|u \otimes \psi(f)\|\, \|v \otimes \psi(g)\|\, \|X\| \left(2 \sqrt{\frac{\lambda^{n+1}\, T^{n+1}}{(n+1)!}} + \frac{\lambda^{n+1}\, T^{n+1}}{(n+1)!} \right) \\ &\leq 3\, max \left(\sqrt{\frac{\lambda^{n+1}\, T^{n+1}}{(n+1)!}}, \frac{\lambda^{n+1}\, T^{n+1}}{(n+1)!} \right) \|u\|\, \|v\|\, e^{\frac{\|f\|^2}{2}}\, e^{\frac{\|g\|^2}{2}}\, \|X\| < \epsilon \end{aligned}$$

from which (3.16) follows. □

Corollary 3.4. *In the notation of Proposition 3*

$$| \, | < u \otimes \psi(f), j_{n,t}(X)\, v \otimes \psi(g) > | - | < u \otimes \psi(f), j_t(X)\, v \otimes \psi(g) > | \, | < \epsilon \tag{3.17}$$

for all $t \in [0, T]$ *and* u, v, f, g *with* $u, v \neq 0$*, provided that*

$$max \left(\sqrt{\frac{\lambda^{n+1}\, T^{n+1}}{(n+1)!}}, \frac{\lambda^{n+1}\, T^{n+1}}{(n+1)!} \right) < \frac{\epsilon}{3\, \|u\|\, \|v\|\, e^{\frac{\|f\|^2}{2}}\, e^{\frac{\|g\|^2}{2}}\, \|X\|} \; .$$

Proof. The proof follows by applying the triangle inequality to (3.16). □

Remark 3.5. In the case of unitary evolutions and quantum flows, to produce a numerical result for the matrix elements symbolic and numerical computations should be run in parallel. That is because, for example, the numerical computation for specific t and u, v, f, g of the matrix element of U_n at the n-th step requires the symbolic calculation, i.e. for general s, of the matrix element at the previous step, with u transformed by the adjoints of K, B, C, D.

4 Example

In (2.4), let $K = \left(i - \frac{1}{2}\right) I$, $B = C = iI$ and $D = 0$, corresponding to $L = iI$ and $W = -H = I$. Then, letting

$$\phi(t) := \langle u \otimes \psi(f), U_t\, v \otimes \psi(g) \rangle \; ; \; \phi_n(t) := \langle u \otimes \psi(f), U_{n,t}\, v \otimes \psi(g) \rangle$$

(2.4) and (2.6) become

$$\phi(t) = \langle u \otimes \psi(f), v \otimes \psi(g) \rangle + \int_0^t \left(-\frac{1}{2} + i(g(s) + \overline{f(s)} + 1) \right) \phi(s)\, ds \quad (4.18)$$

and

$$\phi_n(t) = \langle u \otimes \psi(f), v \otimes \psi(g) \rangle + \int_0^t \left(-\frac{1}{2} + i(g(s) + \overline{f(s)} + 1) \right) \phi_{n-1}(s)\, ds \quad (4.19)$$

respectively.

The exact solution of (4.18) is

$$\phi(t) = \langle u \otimes \psi(f), v \otimes \psi(g) \rangle e^{\int_0^t \left(-\frac{1}{2} + i(g(s) + \overline{f(s)} + 1) \right) ds} .$$

Taking $u = v = e_{n_0}$ where $\{e_n : n = 1, 2, ...\}$ is an orthonormal basis for $\mathcal{H}$, and $f(s) = g(s) = e^{-s}$ we have

$$\langle u \otimes \psi(f), v \otimes \psi(g) \rangle = \langle u, v \rangle \langle \psi(f), \psi(g) \rangle = e^{\frac{1}{2}}$$

so

$$\phi(t) = e^{\frac{1}{2} + 2i} e^{\left(i - \frac{1}{2}\right)t} e^{-2ie^{-t}} .$$

For example, for $T = 2$ we have

$$\|f\|^2 = \|g\|^2 = \frac{1}{4} \; ; \|K\| = \frac{\sqrt{5}}{2} \, ; \, \|B\| = \|C\| = 1 \, ; \, \|D\| = 0 \, ; \, M = \frac{\sqrt{5}}{2} \, ; \, \alpha(2) = 1 \, ; \, \lambda = 3\sqrt{5}e^2$$

and (3.15) becomes

$$\frac{(6\sqrt{5}e^2)^{n+1}}{(n+1)!} < \frac{\epsilon^2}{e^{\frac{1}{2}}}$$

which for $\epsilon = 0.000001$ is seen (using Mathematica) to be satisfied for $n \geq 292$. However, iterating (4.19) we find, for example for $t = 1$,

$$\phi(1) = -0.6391901814352063 + i\, 0.7690487058417224$$

and

$$\phi_{14}(1) = -0.6391900891154059 + i\, 0.7690483524915057$$

with

$$|\phi_{14}(1) - \phi(1)| = 3.65211 * 10^{-7} < 0.000001$$

so, in this case, convergence within the required error bound is actually faster than predicted by formula (3.15).

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References

[1] A. Boukas, Application of Quantum Stochastic Calculus to Feedback Control, Global Journal of Pure and Applied Mathematics, **1,** no. 1, (2005), 69–111.

[2] R. L. Hudson, K. R. Parthasarathy, Quantum Ito's formula and stochastic evolutions, Commun. Math. Phys., **93,** (1984), 301–323.

[3] K. Ito, On stochastic differential equations, Memoirs Amer. Math. Soc., **4,** (1951).

[4] K. R. Parthasarathy, An introduction to quantum stochastic calculus, Birkhauser Boston Inc., 1992.