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On ideals of fuzzy points n-ary semigroups

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Abstract

In this paper, the n-ary semigroup $\underline{S}$ of all fuzzy points of an n-ary semigroup S is considered. The relation between ideals of S and some subsets of $\underline{S}$ will be discussed.

Abstract

1 Introduction and Preliminaries

In 1965, Zadeh introduced the fundamental concept of a fuzzy set in [20]. The applications of fuzzy sets can now be seen in a variety of disciplines. A fuzzy subset of S is a function from S into the closed interval $[0, 1]$. In [16], Rosenfeld introduced the fuzzy groups concept to initiate the study of fuzzy algebraic structures in 1971. In [11, 12, 13, 14], Kuroki defined

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fuzzy semigroups and different kinds of fuzzy ideals in semigroups. In [10], Kim considered the semigroup $\underline{S}$ of the fuzzy points of a semigroup S, and discussed the relation between fuzzy interior ideals of S and the subsets of $\underline{S}$. In [7], the relation between some ideals of a semigroup S and the subsets of $\underline{S}$ was discussed by Hamouda and in [8], he later considered the ternary semigroup $\underline{S}$ of all fuzzy points of a ternary semigroup S and discussed the relation between some fuzzy ideals of a ternary semigroup S and the subsets of $\underline{S}$.

Kasner generalized the classical algebraic structures to n-ary structures for the first time in [9] in 1904. In [17], Sioson showed the properties of regular n-ary semigroups. In [1], Dudek further extended Sioson's study on regular n-ary semigroups and provided proofs on some results of n-ary groups in [2, 3, 4]. He also studied the properties of ideals of some elements of n-ary ($n \geq 3$) semigroups containing an idempotent in [5]. Wang et al. discussed the relation between soft regular n-ary semigroups and regular n-ary semigroups in [19]. Studies on the applications of n-ary systems were made in the fields of automata theory in [6], physics in [15] and [18], among others.

A nonempty set S together with an n-ary operation given by $f : S^n \to S$, where $n \geq 2$, is called an *n-ary groupoid* and is denoted by (S, f). The following sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by x_i^j. We call an n-ary groupoid (S, f) as (i, j)-*associative* if the following holds:

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for every $x_1, x_2, \ldots, x_{2n-1} \in S$. The operation f is *associative* if the above identity holds for every $1 \leq i \leq j \leq n$, and (S, f) is then called an *n-ary semigroup*. For any subset A of an n-ary semigroup S, a fuzzy subset C_A of S is defined by

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

for all $x \in S$. C_A is called the *characteristic function* of A. For any $\alpha \in (0, 1]$ and $x \in S$, a fuzzy subset x_α of S is defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

for all $y \in S$. x_α is called a *fuzzy point* of S. Let g and h be two fuzzy subsets of S. The relation $g \subseteq h$ is defined by $g(x) \leq h(x)$ for all $x \in S$. The fuzzy subsets $g \cap h$ and $g \cup h$ are defined by $(g \cap h)(x) = \min\{g(x), h(x)\}$ and $(g \cup h)(x) = \max\{g(x), h(x)\}$ for all $x \in S$.

A nonempty subset T of S is called an *n-ary subsemigroup* of S if $f(a_1^n) \in T$ for all $a_1, a_2, \ldots, a_n \in T$. A nonempty subset I of S is called an *i-ideal* of S if for every $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n \in S$ with $a \in I$, then $f(x_1^{i-1}, a, x_{i+1}^n) \in I$. A nonempty subset I of S is called an *ideal* of S if I is an i-ideal for every $1 \leq i \leq n$.

In this paper, the n-ary semigroup $\underline{S}$ of the fuzzy points of an n-ary semigroup S is considered. The relation between i-ideals A of S and the subsets $\underline{C_A}$ of $\underline{S}$, and ideals A of S and the subsets $\underline{C_A}$ of $\underline{S}$ will be shown.

2 Main Results

Let $F(S)$ be the set of all fuzzy subsets in an n-ary semigroup S. For each $g_1, g_2, \ldots, g_n \in F(S)$, the product of $g_1, g_2, \ldots, g_n$ is a fuzzy subset $g_1 \circ g_2 \circ \cdots \circ g_n$ defined as follows:

$$(g_1 \circ g_2 \circ \cdots \circ g_n)(x) = \begin{cases} \bigvee\limits_{x=f(a_1,a_2,\ldots,a_n)} \{\bigwedge\limits_{i=1}^{n} g_i(a_i)\} & \text{if } x = f(a_1^n) \\ & a_1, a_2, \ldots, a_n \in S, \\ 0 & \text{otherwise.} \end{cases}$$

for all $x \in S$. Then $F(S)$ is an n-ary semigroup with the product $\circ$.

Let $\underline{S}$ be the set of all fuzzy points in an n-ary semigroup S. Then

$$(a_1)_{\alpha_1} \circ (a_2)_{\alpha_2} \circ \cdots \circ (a_n)_{\alpha_n} = (f(a_1^n))_{\min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}}.$$

Thus, $\underline{S}$ is an n-ary subsemigroup of $F(S)$. For any $g \in F(S)$, $\underline{g}$ denotes the set of all fuzzy points contained in g, that is,

$$\underline{g} = \{x_\alpha \in \underline{S} \mid g(x) \geq \alpha\}.$$

For any $\underline{g_1}, \underline{g_2}, \ldots, \underline{g_n} \subseteq \underline{S}$, we define the product of $\underline{g_1}, \underline{g_2}, \ldots, \underline{g_n}$ as

$$\underline{g_1} \circ \underline{g_2} \circ \cdots \circ \underline{g_n} = \{(a_1)_{\alpha_1} \circ (a_2)_{\alpha_2} \circ \cdots \circ (a_n)_{\alpha_n} \mid (a_i)_{\alpha_i} \in \underline{g_i}\}.$$

Theorem 2.1. *Let $g_1, g_2, \ldots, g_k$ be fuzzy subsets in an n-ary semigroup S. Then*

(1) $\underline{\cup_{i=1}^{k} g_i} = \cup_{i=1}^{k} \underline{g_i}$.

(2) $\underline{\cap_{i=1}^{k} g_i} = \cap_{i=1}^{k} \underline{g_i}$.

Proof. (1) Let $x_\alpha \in \underline{\cup_{i=1}^{k} g_i}$. Then $(\cup_{i=1}^{k} g_i)(x) \geq \alpha$. This implies $g_i(x) \geq \alpha$ for some i. Hence, $x_\alpha \in \underline{g_i}$ for some i. Therefore $x_\alpha \in \cup_{i=1}^{k} \underline{g_i}$. Conversely, let $x_\alpha \in \cup_{i=1}^{k} \underline{g_i}$. Then $x_\alpha \in \underline{g_i}$ for some i. So, $g_i(x) \geq \alpha$ for some i. This implies $(\cup_{i=1}^{k} g_i)(x) \geq \alpha$. Hence, $x_\alpha \in \underline{\cup_{i=1}^{k} g_i}$.

(2) Let $x_\alpha \in \underline{\cap_{i=1}^{k} g_i}$. Then $(\cap_{i=1}^{k} g_i)(x) \geq \alpha$. This implies $g_i(x) \geq \alpha$ for all i. Hence, $x_\alpha \in \underline{g_i}$ for all i. Hence, $x_\alpha \in \cap_{i=1}^{k} \underline{g_i}$. Conversely, let $x_\alpha \in \cap_{i=1}^{k} \underline{g_i}$. Then $x_\alpha \in \underline{g_i}$ for all i. So, $g_i(x) \geq \alpha$ for all i. This implies $(\cap_{i=1}^{k} g_i)(x) \geq \alpha$. Therefore, $x_\alpha \in \underline{\cap_{i=1}^{k} g_i}$. □

Theorem 2.2. *Let $g_1, g_2, \ldots, g_n$ be fuzzy subsets in an n-ary semigroup S. Then* $\underline{g_1} \circ \underline{g_2} \circ \cdots \circ \underline{g_n} \subseteq \underline{g_1 \circ g_2 \circ \cdots \circ g_n}$.

Proof. Let $x_\alpha \in \underline{g_1} \circ \underline{g_2} \circ \cdots \circ \underline{g_n}$. Then $x_\alpha = (a_1)_{\alpha_1} \circ (a_2)_{\alpha_2} \circ \cdots \circ (a_n)_{\alpha_n}$ for some $(a_i)_{\alpha_i} \in \underline{g_i}$. This implies $x_\alpha = (f(a_1^n))_{\min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}}$ and $(g_i)(a_i) \geq \alpha_i$ for all i. So, $x = f(a_1^n)$ and $\alpha = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Therefore, $(g_i)(a_i) \geq \alpha_i \geq \alpha$ for all i. Hence, $(g_1 \circ g_2 \circ \cdots \circ g_n)(x) \geq \alpha$. So, $x_\alpha \in \underline{g_1 \circ g_2 \circ \cdots \circ g_n}$. □

Theorem 2.3. *Let A and B be nonempty subsets of an n-ary semigroup S. Then $A \subseteq B$ if and only if $C_A \subseteq C_B$.*

Proof. Assume that $A \subseteq B$. We consider two cases:
Case 1: $x \notin A$. Then $C_A(x) = 0$. So, $C_A(x) = 0 \leq C_B(x)$.
Case 2: $x \in A$. Since $A \subseteq B, x \in B$. Then $C_B(x) = 1$. Hence, $C_A(x) \leq C_B(x)$. Thus, $C_A \subseteq C_B$. Conversely, assume that $C_A \subseteq C_B$. Let $x \in A$. Then $C_A(x) = 1$. Since $C_A \subseteq C_B, 1 = C_A(x) \leq C_B(x)$. Then $C_B(x) = 1$. Hence, $x \in B$. Thus, $A \subseteq B$. □

Theorem 2.4. *Let A be a nonempty subset of an $n-$ary semigroup S. Then $x_\alpha \in \underline{C_A}$ if and only if $x \in A$.*

Proof. Assume that $x_\alpha \in \underline{C_A}$. Then $C_A(x) \geq \alpha$. Hence, $C_A(x) = 1$. This implies $x \in A$. Conversely, assume that $x \in A$. Then $C_A(x) = 1 \geq \alpha$ for all $\alpha \in (0, 1]$. This implies $x_\alpha \in \underline{C_A}$. □

Theorem 2.5. *For any nonempty subsets A and B of an n-ary semigroup S, then $A \subseteq B$ if and only if $\underline{C_A} \subseteq \underline{C_B}$.*

Proof. Assume that $A \subseteq B$. Let $x_\alpha \in \underline{C_A}$. By Theorem 2.4, $x \in A$. Since $A \subseteq B$, $x \in B$. By Theorem 2.4, $x_\alpha \in \underline{C_B}$. Thus, $\underline{C_A} \subseteq \underline{C_B}$. Conversely, assume that $\underline{C_A} \subseteq \underline{C_B}$. Let $x \in A$. By Theorem 2.4, $x_\alpha \in \underline{C_A}$. Since $\underline{C_A} \subseteq \underline{C_B}$, $x_\alpha \in \underline{C_B}$. By Theorem 2.4, $x \in B$. Thus, $A \subseteq B$. □

Theorem 2.6. *For any fuzzy subsets g and h of an n-ary semigroup S, then $g \subseteq h$ if and only if $\underline{g} \subseteq \underline{h}$.*

Proof. Assume that $g \subseteq h$. Thus, $g(x) \leq h(x)$ for all $x \in S$. Let $x_\alpha \in \underline{g}$. Then $h(x) \geq g(x) \geq \alpha$. Hence, $x_\alpha \in \underline{h}$. Conversely, assume that $\underline{g} \subseteq \underline{h}$. Let $x \in S$. If $g(x) = 0$, then $g(x) \leq h(x)$. Assume $g(x) \neq 0$ and let $\alpha = g(x)$. Then $x_\alpha \in \underline{g}$. So $x_\alpha \in \underline{h}$. Hence, $h(x) \geq \alpha = g(x)$. So $g \subseteq h$. □

A fuzzy subset g of an n-ary semigroup S is called a *fuzzy n-ary subsemigroup* of S if $g(f(a_1^n)) \geq \min\{g(a_1), g(a_2), \ldots, g(a_n)\}$ for all $a_1, a_2, \ldots, a_n \in S$.

Theorem 2.7. *Let g be a nonzero fuzzy subset of S. Then g is a fuzzy n-ary subsemigroup of S if and only if $\underline{g}$ is an n-ary subsemigroup of $\underline{S}$.*

Proof. Assume that g is a fuzzy n-ary subsemigroup of S. Let $(a_1)_{\alpha_1}$, $(a_2)_{\alpha_2}$, $\ldots$,$(a_n)_{\alpha_n} \in \underline{g}$. So, $g(a_i) \geq \alpha_i$ for all i. Then

$$g(f(a_1^n)) \geq \min\{g(a_1), g(a_2), \ldots, g(a_n)\} \geq \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

Hence, $(f(a_1^n))_{\min\{\alpha_1,\alpha_2,\ldots,\alpha_n\}} \in \underline{g}$. Thus, $\underline{g}$ is an n-ary subsemigroup of $\underline{S}$. Conversely, assume that $\underline{g}$ is an n-ary subsemigroup of $\underline{S}$. Let $a_1, a_2, \ldots, a_n \in S$ and we choose $\alpha_i = g(a_i)$ for all i. If $\alpha_i = 0$ for some i, then $\min\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = 0 \leq g(f(a_1^n))$. Assume $\alpha_i \neq 0$ for all i. This implies $(a_i)_{\alpha_i} \in \underline{g}$ for all i. Hence,

$$(f(a_1^n))_{\min\{\alpha_1,\alpha_2,\ldots,\alpha_n\}} \in \underline{g}.$$

Therefore,

$$g(f(a_1^n)) \geq \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = \min\{g(a_1), g(a_2), \ldots, g(a_n)\}.$$

Thus, g is a fuzzy n-ary subsemigroup of S. □

A fuzzy subset g of an n-ary semigroup S is called a *fuzzy i-ideal* of S if $g(f(a_1^n)) \geq g(a_i)$ for all $a_1, a_2, \ldots, a_n \in S$.

Lemma 2.8. *Let g be a nonzero fuzzy subset of S. Then g is a fuzzy i-ideal of S if and only if $\underline{g}$ is an i-ideal of $\underline{S}$.*

Proof. Assume that g is a fuzzy i-ideal of S. Let $a_{\alpha_i} \in \underline{g}$ and $(x_1)_{\alpha_1}, \ldots, (x_{i-1})_{\alpha_{i-1}}, (x_{i+1})_{\alpha_{i+1}}, \ldots, (x_n)_{\alpha_n} \in \underline{S}$. Then $g(a) \geq \alpha_i$. Therefore,

$$g(f(x_1^{i-1}, a, x_{i+1}^n)) \geq g(a) \geq \alpha_i \geq \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

Hence, $f(x_1^{i-1}, a, x_{i+1}^n)_{\min\{\alpha_1,\alpha_2,\ldots,\alpha_n\}} \in \underline{g}$. Then $\underline{g}$ is an i-ideal of $\underline{S}$. Conversely, assume that $\underline{g}$ is an i-ideal of $\underline{S}$. Let $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, a \in S$. If $g(a) = 0$, then $g(f(x_1^{i-1}, a, x_{i+1}^n)) \geq 0 = g(a)$. Assume that $g(a) \neq 0$ and let $\alpha = g(a)$. This implies $a_\alpha \in \underline{g}$. By assumption, $f(x_1^{i-1}, a, x_{i+1}^n)_\alpha \in \underline{g}$. Therefore, $g(f(x_1^{i-1}, a, x_{i+1}^n)) \geq \alpha = g(a)$. Thus, g is a fuzzy i-ideal of S. □

Lemma 2.9. *Let A be a nonempty subset of S. Then A is an i-ideal of S if and only if C_A is a fuzzy i-ideal of S.*

Proof. Assume that A is an i-ideal of S. Let $a_1, a_2, \ldots, a_n \in S$.
Case 1: $a_i \in A$. Then $f(a_1^n) \in A$. Hence, $C_A(f(a_1^n)) = 1 \geq C_A(a_i)$.
Case 2: $a_i \notin A$. Then $C_A(a_i) = 0 \leq C_A(f(a_1^n))$.
Thus, C_A is a fuzzy i-ideal of S. Conversely, assume that C_A is a fuzzy i-ideal of S. Let $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in S$ and $a \in A$. So, $C_A(a) = 1$. Then $C_A(f(x_1^{i-1}, a, x_{i+1}^n)) \geq C_A(a) = 1$. Hence, $f(x_1^{i-1}, a, x_{i+1}^n) \in A$. Thus, A is an i-ideal of S. □

Theorem 2.10. *Let A be a nonempty subset of S. Then A is an i-ideal of S if and only if $\underline{C_A}$ is an i-ideal of $\underline{S}$.*

Proof. By Lemma 2.8 and Lemma 2.9. □

Theorem 2.11. *Let A be a nonempty subset of S. Then A is an ideal of S if and only if $\underline{C_A}$ is an ideal of $\underline{S}$.*

Proof. By Theorem 2.10. □

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