International Journal of Mathematics and
Computer Science, **13**(2018), no. 2, 171–178

On Analytic Properties of a Sigmoid Function

Uzoamaka A. Ezeafulukwe[1], **Maslina Darus** [2], **Olubunmi Abidemi Fadipe-Joseph**[3,4]

[1]Mathematics Department
Faculty of Physical Sciences
University of Nigeria
Nsukka, Nigeria

[1,2]School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600, Bangi, Selangor, Malaysia

[3]Department of Mathematics
Faculty of Physical Sciences
University of Ilorin
P. M. B. 1515, Ilorin, Nigeria

[4]The Abdus Salam International Center for Theoretical Physics
Trieste, Italy

email: uzoamaka.ezeafulukwe@unn.edu.ng, maslina@ukm.edu.my, famelov@unilorin.edu.ng

(Received January 18, 2018, Accepted February 21, 2018)

Abstract

In this article, we look at certain properties of a sigmoid function and determine the starlikeness and the convexity of this function.

Key words and phrases: sigmoid function, Bernoulli function, starlike, convex.
AMS (MOS) Subject Classification: 30C45.
ISSN 1814-0432, 2018, http://ijmcs.future-in-tech.net

1 Introduction, Definitions and Preliminaries

1.1 Introduction

Let $\mathcal{S}$ denote the class of functions, which are analytic, univalent in the unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and of the normalized form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

This class of univalent functions and some of its subclasses are defined by geometric conditions. We recall the following definitions for subclasses of $\mathcal{S}$ which are classes of starlike and convex functions $\mathcal{S}^*$ and $\mathcal{S}^c$ respectively:

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \ \mathcal{R}e\{\frac{zf'(z)}{f(z)}\} > 0, \quad z \in \mathcal{U} \right\},$$

$$\mathcal{S}^c = \left\{ f \in \mathcal{A} : \mathcal{R}e(1 + \frac{zf''(z)}{f'(z)}) > 0, \ z \in \mathcal{U} \right\}.$$

The geometric properties of the following examples of U had being studied since the 19^{th} century viz:

i $f(z) = \frac{z}{1-z} \in \mathcal{S}^c$,

ii $f(z) = \frac{z}{(1-z)^2} \in \mathcal{S}^*$,

iii $f(z) = \frac{1}{2}\log(\frac{1+z}{1-z}) \in \mathcal{S}^c$.

These are among few to mention.

The "largest" of this class of functions, the Koebe function $k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}[(\frac{1+z}{1-z})^2 - 1]$ named after the German Mathematician P. Koebe added a great breakthrough to the univalent theory of functions. Koebe's great achievement includes the first correct proof of the Riemann Mapping Theorem. Bieberbach's guess [1] that the modulus of the coefficients of $f(z)$ in equation (1.1) is less than or equal to k, $k \in \mathbb{N}$. It was later proved by de Brange [2] after seventy years of its first appearance. The proof of this conjecture paved the way to geometric function theory, of which this article falls under although we had not yet calculated the coefficient problem.

In order to study the sigmoid fuction, we recall first Mocanu's [6] claim by showing that the Bernoulli function

$$\upsilon(z) = \frac{z}{e^z - 1}$$

which is analytic in the disc $\mathcal{U}_b = \{z \in \mathbb{C} : |z| < 2\pi\}$ with series expansion

$$f(z) = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n},$$

where $\mathcal{B}_{2n}$ in the Bernoulli numbers, is convex in $\mathcal{U}$.

Later, Şerb [7] determined the radius of convexity of the inverse of the Bernoulli function $\upsilon(z)$. Then, Fadipe-Joseph et al [3] studied the modified sigmoid function

$$g(z) = \frac{2}{1 + e^{-z}}.$$

They showed that $\mathcal{R}e\{g(z)\} > 0$ and $\mathcal{R}e\{g'(z)\} > 0$ in the disc

$$\mathcal{U}_s = \left\{z \in \mathbb{C} : |z| < \frac{\pi}{2}\right\} \supset \mathcal{U} \left\{z \in \mathbb{C} : |z| < 1\right\}.$$

They also calculated a series of the modified sigmoid function and obtained

$$g(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right]^m.$$

They solved some coefficient problem of $g(z)$.

In this article we calculate the starlikeness and convexity of a sigmoid function

$$G(z) = \frac{1}{1 + e^{-z}}, z \in \mathbb{C}.$$

We also show some consequences of starlikeness and convexity of the sigmoid function. We need the following lemmas and definitions to establish our results.

1.2 Definitions and Preliminaries

Let $G(z)$ be the sigmoid function

$$G(z) = \frac{1}{1+\exp(-z)},\ z \in \mathbb{C}, \tag{1.2}$$

then $G(z)$ is analytic in the disc $\mathcal{U}_{sp} = \{z \in \mathbb{C} : |z| < \pi\}$.

Lemma 1.1. *[4] Suppose that the function $\mathcal{H} : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition*

$$\mathcal{R}e\{\mathcal{H}(is,t)\} \leq 0$$

for all real s and for all

$$t \leq -\frac{k(1+s^2)}{2},\ k \in \mathbb{N}.$$

If the function

$$p(z) = 1 + p_k z^k + \dots$$

is analytic in $\mathcal{U}$ and

$$\mathcal{R}e\{\mathcal{H}(p(z), zp^{'}(z))\} > 0 \quad z \in \mathcal{U},$$

then

$$\mathcal{R}e\{p(z)\} > 0, \quad z \in \mathcal{U}.$$

Lemma 1.2. *[5] Let $\varsigma, \gamma \in \mathbb{C}$ with $\varsigma \neq 0$ and let $h \in \mathcal{H}(\mathcal{U})$ with $h(0) = c$. If $\mathcal{R}e\{\varsigma h(z) + \gamma\} > 0, \quad z \in \mathcal{U}$, then the solution of the following differential equation:*

$$q(z) + \frac{zq^{'}(z)}{\varsigma q(z) + \gamma} = h(z), \quad z \in \mathcal{U}; \quad q(0) = c$$

is analytic in $\mathcal{U}$ and satisfies the inequality given by $\mathcal{R}e\{\varsigma q(z) + \gamma\} > 0$, $z \in \mathcal{U}$.

2 Starlikeness and Convexity of a sigmoid function

We apply the technique used in [8] to prove the following theorem:

Theorem 2.1. *Let* $F(z) = Log(1+e^z)$ *with* $z \in U$. *If*

$$\mathcal{R}e\{1 + \frac{z\chi''(z)}{\chi'(z)}\} > -\delta, \tag{2.1}$$

where $\chi(z) = F(z) + \frac{z}{\tau}F'(z)$ *with* $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$, *for* $z = r\exp(i\theta)$ *and* $(\tau > 2\delta > 0)$, *then* $F(z)$ *is convex in* $\mathcal{U}$.

Proof. Let

$$\Phi(z) = 1 + \frac{zF''(z)}{F'(z)} = 1 + \frac{z}{1+e^z}.$$

We want to show that $\mathcal{R}e\{\Phi(z)\} > 0, \quad z \in \mathcal{U}$.

Since $\chi(z) = F(z) + \frac{z}{\varsigma}F'(z)$ and $\chi'(z) = (1+\frac{1}{\varsigma})F'(z) + \frac{1}{\varsigma}zF''(z)$,

$$1 + \frac{z\chi''(z)}{\chi'(z)} = \Phi(z) + \frac{z\Phi'}{\Phi(z)+\varsigma} := h(z) \tag{2.2}$$

and

$$\mathcal{R}e\{h(z)+\delta\} > 0, \quad z \in \mathcal{U}. \tag{2.3}$$

By Lemma 1.2, the differential equation (2.2) has a solution $\Phi \in \mathcal{H}(\mathcal{U})$ with $h(0) = \Phi(0) = 1$. Let $\mathcal{H}(\vartheta, \eta) = \vartheta + \frac{\eta}{\vartheta+\varsigma} + \delta$ where $\varsigma > \delta > 0$.

Now, from $\mathcal{R}e\{\mathcal{H}(\Phi(z)), z\Phi'(z)\} > 0, z \in \mathcal{U}$, we need to verify that, $\mathcal{R}e\{\mathcal{H}(is,t)\} \le 0 \quad (s \in \mathbb{R}, t \le \frac{-(1+s)}{2})$, and

$$\mathcal{R}e\{\mathcal{H}(is,t)\} = \frac{-\tau(1+\delta^2)}{2(\tau^2+\delta^2)} + \delta \le \frac{-\Upsilon(\tau,s)}{|\tau+is|},$$

where

$$\Upsilon(\tau,s) = (\tau-2\delta)s^2 - (2\delta\tau^2 - 4\delta^2\tau)s + \tau^3\delta^2 - 2\tau^2\delta^3 = (\tau-2\delta)(s-\delta\tau)^2.$$

Since $\tau > 2\delta > 0, \Upsilon(\tau,s) \ge 0$, by Lemma 1.1, we conclude that $\mathcal{R}e(\Phi(z)) > 0, z \in \mathcal{U}$. This ends the proof of Theorem 2.1.

Theorem 2.2. *Let* $G(z) = \frac{1}{1+e^{-z}},\ z \in \mathbb{C}$. *Then* $G(z)$ *is starlike in* $\mathcal{U}$.

Proof. Let $G(z) = \frac{1}{1+e^{-z}}$. Then

$$\mathcal{R}e\left\{\frac{zG^{'}(z)}{G(z)}\right\} = \frac{\cos\theta + \imath\cos\theta\cos(\varrho) + \imath\varrho\sin(\varrho)}{(1+\imath^2+2\imath\cos(\varrho))}, \tag{2.4}$$

where

$$\varrho = \sin\theta,\ \imath = \exp(\cos\theta),\ \frac{-\pi}{2} < \theta < \frac{\pi}{2}.$$

Since $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$, $\mathcal{R}e\left\{\frac{zG^{'}(z)}{G(z)}\right\} > 0$.

Theorem 2.3. *Let* $G(z) = \frac{1}{1+e^{-z}}, z \in \mathbb{C}$, *then* $G(z)$ *is convex in* $\mathcal{U}$.

Proof.

$$\left\{1 + \frac{zG^{''}(z)}{G^{'}(z)}\right\} = \left\{1 - z + \frac{2z}{1+e^z}\right\} = 1 - \cos\theta + \frac{2\left(\cos\theta + \imath\cos\theta\cos(\varrho) + \imath\varrho\sin(\varrho)\right)}{(1+\imath^2+2\imath\cos(\varrho))}, \tag{2.5}$$

where

$$\varrho = \sin\theta,\ \imath = \exp(\cos\theta),\ \frac{-\pi}{2} < \theta < \frac{\pi}{2}.$$

Since $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$, $\cos\theta \leq 1$ and $\mathcal{R}e\left\{\frac{zG^{'}(z)}{G(z)}\right\} > 0$, then

$$\left\{1 + \frac{zG^{''}(z)}{G^{'}(z)}\right\} > 0.$$

We conclude that $G(z)$ is convex.

3 Consequences of starlikeness and convexity of a sigmoid function

In this section we show the consequences of the results above.

Theorem 3.1. *Let* $F(z) = \log(1 + e^z)$, $G(z) = \frac{1}{1+e^{-z}}$, $z \in \mathcal{U}$. *Then the following are equivalent:*
(i) $\{\mathcal{R}eG(z)\} > 0$,
(ii) $\mathcal{R}e\{1 + \frac{zF''(z)}{F'(z)}\} > 0$,
(iii) $\mathcal{R}e\{1 + \frac{zG''(z)}{G'(z)}\} > 0$.

Proof. Let $G(z) = \frac{1}{1+e^{-z}}$, $\Phi(z) = 1 + \frac{zF''(z)}{F'(z)}$, $z \in \mathcal{U}$.

Assume $G(z) > 0$; that is, $\mathcal{R}e\{\frac{e^z}{1+e^z}\} > 0$. Then $\mathcal{R}e\{\frac{z}{1+e^z}\} > 0$, $z \in \mathcal{U}$. So (ii) is true.

Assume $\mathcal{R}e\{\Phi(z)\} > 0$. By Theorem 2.3, (iii) is true.

Assume (iii) is true; that is, $\mathcal{R}e\{\frac{z}{1+e^z}\} \equiv \mathcal{R}e\{z - \frac{z}{1+e^{-z}}\} > 0, z \in \mathcal{U}$.
But

$$\mathcal{R}e\{1 + \frac{1}{1 + e^{-z}}\} > \mathcal{R}e\{\frac{z}{1 + e^z}\}$$

in $\mathcal{U}$. Then $\mathcal{R}e\{\frac{1}{1+e^{-z}}\} > 0$. This finishes the proof of the theorem.

Acknowledgement: The work here is fully supported by UKM grant: GUP-2017-064.

References

[1] L. Bieberbach, Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, Sitzungsberichte Preussische Akademmie der Wissenschaften,(1916), 940–955.

[2] L. de Branges, A proof of the Bieberbach conjecture, Act. Math., **154,** (1985), 137–152.

[3] O. A. Fadipe-Joseph, A. T. Oladipo, U. A. Ezeafulukwe, Modified sigmoid function in univalent function theory, Int. J. Math. Sci. Engg. App., **7,** (2013), 313–317.

[4] S. S. Miller, P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J., **28,** (1981), 157–172.

[5] S. S. Miller, P. T. Mocanu, Univalent solutions of Briot-Bouquet differential equations, J. Differential Equations, **56,** (1985), 297–309.

[6] P. T. Mocanu, Convexity of some particular functions, Studia Univ. Babes-Bolyai Math., **29,** (1984), 70–73.

[7] I. Serb, The radius of convexity and starlikeness of a particular function, Mathematica Montisnigri, **7,** (1996), 65–69.

[8] R.-G. Xiang, Z.-G. Wang, M. Darus, A family of integral operators preserving subordination and superordination, Bull. Malaysian Math. Sci. Soc., **2,** no. 1, (2010), 121–131.