International Journal of Mathematics and
Computer Science, **13**(2018), no. 2, 187–205

Sum and product of Fuzzy ideals of a ring

Rabah Kellil

College of Science Al Zulfi
Majmaah University
Saudi Arabia

email: r.kellil@mu.edu.sa

(Received February 15, 2018, Accepted March 27, 2018)

Abstract

In this paper we continue our investigations about some fuzzy algebraic structures started in [12], [13], [14]. As in [12], these studies had risen from the study of [1], [3], [4], [5], [6]. We introduce a new definition of a fuzzy ideal which, we think, is more realistic than the classical one. This new definition permits us to define the notion of the sum of two fuzzy ideals. Moreover, we prove that given two classical fuzzy ideals μ, β, there exists a fuzzy ideal, called the product of μ and β, extending the notion of product of ideals in the crisp case. We also construct a crisp ideal consisting of elements x such that $\mu(y) \leq \beta(xy),\ \forall y \in R$ and give an example of such ideal in the case of the ring $\mathbb{Z}$. Our principal motivation for introducing these definitions is to state some possible generalization of the classical and well-known ones of the crisp algebra.

1 Introduction

Fuzzy mathematics has as one of its principal origin the fabulous seminal paper [24] on fuzzy sets written by Lotfi Asker Zadeh. Actually the concepts introduced form an important branch of mathematics. Initially its topics were related to fuzzy set theory and fuzzy logic. In the last two decades interest has shifted to the development of fuzzy algebra in view of

Key words and phrases: Fuzzy operation, fuzzy ideal, fuzzy prime ideal, fuzzy maximal ideal.
AMS (MOS) Subject Classifications: 08A72, 03E72, 20N25, 08A05.
ISSN 1814-0432, 2018, http://ijmcs.future-in-tech.net

the generalization of the well-known rules of algebraic structures. Among these algebraic fuzzy structures and fuzzy ideals of a ring took a large part of that interest in mathematical research activity. In [20] Azriel Rosenfeld introduced the concept of Fuzzy Groups. A detailed study has been done by different researchers and in various topics of algebraic systems. More exactly, the concept of a fuzzy ideal of a ring was introduced by Liu [18] and in [19]. On the other hand, Malik ([21], [22]) introduced the concept of prime fuzzy ideal, maximal fuzzy ideal and radical of a fuzzy ideal. In [23], Mukherjee and Sen studied the fuzzy ideals to characterize regular and Noetherian rings and then determined all the prime fuzzy ideals of the ring $\mathbb{Z}$. The notion of fuzzy quotient ring was introduced by Kumar and Kuroaka in [15] and by Kuroki in [16]. By the notion of fuzzy quotient ring, Lee [17] characterized the Artenian and Noetherian ring.

In the present paper, we introduce a new definition of fuzzy ideal to define the sum of fuzzy ideals. We first recall some basic concepts for the sake of completeness and then show why our definition of fuzzy ideal is more realistic than the classical one.

2 Preliminaries

This section contains some definitions and known properties of the fuzzy subgroups and fuzzy ideals. All the rings are supposed to be commutative and unitary.

Definition 2.1. *Let $(G, \star)$ be a group and e its identity. A fuzzy subset μ of G is a fuzzy subgroup of G if and only if*

1. $\mu(e) = 1$,
2. $\mu(a \star b) \geq min\{\mu(a), \mu(b)\}; \quad \forall a; b \in G$,
3. $\forall a \in G;\ \mu(a) = \mu(a^{-1})$.

It is called normal if in addition $\mu(a \star b \star a^{-1}) = \mu(b), \forall a \in G$.

2.1 Fuzzy Operations

Definition 2.2. *A fuzzy operation on a set E is a mapping $f : E \times E \times E \longrightarrow [0, 1]$ such that, for $x, y \in E$ there exists a unique element $z \in E$ satisfying $f(x, y, z) = 1$.*

If f is a fuzzy operation on a set E, then

1. *f is commutative if $\forall x, y, z \in E$, $f(x, y, z) = f(y, x, z)$.*

2. *f is associative if $\forall x, y, z, a, b \in E$, $f(x, y, a) > 0$, $f(y, z, b) > 0$ and $f(a, z, \alpha) = f(x, b, \beta)$ imply $\alpha = \beta$.*

3. *$e \in E$ is an identity of E for f if $f(x, e, x) = f(e, x, x)$, $\forall x \in E$.*

Proposition 2.3. *If E has an identity e for f and $f(x, e, x) = 1$, $\forall x \in E$, then this identity is unique.*

Proof. Suppose that e, e' are two identities of E for f. Then $1 = f(e', e, e') = f(e, e', e')$ and $1 = f(e, e', e) = f(e', e, e)$ and so $f(e', e, e') = f(e', e, e) = 1$. But there exists a unique element x such that $f(e', e, x) = 1$. Consequently, $e = e'$. □

Definition 2.4. *Let $f : E \times E \times E \longrightarrow [0, 1]$ be a fuzzy operation on a set E and e be the unique identity of E for f. An element $x \in E$ is symmetrizable if $f(x, x', e) = f(x', x, e)$ for some $x' \in E$.*

Definition 2.5. *Let $f : E \times E \times E \longrightarrow [0, 1]$ be a fuzzy operation on a set E and e be the unique identity of E for f. An element $x' \in E$ if it exists such that $f(x, x', e) = f(x', x, e) = 1$ is called a symmetric element of $x \in E$.*

Definition 2.6. *Let $f : E \times E \times E \longrightarrow [0, 1]$ be a fuzzy operation on a set E. An element $a \in E$ is left (resp. right) regular or cancelable if for any elements $x, y, z \in E$, the equality $f(a, x, z) = f(a, y, z)$ (resp. $f(x, a, z) = f(y, a, z)$) implies $x = y$. It is regular or cancelable if it is left and right regular.*

Proposition 2.7. *If f a fuzzy operation on E has an identity e. Any left (resp. right) regular element has at most one symmetric element.*

Proof. Suppose that x is left regular and has two symmetric elements x' and x''. Then $f(x, x', e) = f(x', x, e) = f(x, x'', e) = f(x'', x, e) = 1$ and so $f(x, x', e) = f(x, x'', e) = 1$ which implies the equality $x' = x''$. For the right regularity the proof is trivial. □

2.2 Fuzzy ideals

Let us recall the classical definition of a fuzzy ideal of a ring.

Definition 2.8. *(classical definition) Let $(R, +, \times)$ be a ring and 0_R its identity for $+$. A fuzzy subset μ of R is called a fuzzy ideal of R if*

1. $\mu(0_R) = 1$,
2. $\mu(a) = \mu(-a)$,
3. $\mu(a + b) \geq min\{\mu(a), \mu(b)\}, \quad \forall a, b \in R$,
4. $\mu(a \times b) \ \geq \max\{\mu(a), \mu(b)\}, \quad \forall a, b \in R$.

By changing the last condition in the above definition we get another definition of a fuzzy ideal of a ring as follows

Definition 2.9. *(New definition) Let $(R, +, \times)$ be a ring and 0_R its identity for $+$, a fuzzy subset μ of R is a right (respectively left) fuzzy ideal of R if*

1. $\mu(0_R) = 1$,
2. $\mu(a) = \mu(-a)$,
3. $\mu(a + b) \geq min\{\mu(a), \mu(b)\}, \quad \forall a, b \in R$,
4. $\mu(a) > \ 0 \implies \mu(a \times b) \ > \ 0$ *(respectively* $\mu(b \times a) > \ 0), \quad \forall a, b \in R$.

It is called proper if $\mu(1) = 0$.

Remark 2.10. 1. *It is easy to prove that if μ is a fuzzy ideal in the sense of the new definition, it is also a fuzzy ideal in a classical meaning.*

2. *It is clear that if μ is an ideal, $\mu(1) \geq \theta \in [0, 1]$ implies $\mu(x) = \mu(x.1) \geq \sup\{\mu(1), \mu(x)\} \geq \theta$, for all $x \in R$. So to avoid the trivial case, in the sequel, the terminology of proper ideal means that $\mu(1) \neq 1$.*

3. *In the classical definition of fuzzy ideal the product $a.b$ is closer to the ideal that each of the elements a and b if we consider that the membership function μ indicates the behavior of the elements of the ring with respect to the ideal. On the other hand, in our definition, the element $a.b$ is close to the ideal (i.e. $\mu(a.b) > \ 0$) if one of the elements a or b is close to the ideal.*

Definition 2.11. *A fuzzy subset μ of R is a two sided fuzzy ideal or simply fuzzy ideal of R if*

1. $\mu(0_R) = 1$,
2. $\mu(a) = \mu(-a)$,

3. $\mu(a+b) \geq min\{\mu(a), \mu(b)\}, \quad \forall a, b \in R,$

4. $\mu(a) > \ 0 \implies \mu(a \times b) = \mu(b \times a) \ > \ 0, \quad \forall a, b \in R.$

Definition 2.12. *A fuzzy ideal is called prime if* $\mu \neq \underline{1}, \mu(a \times b) > 0 \implies \mu(a) > 0$ *or* $\mu(b) > 0, \ \forall a, b \in R.$

Since $\mu \neq \underline{1}$, *it is clear that* $\mu(1) = 0$.

Remark 2.13. *In the sequel, since the rings are assumed to be commutative, the term fuzzy ideal means two sided fuzzy ideal.*

The following illustration gives an example of a fuzzy prime ideal.

Example 1. *Let* $\theta \in]0,1[$. *If* μ *is a fuzzy subset of the ring* R *satisfying,*

$$\mu(x) = \begin{cases} 1 & if \quad x = 0 \\ 0 & if \quad x = 1 \\ 0 & if \quad \mu(1-x) \neq 0 \\ 0 & if \quad x \text{ is invertible} \\ \theta & otherwise \end{cases}$$

is a prime fuzzy ideal of the ring $(R, +, \times)$.

Now we introduce the notion of maximal fuzzy ideal that we want to coincides with the definition of maximum crisp ideal when the membership function μ takes its values in $\{0, 1\}$.

Definition 2.14. *Let* $(R, +, \times)$ *be a ring. A proper fuzzy ideal* μ *of* R *is said to be maximal if for any fuzzy ideal* ν *of* R *the following implication holds:*

$$\mu \leq \nu \quad \implies \quad \nu = \mu \quad or \quad \nu = \underline{1},$$

where $\underline{1}$ *is the constant fuzzy subset defined by* $\underline{1}(x) = 1, \quad \forall x \in R$.

To close this section we give some results obtained in [12] (the reader interested in their proofs can consult the cited paper).

Proposition 2.15. [12]. *If* μ *is a proper fuzzy ideal of* R, *then for all* $\theta \in [0,1]$ *the set* $I = \{x \in R \ \mid \ \mu(x) \geq \theta\}$ *is an ideal of the ring* $(R, +, \times)$.

In addition, if $\forall a, b \in R, a \times b \in I$ *implies* $\mu(a \times b) = \mu(a).\mu(b)$, *then* I *is a prime ideal of the ring* R.

Let us give an example of a fuzzy prime ideal which is not maximal.

Example 2. *As an application of the above example, let R be the ring $\mathbb{Z}/8\mathbb{Z}$. Then the mapping μ given by,*

a	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
$\mu(a)$	*1*	*0*	*0.5*	*0*	*0.6*	*0*	*0.5*	*0*

,

is as required in the above example so it is a fuzzy prime ideal. We can also prove it directly. For this, to see that $\mu(a.b) \geq \max\{\mu(a), \mu(b)\}$ and $\mu(a.b) > 0$ imply $\mu(a) > 0$ or $\mu(b) > 0$, it suffices to draw the multiplication table and affect each cell with the value of element obtained in the corresponding cell by the product.

If we change the value of $\mu(4) = 0.6$ in the example to $\mu(4) = 0.7$, the resulting function is still a fuzzy ideal greater than the preceding one and different from the constant fuzzy ideal $\underline{1}$.

Proposition 2.16. *Let $(R, +, \times)$ be a ring and 0_R its identity under addition. Then the intersection $\alpha \wedge \mu$ of two fuzzy ideals α and μ of R is a fuzzy ideal of R.*

In the sequel, if α is a fuzzy set of R, we define for any $n \in \mathbb{N}$, α^n by $\forall x \in R,\ \alpha^n(x) = (\alpha(x))^n$.

Proposition 2.17. [12]. *Let μ be a fuzzy ideal of a ring R and let*

$$\beta \in \sqrt{\mu} = \{\alpha : R \longrightarrow [0,1] \ \mid\ \exists n \in \mathbb{N} \ \text{ and } \ \alpha^n = \mu\}, \quad \text{then}$$

1. *β is an ideal,*
2. *if μ is prime, then β is also prime,*
3. *if μ is maximal, so is β.*

Proposition 2.18. [12]. *Let μ be a fuzzy ideal on a ring R and a be an element of the center of R. The fuzzy set $\mu_a : R \longrightarrow [0,1]$ defined by $\mu_a(x) = \mu(a.x)$ is a fuzzy ideal of R.*

If μ is prime and $\mu(a) = 0$, then μ_a is prime.

If μ is maximal and the mapping $b \longmapsto ab$, from R to R is surjective, the ideal μ_a is also maximal.

Example 3. [12]. *As an example of a fuzzy ideal, let us denote by $r_a \in \{0, 1, 2\}$ the remainder of the Euclidean Division (long division) of a by 3 in $\mathbb{Z}$. For any element $a \in \mathbb{Z}$, we set:*

$$\mu(a) = \begin{cases} 1 & if \ \ r_a = 0 \\ 1/3 & if \ \ r_a \neq 0 \end{cases}$$

We have seen that μ is a fuzzy subgroup of $\mathbb{Z}$. Indeed, for all $a, b \in \mathbb{Z}$ and from the properties of the remainders and by performing all the possible values of r_a and r_b, we have $\mu(ab) = r_a r_b \geq \max\{r_a, r_b\}$ and then μ is a fuzzy ideal of the ring $(\mathbb{Z}, +, \times)$.

Proposition 2.19. [12]. *Let $\theta \in]0,1]$ be fixed, a be an element of a commutative ring $(R, +, .)$ and*
$\mu : R \longrightarrow [0,1]$ be a mapping satisfying the following conditions.

1. $\theta \geq \mu(a) \geq \dfrac{\theta}{2}$,
2. $\mu(0) = 1$,
3. $\mu(x) = \begin{cases} a & If\ x = a.y\ for\ some\ y \in R \\ \theta - \mu(a) & otherwise \end{cases}$

   Then μ is an ideal.

3 Main results

We start this section by introducing the following notion of a fuzzy ideal generated by an element.

Definition 3.1. *Let a be an element of a commutative ring R. The ideal generated by a is the least ideal $\mu : R \longrightarrow [0,1]$ such that*

$$\mu(a) > 0,\ and$$

$$\mu(x) = \mu(a)\ \ if\ \ and\ \ only\ \ if\ \ x = ay\ \ for\ \ some\ \ y \in R \setminus \{0\}.$$

We then get the following:

Proposition 3.2. *Let $a \in R \setminus \{0\}$ and $\mu(a)$ be fixed in $(0,1]$, the fuzzy set μ_a defined by*

$$\mu_a(x) = \begin{cases} 1 & if \quad x = 0 \\ \mu(a) & if \quad x = ay,\ y \in R \setminus \{0\} \\ 0 & otherwise \end{cases}$$

is an ideal satisfying the required conditions of Definition 21.

Proof. Indeed we have

1. either $x = at$ and then $-x = a(-t)$ or $x \neq at$ and $-x \neq at'$ and all the two cases one has $\mu(x) = \mu(-x)$.

2. For $x, y \in R$, we have

$$\mu_a(x+y) = \begin{cases} 1 & if \quad x+y=0 \\ \mu(a) & if \quad x+y = at, t \neq 0 \\ 0 & otherwise \end{cases}$$

   More exactly, in the case where $x + y = 0$ we always have $\mu_a(x + y) = 1 \geq \inf\{\mu_a(x), \mu_a(y)\}$.

   If $x+y = at,\ t \neq 0$ then $\mu_a(x+y) = \mu_a(a) \geq \inf\{\mu_a(x), \mu_a(y)\}$ since the elements x and y can not be simultaneously null in which case $\mu_a(x+y)$ will be less than $1 = \inf\{\mu_a(x), \mu_a(y)\}$.

   If $x + y \neq at, \quad t \in R \setminus \{0\}$, then either $\mu_a(x + y) = 1\ if\ x + y = 0$ or $\mu_a(x + y) = 0$. In the first case, the result follows and in the second one, at least one of the elements x or y can not be a multiple of a so then $\inf\{\mu_a(x), \mu_a(y)\} = 0$ and the result follows.

3. Let $x, y \in R$ then

$$\mu_a(x.y) = \begin{cases} 1 & if \quad x.y = 0 \\ \mu(a) & if \quad x.y = at, t \neq 0 \\ 0 & otherwise \end{cases}$$

   If the first case it is clear that if $x.y = 0$, then $\mu_a(x.y) = 1 \geq \max\{\mu_a(x), \mu_a(y)\}$ either R is an integral domain or not.

   If $x.y = at,\ t \neq 0$, then neither x nor y are null and so $\mu_a(x.y) = \mu_a(a) \geq \max\{0, \mu_a(a)\} \geq \max\{\mu_a(x), \mu_a(y)\}$.

   If $x.y \neq at,\ \forall\ t \in R$, then neither x nor y are multiple of a and so $\mu_a(x) = 0 = \mu_a(y)$ and

   $\mu_a(x.y) = 0 \geq \max\{\mu_a(x), \mu_a(y)\} = 0$. From Remark 1, we get axiom 4 of Definition 10.

□

Remark 3.3. *In the case of crisp algebra, the intersection of ideals is an ideal, but due to the fact that* $\mu_i(a) > 0,\ i \in I$ *doesn't imply necessarily that* $\bigwedge_{i \in I} \mu_i(a) > 0$, *the intersection of fuzzy ideals is not necessary an ideal. We introduce the following definition to solve this anomalous and then define the notion of sum of ideals.*

Definition 3.4. *A ring R is called fuzzy Artenian if any nonempty set of fuzzy ideals has a least element with respect to the inclusion. A fuzzy set μ is included in a fuzzy set β if for any element x of the universe R we have $\mu(x) \leq \beta(x)$.*

With the above definition we get the following:

Proposition 3.5. *Let μ, β be two ideals of a fuzzy Artenian ring R. The meet of ideals α_i of R such that $\mu \vee \beta \leq \alpha_i$ is an ideal called the sum of the ideals μ and β and will be denoted by $\mu + \beta$.*

Proof. Let $M = \{fuzzy\ ideals\ \alpha_i,\ i \in I\ \ and\ \ \mu \vee \beta \leq \alpha_i\}$. This set is not empty since $\underline{1} \in M$. So it has a least element say α.

$\alpha \in M \implies \mu \vee \beta \leq \alpha$. On the other hand, $\bigwedge_{i \in I} \alpha_i \leq \alpha$ and from the definition of α we have $\alpha \leq \alpha_i$ which implies $\alpha \leq \bigwedge_{i \in I} \alpha_i$. Finally $\alpha = \bigwedge_{i \in I} \alpha_i$ and the proof follows.

□

Remark 3.6. *In the above proposition we can replace the condition, R is a fuzzy artenian by the following condition:*

any fuzzy ideal is included in a finite number of fuzzy ideals.

Example 4. *Let a, b two constants in $]0, 1]$ and μ_4, μ_6 be the fuzzy ideals of the ring $\mathbb{Z}$ generated by 4 and 6 respectively. that is*

$$\mu_4(x) = \begin{cases} 1 & if \quad x = 0 \\ a & if \quad x \in 4\mathbb{Z} \setminus \{0\} \\ 0 & \quad otherwise \end{cases},$$

$$\mu_6(x) = \begin{cases} 1 & if \quad x = 0 \\ b & if \quad x \in 6\mathbb{Z} \setminus \{0\} \\ 0 & \quad otherwise \end{cases}.$$

We want to prove that $\mu_4 + \mu_6 \leq \mu_2$, where μ_2 is defined by

$$\mu_2(x) = \begin{cases} 1 & if \quad x = 0 \\ c & if \quad x \in 2\mathbb{Z} \setminus \{0\} \\ 0 & \quad otherwise \end{cases}, \quad with \quad c \geq \max\{a, b\}.$$

Let $x \in \mathbb{Z}$, one has

$$(\mu_4 \vee \mu_6)(x) = \begin{cases} 1 & if \quad x = 0 \\ \max\{a,b\} & if \quad x \in 12\mathbb{Z} \setminus \{0\} \\ a & if \quad x \in 4\mathbb{Z} \setminus 6\mathbb{Z} \\ b & if \quad x \in 6\mathbb{Z} \setminus 4\mathbb{Z} \\ 0 & otherwise \end{cases},$$

So it is easy to see that $(\mu_4 \vee \mu_6)(x) \leq \mu_2(x), \quad \forall x \in \mathbb{Z}$.

On the other hand if $\mu \geq (\mu_4 \vee \mu_6)$, *then* $supp(\mu) \supset supp[(\mu_4 \vee \mu_6)]$ *and as the support of a fuzzy ideal is a crisp ideal, necessary* $supp(\mu) = 2\mathbb{Z}$ *or* $supp(\mu) = \mathbb{Z}$. *Since* $(\mu_4 \vee \mu_6)(x) \leq \mu_2(x)$ *it cannot be* $\mathbb{Z}$. *So if* μ *is the least ideal satisfying the required condition it must be as follow:*

$$\mu(x) = \begin{cases} 1 & if \quad x = 0 \\ \max\{a,b\} & if \quad x \in 12\mathbb{Z} \setminus \{0\} \\ a & if \quad x \in 4\mathbb{Z} \setminus 6\mathbb{Z} \\ b & if \quad x \in 6\mathbb{Z} \setminus 4\mathbb{Z} \\ f(x) & if \quad x \in 2\mathbb{Z} \setminus (4\mathbb{Z} \cup 6\mathbb{Z}) \\ 0 & othewise \end{cases},,$$

where the function $f(x)$ *is non null, minimal and satisfying the conditions required such that* μ *is a fuzzy ideal.*

Remark 3.7. *The function* $f(x)$ *exists by the proof of the example, but due to the fact that we haven't required that our ring is fuzzy artenian or that any fuzzy ideal is included in a finite number of fuzzy ideals, we are not able to determine it explicitly.*

Remark 3.8. *From the example above, if we set for any* $c \geq \max\{a, b\}$

$$\mu_2^c(x) = \begin{cases} 1 & if \quad x = 0 \\ c & if \quad x \in 2\mathbb{Z} \\ 0 & otherwise \end{cases},$$

it is easy to see that $(\mu_4 \vee \mu_6)(x) \leq \mu_2^c(x), \quad \forall x \in \mathbb{Z}$. *So if* $b \geq a$, *the least fuzzy ideal of the form* $\mu_2^c(x), c \geq \max\{a, b\}$ *is exactly* $\mu_2^b(x)$. *So a question rises up from this remark: Is* $\mu_2^b(x)$ *the least fuzzy ideal which contains* $(\mu_4 \vee \mu_6)$*?*

Proposition 3.9. *Let* μ *be a proper ideal of a ring* R. *For any invertible element* a *of* R *we have* $\mu(a) = 0$.

Proof. If $\mu(a) > 0$, then we have $0 = \mu(1) = \mu(aa^{-1}) > 0$ which is a contradiction. □

Remark 3.10. *In the general case if μ and ν are two fuzzy ideals of a commutative ring R, any fuzzy set α such that for all $x, y \in R$, $\alpha(-x) = \alpha(x)$, $\alpha(x+y) \geq \min\{\alpha(x), \alpha(y)\}$ and*

$$max\{\mu(x), \nu(x)\} \leq \alpha(x) \leq \alpha(xy) \quad \forall\ x, y \in R \setminus \mathcal{U}(R).$$

is a fuzzy ideal which contains $\mu + \nu$. Indeed,

1. *$\alpha(0) \geq max\{\mu(0), \nu(0)\} = max\{1, 1\} = 1$ so $\alpha(0) = 1$,*
2. *if $\alpha(x) > 0$, then $x \notin \mathcal{U}(R)$ and xy is so for any $x, y \in R$ and by the hypothesis we get $\alpha(xy) > 0$.*
3. *For $x, y \in R$, by our assumptions, we have the two other axioms, that is $\alpha(-x) = \alpha(x)$ and $\alpha(x+y) \geq \min\{\alpha(x), \alpha(y)\}$ for all $x, y \in R$.*

In the rest of the paper, we return to the classical notion of fuzzy ideal as in Definition 8 and as it is said that the axiom 4 in this definition is stronger than the axiom 4 in Definition 10, the results are still valid for our definition.

In ring theory we know that any maximal ideal is a prime ideal. We want to know if this result is still valid for the fuzzy ideals. We start by an example where it is shown that this is true for the ideals given in the example.

Example 5. *Let R be a ring, where we assume that the sum of two elements is invertible if and only if one of the elements is invertible and let $\mu : R \longrightarrow [0, 1]$ be such that,*

$$\mu(x) = \begin{cases} 0 & if \quad x \in \mathcal{U}(R) \\ 1 & \quad otherwise \end{cases}$$

Then μ is both maximal and prime fuzzy ideal.

Proof. 1. 0 is not invertible. So $\mu(0) = 1$.

2. We have a invertible if and only if $-a$ is invertible. So $\forall x \in R \;\; \mu(x) = \mu((-x))$.
3. Suppose that $\mu(x+y) = 0$. Then $x+y$ is invertible and by our hypothesis both x and y are invertible. Then $\mu(x) = 0$ or $\mu(y) = 0$ and we conclude that $\forall x, y \in R, \quad \mu(x+y) \geq \min\{\mu(x), \mu(y)\}$.
4. let $x, y \in R$. To prove that $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$, one can distinguish 3 cases:

(a) If x and y are invertible, then xy is also invertible and $\mu(xy) = 0 \geq 0 = \max\{\mu(x), \mu(y)\}$.

(b) If x is invertible and y is not invertible, then $z = xy$ is not invertible for otherwise $y = zx^{-1}$ the product of two invertible becomes invertible, a contradiction. In this case $\mu(xy) = 1 = \mu(y) \geq \max\{\mu(x), \mu(y)\}$.

(c) Suppose x and y are not invertible. Suppose that xy is invertible. Then there exists $z \in R$ such that $(xy)z = 1$ but $(xy)z = 1 \iff x(yz) = 1$. Then x is invertible, a contradiction. So if x and y are not invertible, so is xy and consequently also in this case $\mu(xy) = 1 \geq \max\{\mu(x), \mu(y)\}$.

5. Suppose now that ν is an ideal such that $\mu \leq \nu$ and $\mu \neq \nu$. Then there exists at least one element $x \in R$ such that $\mu(x) < \nu(x)$. From the definition of μ we have $\mu(x) = 0$, So x is invertible. On the other hand, $\nu(1) = \nu(xx^{-1}) \geq \eta(x) > 0$. By our hypothesis on fuzzy subsets of a ring we have $\nu(1) = 1$. For any element $a \in R$, $\nu(a) = \nu(1a) \geq \nu(1) = 1$ and then $\nu = \underline{1}$. This proves that μ is maximal.

6. It is easy to prove that μ is also prime.

□

Theorem 6. *Let μ and β be two fuzzy ideals of a ring R. For a decomposition of an element x as a finite sum, $x = \sum a_i b_i$ where $a_i, b_i \in R$, we set $s(x) = \min_i\{min(\mu(a_i), \beta(b_i))\}$. Denote by S_x the set of all the possible decompositions of x as above. If any element admits a finite number of decompositions,*

$$\eta(x) = \max_{S_x} s(x)$$

is an ideal, called in the sequel the product of the fuzzy ideals μ and β

Proof. 1. It is clear that among the different decompositions of 0 we have the decomposition $0 = 0.0$ So $\eta(0) = \min\{\mu(0), \beta(0)\} = 1$.

2. If $\sum a_i b_i$ is a decomposition of x, then $\sum(-a_i)b_i$ and $\sum a_i(-b_i)$ are also two decompositions of $-x$. On the other hand, for any decomposition of $-x$ there corresponds two decompositions of x. Since $\mu(-a_i) = \mu(a_i)$ and $\beta(-a_i) = \beta(a_i)$, $\eta(x) = \eta(-x)$.

3. For the third axiom, it suffices to prove it for two decompositions $x = ab$ and $x = a'b'$ of x and two similar decompositions of y as $y = cd$ and $y = c'd'$.

   It suffices to treat three cases.

   (a) First case.
   Suppose that $\mu(a) \leq \beta(b), \beta(b') \leq \mu(a'), \mu(c) \leq \beta(d), \mu(c') \leq \beta(d'), \mu(a) \geq \mu(a'), \mu(c) \leq \mu(c')$ and $\mu(c) \leq \mu(a) \leq \mu(c')$ then $\eta(x) = \mu(a)$ and $\eta(y) = \mu(c')$. On the other hand

   $$\min\{\min(\mu(a), \beta(b)), \min(\mu(c), \beta(d))\} = \min\{\mu(a), \mu(c)\} = \mu(c),$$

   $$\min\{\min(\mu(a), \beta(b)), \min(\mu(c'), \beta(d'))\} = \min\{\mu(a), \mu(c')\} = \mu(a),$$

   $$\min\{\min(\mu(a'), \beta(b')), \min(\mu(c), \beta(d))\} = \min\{\beta(b'), \mu(c)\} = \mu(c) \textit{ or } \beta(b'),$$

   $$\min\{\min(\mu(a'), \beta(b')), \min(\mu(c'), \beta(d'))\} = \min\{\beta(b'), \mu(c')\} = \beta(b').$$

   As $\mu(a)$ is great than $\beta(b')$ and $\mu(c)$ we get

   $$\eta(x + y) = \mu(a) \geq \mu(a) = \min\{\eta(x), \eta(y)\}.$$

   (b) Second case.
   Suppose that $\mu(a) \leq \beta(b), \beta(b') \leq \mu(a'), \mu(c) \leq \beta(d), \mu(c') \leq \beta(d'), \mu(a) \geq \mu(a'), \mu(c) \leq \mu(c')$ and $\mu(a) \leq \mu(c)$ then $\eta(x) = \mu(a)$ and $\eta(y) = \mu(c')$. On the other hand

   $$\min\{\min(\mu(a), \beta(b)), \min(\mu(c), \beta(d))\} = \min\{\mu(a), \mu(c)\} = \mu(a),$$

   $$\min\{\min(\mu(a), \beta(b)), \min(\mu(c'), \beta(d'))\} = \min\{\mu(a), \mu(c')\} = \mu(a),$$

   $$\min\{\min(\mu(a'), \beta(b')), \min(\mu(c), \beta(d))\} = \min\{\beta(b'), \mu(c)\} = \beta(b'),$$

   $$\min\{\min(\mu(a'), \beta(b')), \min(\mu(c'), \beta(d'))\} = \min\{\beta(b'), \mu(c')\} = \beta(b').$$

   As $\mu(a)$ is great than $\beta(b')$ we get

   $$\eta(x + y) = \mu(a) \geq \mu(a) = \min\{\eta(x), \eta(y)\}.$$

(c) Third case.

Suppose that $\mu(a) \leq \beta(b), \beta(b') \leq \mu(a'), \mu(c) \leq \beta(d), \mu(c') \leq \beta(d'), \mu(a) \geq \mu(a'), \mu(c) \leq \mu(c')$ and $\mu(c') \leq \mu(a)$ then $\eta(x) = \mu(a)$ and $\eta(y) = \mu(c')$. On the other hand

$$\min\{\min(\mu(a), \beta(b)), \min(\mu(c), \beta(d))\} = \min\{\mu(a), \mu(c)\} = \mu(c),$$

$$\min\{\min(\mu(a), \beta(b)), \min(\mu(c'), \beta(d'))\} = \min\{\mu(a), \mu(c')\} = \mu(c'),$$

$$\min\{\min(\mu(a'), \beta(b')), \min(\mu(c), \beta(d))\} = \min\{\beta(b'), \mu(c)\} = \mu(c) \ or \ \beta(b'),$$

$$\min\{\min(\mu(a'), \beta(b')), \min(\mu(c'), \beta(d'))\} = \min\{\beta(b'), \mu(c')\} = \mu(c') \ or \ \beta(b').$$

As $\mu(a)$ is great than $\beta(b')$ and $\mu(c')$ we get

$$\eta(x+y) = \beta(b') \geq \mu(c') = \min\{\eta(x), \eta(y)\} \ \ if \ \ \beta(b') \geq \mu(c')$$

or

$$\eta(x+y) = \mu(c') \geq \mu(c') = \min\{\eta(x), \eta(y)\} \ \ if \ \ \beta(b') \leq \mu(c').$$

We can then conclude that

$$\forall x, y \in R, \quad \eta(x+y) \geq \min\{\eta(x), \eta(y)\}.$$

4. For the fourth axiom, it suffices to prove it for two decompositions $x = ab$ and $x = a'b'$ of x and two similar decompositions of y as $y = cd$ and $y = c'd'$. The product xy can then be written as $abcd = acbd$ or $abc'd' = ac'bd'$ or $a'b'cd = a'cb'd$ or $a'b'c'd' = a'c'b'd'$.

   It suffices to treat three cases.

   (a) First case.

   Suppose that $\mu(a) \leq \beta(b), \beta(b') \leq \mu(a'), \mu(c) \leq \beta(d), \mu(c') \leq \beta(d'), \mu(a) \geq \mu(a'), \mu(c) \leq \mu(c')$ and $\mu(c) \leq \mu(a) \leq \mu(c')$ then $\eta(x) = \mu(a)$ and $\eta(y) = \mu(c')$,so $\max\{\eta(x), \eta(y)\} = \mu(c')$. On the other hand

$$A = \min(\mu(ac), \beta(bd)) \geq \min(\max(\mu(a), \mu(c)), \max(\beta(b), \beta(d))) = \mu(a)$$

$$B = \min(\mu(ac'), \beta(bd')) \geq \min(\max(\mu(a), \mu(c')), \max(\beta(b), \beta(d'))) =$$

$$\begin{cases} \min(\mu(c'),\beta(b)) = \mu(c') & if \quad \mu(c') \geq \mu(a) \ \ and \ \ \beta(b) \geq \beta(d') \\ \min(\mu(c'),\beta(b)) = \mu(a) & if \quad \mu(c') \leq \mu(a) \ \ and \ \ \beta(b) \geq \beta(d') \\ \min(\mu(c'),\beta(d')) = \mu(c') & if \quad \beta(b) \leq \beta(d') \end{cases}$$

$$C = \min(\mu(a'c),\beta(b'd)) \geq \min(\max(\mu(a'),\mu(c)),\max(\beta(b'),\beta(d))) =$$

$$\begin{cases} \min(\mu(a'),\beta(b')) = \mu(a') & if \quad \mu(a') \geq \mu(c) \ \ and \ \ \beta(b') \geq \beta(d) \\ \min(\mu(c),\beta(b')) = \beta(b') & if \quad \mu(a') \leq \mu(c) \ \ and \ \ \beta(b') \geq \beta(d) \\ \min(\mu(a'),\beta(d)) = \mu(a') & if \quad \mu(a') \geq \mu(c) \ \ and \ \ \beta(b') \leq \beta(d) \\ \min(\mu(c),\beta(d)) = \mu(c) & if \quad \mu(a') \leq \mu(c) \ \ and \ \ \beta(b') \leq \beta(d) \end{cases}$$

$$D = \min(\mu(a'c'),\beta(b'd')) \geq \min(\max(\mu(a'),\mu(c')),\max(\beta(b'),\beta(d'))) =$$

$$\min(\mu(c'),\beta(d')) = \mu(c').$$

We conclude that

$$\max\{A,B,C,D\} = \mu(c') \geq \mu(c') = \max\{\eta(x),\eta(y)).$$

(b) Second case.

Suppose that $\mu(a) \leq \beta(b), \beta(b') \leq \mu(a'), \mu(c) \leq \beta(d), \mu(c') \leq \beta(d'), \mu(a) \geq \mu(a'), \mu(c) \leq \mu(c')$ and $\mu(a) \leq \mu(c)$ then $\eta(x) = \mu(a)$ and $\eta(y) = \mu(c')$,so $\max\{\eta(x),\eta(y)\} = \mu(c')$. On the other hand

$$A = \min(\mu(ac),\beta(bd)) \geq \min(\max(\mu(a),\mu(c)),\max(\beta(b),\beta(d))) =$$

$$\begin{cases} \min(\mu(c),\beta(b)) = \beta(b) & if \quad \beta(b) \leq \mu(c) \\ \min(\mu(c),\beta(b)) = \mu(c) & otherwise \end{cases}$$

$$B = \min(\mu(ac'),\beta(bd')) \geq \min(\max(\mu(a),\mu(c')),\max(\beta(b),\beta(d'))) = \mu(c')$$

$$C = \min(\mu(a'c),\beta(b'd)) \geq \min(\max(\mu(a'),\mu(c)),\max(\beta(b'),\beta(d))) = \mu(c)$$

$$D = \min(\mu(a'c'),\beta(b'd')) \geq \min(\max(\mu(a'),\mu(c')),\max(\beta(b'),\beta(d'))) = \mu(c').$$

We conclude taking in account the above conditions that

$$\max\{A,B,C,D\} = \mu(c') \geq \mu(c') = \max\{\eta(x),\eta(y)).$$

(c) Third case.

Suppose that $\mu(a) \leq \beta(b), \beta(b') \leq \mu(a'), \mu(c) \leq \beta(d), \mu(c') \leq \beta(d'), \mu(a) \geq \mu(a'), \mu(c) \leq \mu(c')$ and $\mu(c') \leq \mu(a)$ then

$$A = \min(\mu(ac),\beta(bd)) \geq \min(\max(\mu(a),\mu(c)),\max(\beta(b),\beta(d))) = \mu(a)$$

$$B = \min(\mu(ac'), \beta(bd')) \geq \min(\max(\mu(a), \mu(c')), \max(\beta(b), \beta(d'))) = \mu(a)$$

$$C = \min(\mu(a'c), \beta(b'd)) \geq \min(\max(\mu(a'), \mu(c)), \max(\beta(b'), \beta(d))) =$$

$$\begin{cases} \min(\mu(a'), \beta(b')) = \beta(b') & if \quad \beta(b') \geq \beta(d) \\ \min(\mu(a'), \beta(d)) = \alpha & if \quad \mu(a') \geq \mu(c) \;\; and \;\; \beta(b') \leq \beta(d) \\ \min(\mu(c), \beta(d)) = \mu(c) & if \quad \mu(a') \leq \mu(c) \end{cases}$$

$$D = \min(\mu(a'c'), \beta(b'd')) \geq \min(\max(\mu(a'), \mu(c')), \max(\beta(b'), \beta(d'))) =$$

$$\begin{cases} \min(\mu(a'), \beta(b')) = \beta(b') & if \quad \beta(b') \geq \beta(d') \\ \min(\mu(c'), \beta(d')) = \mu(c') & if \quad \mu(a') \leq \mu(c') \\ \min(\mu(a'), \beta(d')) = \gamma & if \quad \mu(a') \geq \mu(c') \;\; and \;\; \beta(b') \leq \beta(d') \end{cases}$$

Taking in account the above conditions and considering the possible values of α and γ that

$$\max\{A, B, C, D\} = \mu(a) \geq \mu(c') = \max\{\eta(x), \eta(y)).$$

□

Proposition 3.11. *Let μ and β be two fuzzy ideals of a commutative ring R. The set*

$$I = \{x \in R \mid \mu(y) \leq \beta(xy) \;\; \forall y \in R\}$$

is an ideal of the ring R.

Proof.

1. For all $y \in R$ we have $\mu(y) \leq 1 = \beta(0) = \beta(0y)$.
2. Let $x, x' \in I$. Then for all $y \in R$, one has $\mu(y) \leq \beta(xy)$ and $\mu(y) \leq \beta(x'y)$, so $\mu(y) \leq \min\{\beta(xy), \beta(x'y)\}$. But $\min\{\beta(xy), \beta(x'y)\} \leq \beta(xy - x'y)$ and the last expression is just the value $\beta((x - x')y)$ and then $x - x' \in I$.
3. Let now $x \in I$ and $a \in R$. Then for any $y \in R$ we have $\mu(ay) \geq \max\{\mu(a), \mu(y)\}$ so $\mu(y) \leq \mu(ay) \leq \beta(x(ay)) = \beta((xa)y)$ so $xa \in I$ and the conclusion follows.

□

Remark 3.12. *If the fuzzy ideals in the previous proposition are such $\mu \leq \beta$, from the relation $\beta(xy) \geq \max\{\beta(x), \beta(y)\} \geq \beta(y)$ we deduce that $I = R$.*

Proposition 3.13. *Let μ_4 and μ_6 as in Example 24, where we suppose that $a \leq b$. The ideal*

$$I = \{x \in \mathbb{Z} \mid \mu_4(y) \leq \mu_6(xy) \;\; \forall y \in \mathbb{Z}\} = 3\mathbb{Z}.$$

Proof. 1. $0 \in I$ as proved above,

2. if $x = 3k \neq 0$ and since 4 divides y implies that 6 divides xy, so if $\mu_4(y) = a$ then $\mu_6(3ky) = b$ and if $\mu_4(y) = 0$ then $\mu_6(3ky) = 0$ and the result follows.

3. if $x = 3k + r, r \in \{1, 2\}$ we can always find an element y in $\mathbb{Z}$ such that $yx \in 4\mathbb{Z} \setminus 6\mathbb{Z}$.

   (a) For $x = 3k + 1$, it suffices to take $y = 4$. In this case, $\mu_4(y) = a$ and as $xy = 6(2k) + 4 \notin 6\mathbb{Z}$, we have $\mu_6(xy) = 0$.

   (b) For $x = 3k + 2$ it suffices to take $y = 4$. In this case $\mu_4(y) = a$ and as $xy = 6(2k + 1) + 2 \notin 6\mathbb{Z}$, we have $\mu_6(xy) = 0$.

□

4 Conclusion and Future Works

From the present paper two questions may be of interest. The first one is: can we generalize the previous proposition to the fuzzy ideals of the form $\mu_a,\ a \in \mathbb{N}$? The second deals with the determination of the function $f(x)$ of Example 4.

Acknowledgement

This research was supported by the D.S.R of Majmaah University, Saudi Arabia.

References

[1] M. T. Abu Osman, Some properties of fuzzy subgroups, J. Sains Malaysiana, **12,** no. 2, (1984), 155–163.

[2] M. T. Abu Osman, On the direct product of fuzzy subgroups,Fuzzy Sets and Systems, **12,** (1984), 87–91.

[3] M. T. Abu Osman, On normal fuzzy subgroups, Proc. Second Malaysian National Mathematical Science Symposium, (1986), 186–195 (in Malay).

[4] J. M. Anthony, H. Sherwood, Fuzzy group redefined, J. Math. Anal. Appl., **69,** (1979), 124–130.

[5] J. M. Anthony, H. Sherwood, A characterization of fuzzy subgroups, Fuzzy Sets and Systems, **7,** (1982), 297–305.

[6] D. Boixader, J. Jacas, J. Recasens, Fuzzy equivalence relations: advanced material, in: P. Dubois (Ed.), Fundamentals of Fuzzy Sets, Kluwer, Dordrecht, 2000, 261–290.

[7] M. Demirci, Vague groups, J. Math. Anal. Appl., **230,** (1999), 142–156.

[8] M. Demirci, Fuzzy functions and their fundamental properties, Fuzzy Sets and Systems, **106,** (1999), 239–246.

[9] M. Demirci, Fuzzy functions and their applications, J. Math. Anal. Appl., **252,** (2000), 495–517.

[10] J. Jacas, J. Recasens, Fuzzy numbers and equality relations, Proc. FUZZY-IEEE, **93,** San Francisco, 1993, 1298–1301.

[11] J. Jacas, J. Recasens, Fuzzy T-transitive relations: eigenvectors and generators, Fuzzy Sets and Systems, **72,** (1995), 147–154.

[12] R. Kellil, New approaches on some fuzzy algebraic structures, J. Intell. Fuzzy Systems, **32,** no. 1, (2017), 579–587.

[13] R. Kellil, On the set of subhypergroups of certain canonical hypergroups C(n), JP Journal of Algebra, Number Theory and Applications, **38,** no. 2, (2016), 109–211.

[14] R. Kellil, On 2-Absorbing Fuzzy Ideals,JP Journal of Algebra, Number Theory and Applications, **39,** no. 4,(2017), 479–505.

[15] R. Kumar, Fuzzy subgroups, fuzzy ideals, and fuzzy cosets: Some properties, Fuzzy Sets and Systems, **48,** (1992), 267=-274.

[16] T. Kuraoka, N. Kuroki, On fuzzy quotient rings induced by fuzzy ideals, Fuzzy Sets and Systems, **47,** (1992), 381–386

[17] K. H. Lee, On fuzzy quotient rings and chain conditions, J. Korean Soc. Math. Educ. Ser.B: Pure Appl. Math., **7,** no. 1, (2000), 33–40.

[18] W. J. Liu, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems, **8,** (1982), 133–139.

[19] W. J. Liu, Operations on fuzzy ideals, Fuzzy Sets and Systems, **11,** (1983), 31–41.

[20] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., **35,** (1971), 512–517.

[21] D. S. Malik, Fuzzy ideals of Artinian rings, Fuzzy Sets and Systems, **37,** (1990), 111–115.

[22] D. S. Malik, J. N. Mordeson, Fuzzy maximal, radical, and primary ideals of a ring, Inform. Sci., **53,** (1991), 237–250.

[23] T. K. Mukherjee, M. K. Sen, Rings with chain conditions, Fuzzy Sets and Systems, **39,** (1991), 117–123.

[24] L.A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci., **3,** (1971), 177–200.