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Strongly r-Clean Rings

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Abstract

In this article we introduce the concept of strongly r-clean ring. The ring R is said to be strongly r-clean ring if every element of R is sum of a regular element and an idempotent which commute with each other which is a generalization of strongly r-clean ring. Further, we study various properties of the strongly r-clean rings. Furthermore, we show that the triangular matrix representation of strongly r-clean need not be strongly r-clean. Moreover, we prove that if R is abelian then strongly r-clean ring is equivalent to the well known ring theoretic properties such as: strongly clean rings, clean rings and r-clean rings

1 Introduction

Throughout this article, R denotes an associative ring with identity. Clean rings were introduced by Nicholson in [10], where an element in a ring R is called clean if it is a sum of an idempotent and a unit. Further in 1999, Nicholson [9] called an element of a ring R as strongly clean if it is the sum

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of a unit and an idempotent that commute with each other that means if $a = e + u$, where $e^2 = e$ and u is a unit of R such that $eu = ue$ and R is strongly clean if each of its elements is strongly clean. Again clearly from [9], a strongly clean ring is clean, and the converse holds for an abelian ring (that is, all idempotents in the ring are central). Local rings and strongly π-regular rings are well-known examples of the strongly clean rings. After that, Han and Nicholson [7], studied some different types of extensions of clean rings. Since then many people have worked on studying the properties of clean rings and strongly clean rings example [1, 5, 10]. Various properties of regular rings were studied in [6].The clean rings were further extended to r-clean rings and the r-clean rings were introduced by Ashrafi and Nasibi [2, 3] and they defined that an element x of a ring R is r-clean if $x = r + e$, where $r \in Reg(R)$ and $e \in Id(R)$. A ring R is said to be r-clean if each of its element is r-clean. In [2] the authors proved many important results like homomorphic image of a r-clean ring is r-clean, direct product of R is r-clean if and only if R is r-clean and many more. Further in [3] they proved that every abelian r-clean ring is clean. Also they have shown that every r-clean ring with 0 and 1 as the only idempotent is exchange. From [3] we have learnt numerous properties about r-clean rings such as for an Abelian ring R, R is r-clean if and only if R is clean. Let R be a ring such that 0 and 1 are the only idempotents in R, then R is r-clean if and only if its Von-Neumann. Motivated by all above studies, in this paper, we introduce the concept of new class of rings, strongly r-clean rings which is generalization of the class of strongly clean rings and stronger class of r-clean rings. Further, we prove that the existence of this class is not trivial and also prove the triangular matrix representation of strongly r-clean need not be strongly r-clean. After that, we study various properties of the strongly r-clean rings. Moreover, We give the necessary and sufficient condition when the clean rings, strongly clean rings r-clean rings and strongly r-clean rings become equivalent. Throughout the paper $Id(R)$, $Reg(R)$ and $U(R)$ always stand for the set of all idempotent, regular and units of R respectively.

2 Strongly-r-Clean Rings

In this paper, we introduce the concept of strongly r-clean ring which is a generalization of strongly clean ring. Further, we show that the triangular matrix representation of strongly r-clean need not be strongly r-clean. After that, we study some basic properties of strongly r-clean rings. Moreover, we prove that if R is strongly r-clean and 0 and 1 are the only idempotents in R, then R is strongly clean. Finally, we prove that R is strongly r-clean if and only if R/I is strongly r-clean when I be a regular ideal of R and all the idempotents can be lifted modulo I.

Definition 2.1. *An element a is said to be strongly r-clean if $a = r + e$, where $r \in Reg(R)$ and $e \in Id(R)$ and $re = er$. A ring R is called strongly r-clean if every element of R is strongly r-clean.*

From the above definition it is clear that every strongly clean is strongly r-clean. And also clear that every strongly r-clean is r-clean. Now, in the following examples we show that converse need not be true.

Example 2.2. $R = \left[\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in Z_2\right]$ *be a ring then R is r-clean but not strongly r-clean.*

Proof. Since Z_2 is r-clean so R is also r-clean from [3, Theorem 2.14] the elements of R: $M_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $M_5 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $M_6 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $M_7 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $M_8 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus the set of idempotents $Id(R) = e_1 = M_8$, $e_2 = M_2$, $e_3 = M_3$, $e_4 = M_4$, $e_5 = M_5$, $e_6 = M_7$. Also the regular elements are $Reg(R) = r_1 = M_8$, $r_2 = M_2$, $r_3 = M_3$, $r_4 = M_4$, $r_5 = M_5$,. Now, we prove that R has at least one element $x = r + e$ such that $re \neq er$ for all possible $e \in Id(R)$ and $r \in Reg(R)$. Take $M_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$ so all possible representations of M_1 with commutative condition are $M_1 = e_5 + r_2$ but $e_5 r_2 \neq r_2 e_5$, $M_1 = e_4 + r_3$ but $e_4 r_3 \neq r_3 e_4$, $M_1 = e_3 + r_4$ but $e_3 r_4 \neq r_4 e_3$ also $M_1 = e2 + r_5$ but $e_2 r_5 \neq r_5 e_2$. Thus there exists $M_1 \neq 0$ such that M_1 is r-clean but not strongly r-clean.

Theorem 2.3. *The triangular matrix representation of strongly r-clean need not be strongly r-clean.*

Proof. Clear from the example 2.2.

In [2], Ashrafi and Nasibi proved that homomorphic image of an r-clean ring is r-clean. Here, we prove the above result for the strongly r-clean rings:

Theorem 2.4. *Any homomorphic image of a strongly r-clean ring is strongly r-clean.*

Proof. Let R be r-clean and I be an Ideal of R. Let $\bar{x} = x + I \in R/I$. Since R is strongly r-clean therefore $x = r + e$ and $re = er$, where $r \in Reg(R)$, $e \in Id(R)$. Let $\bar{x} = x + I = \bar{r} + \bar{e} \in R/I$. We show that $\bar{r}$ is regular, $\bar{e}$ is idempotent and they commute. As r is regular there exist $y \in R$ such that $ryr = r$, therefore $\bar{r}\bar{y}\bar{r} = \bar{r}$, so $\bar{r}$ is regular and $\bar{e} \in Id(R/I)$. Consider $\bar{r}\bar{e} = (r+I)(e+I) = (re+I) = (er+I)$ since R is strongly r-clean.It follows that $(e+I)(r+I) = \bar{e}\bar{r}$. This implies R/I is strongly r-clean. □

Theorem 2.5. *A direct product $R = \prod R_i$ of rings R_i is strongly r-clean if and only if the same is true for each R_i.*

Proof. Let $x = (x_i) \in R_i$. Each R_i is strongly r-clean if and only if for each i, $x_i = e_i + r_i$ such that $e_i \in Id(R_i)$, $r_i \in Reg(R_i)$ and $e_i r_i = r_i e_i$. Since $r_i \in Reg(R_i)$, there exists $y_i \in R_i$ such that $r_i = r_i y_i r_i$ thus $x = (r_i) + (e_i)$ where $(r_i)_{i\in I} \in Reg(\prod_{i\in I} R_i)$ and $(e_i)_{i\in I} \in Id(\prod_{i\in I} R_i)$ and $(e_i)(r_i) = (r_i)(e_i)$. for each $i \in I$. Thus R is strongly r-clean. Converse, If $R = \prod_{i\in I} R_i$ is strongly r-clean. This implies R_i is strongly r-clean from Theorem 2.4, since R_i is a homomorphic image of R. □

Theorem 2.6. *Let R be a ring, then $x \in R$ is strongly r-clean if and only if $1 - x$ is strongly r-clean.*

Proof. Let $x \in R$ be strongly r-clean then $x = e+r$,$er = re$ where $e \in Id(R)$, $r \in Reg(r)$. Consider $1 - x = 1 - (r + e) = -r + (1 - e)$. Also r is regular, therefore there exists $y \in R$ such that $ryr = r$. We can write $(-r) = (-r)(-y)(-r)$, $-r \in Reg(R)$. Also $(1-e)^2 = (1-e)(1-e) = 1-e$ therefore $(1-e) \in Id(R)$. Now consider $(-r)(1-e) = (-r+re) = (-r+er) = (1-e)(-r)$ therefore $(1-x)$ is strongly r-clean.Conversely, if $(1-x)$ is strongly r-clean, write $1-x = r+e$ and $er = re$, where $r \in Reg(R)$, $e \in Id(R)$. Also, $(-r)(1-e) = (1-e)(-r)$ therefore x is strongly r-clean. □

We know that strongly r-clean is generalization of strongly clean ring and also know that the converse of it is not true. Now, in the following result we prove that every strongly r-clean is equivalent to strongly clean if and only if R has only two idempotents 0 and 1.

Theorem 2.7. *If $R \neq 0$ is strongly r-clean ring and 0 and 1 are the only idempotents in R, then R is strongly clean.*

Proof. Since R is strongly r-clean, each $x \in R$ has form $x = e+r$ and $er = re$ where $e \in Id(R), r \in Reg(R)$.If $r = 0$ then $x = e = (2e-1) + (1-e)$. Now $(2e-1) \in U(R)$ and $(1-e)^2 = 1-e$ which means $(1-e) \in Id(R)$. Also

$(2e-1)(1-e) = (1-e)(2e-1)$ which means that x is strongly clean. Hence R is strongly clean.If $r \neq 0$, then there exists $y \in R$ such that $ryr = r$ therefore $ry \in Id(R)$ which is 0 or 1.If $ry = 0$ then $r = ryr = 0$ which is a contradiction therefore $ry = 1$. Similarly $yr = 1$ which means $r \in U(R)$. Thus R is clean. Also $er = re$ since R is strongly r-clean. Hence R is strongly clean. □

Recall that Idempotents lift modulo an ideal I of a ring R if whenever $a^2 - a \in I$ there exists $e^2 = e \in R$ such that $e - a \in R$. In this case we say that e lifts a. We use this definition in the following result:

Theorem 2.8. *Let I be a regular ideal of ring R and suppose that idempotent can be lifted modulo I. Then R is strongly r-clean if and only if R/I is strongly r-clean.*

Proof. If R/I is strongly r-clean, then for any $a \in R$, $a + I$ is strongly r-clean. Thus $\exists$ $(e + I) \in Id(R/I)$ and $(a - e) + I \in Reg(R/I)$ such that $[((a-e)+I)(x+I)((a-e)+I)] = (a-e)+I$ for some $x \in R$, $x+I \in R/I$. Since $(a-e)x(a-e)+I = (a-e)+I \Longrightarrow (a-e)x(a-e)-(a-e) \in I$. Since I is regular so $(a - e)$ is also regular and $(a - e) \in Reg(R)$. Now, we show commutative property, $(e + I)((a - e) + I) = e(a - e) + I = ((a - e) + I)(e + I)$. Since idempotents can be lifted modulo I, we may assume that e is an idempotent of R, therefore a is strongly r-clean. Converse of above theorem is true, since homomorphic image of strongly r-clean is also strongly r-clean. □

3 Strongly r-Clean Rings and Abelian Rings

In this section we find the connections of the strongly r-clean ring with some well known ring theoretic properties such as: strongly clean, clean and r-clean. Further, we obtain that all above ring theoretic properties are equivalent if R is abelian. Moreover, we prove that R is strongly r-clean if and only if $[[x]]$ is strongly r-clean, if whenever R is abelian.

Theorem 3.1. *Let R be a strongly r-clean ring and abelian. Then eRe is also strongly r-clean.*

Proof. Let $a \in eRe \subset R$ which implies $a = e_1 + r$ and $e_1 r = re_1$ where $e_1 \in Id(R)$, $r \in Reg(R)$ as R is strongly r-clean. Since $a \in eRe$, $a = ee_1e + ere$ it follows that $a = e_1e + re$. Now we need to show that re is regular, e_1e is an idempotent and they commute. For this consider $(e_1e)^2 = (e_1e)(e_1e) = e_1(ee_1)e = e_1(e_1e)e = (e_1e_1)(ee) = e_1^2e^2 = e_1e$, therefore e_1e is an idempotent.

Now Consider $eye \in eRe$, $(re)(eye)(re) = (re)(eye)(er) = (re)(eye)(r) = (re)y(er) = (er)y(re) = e(ryr)e = ere \in eRe$, hence re is regular. Now we show the commutativity of R, $(e_1e)(re) = e_1ree = re_1ee = ree_1e = (re)(e_1e)$. Therefore it is a strongly r-clean Ring. □

In the following result we obtain a condition at which strongly r-clean and strongly clean are equivalent.

Theorem 3.2. *Let R be an Abelian Ring. Then R is strongly r-clean if and only if R is strongly clean.*

Proof. Let R be a strongly r-clean and $x \in R$ then $x = r + e'$ and $e'r = re'$ where $e' \in Id(R)$ $r \in Reg(R)$.Therefore $\exists\ y \in R$ such that $ryr = r$, where $e = ry$ and yr are idempotents. Consider $(re + (1 - e))(ye + (1 - e)) = 1$, $(ye + (1 - e))(re + (1 - e)) = 1$. So $u = (re + (1 - e))$ is a unit and $r = eu$. Now set $f = 1 - e$ then $eu + f$ and hence $-(eu + f)$ is a unit and f is an idempotent. Also $r = eu$, $-r = -eu = f - f - eu = f - (f + eu)$. Therefore $-r$ is clean and $\Longrightarrow r + e'$ is clean. Consider $-f(f + eu) = -f^2 - efu = -f - e(1 - e)u = -f, f + eu)(-f) = -f^2 - euf = -f - eu(1 - e) = -f$. Therefore $-r$ is strongly clean. It follows by [1, Lemma 2.1] that x is strongly clean as required. □

A ring R is called to be exchange if the right regular module RR has finite exchange property.We prove the following corollary for Exchange rings:

Corollary 3.3. *Let R be an Abelian ring. Then R is strongly r-clean if and only if R is exchange.*

We also know that a Ring is said to be reduced if it has no (nonzero) nilpotent elements. These rings are called abelian. Therefore we also have the following result.

Corollary 3.4. *Let R be a reduced ring. Then R is strongly r-clean if and only if R is strongly clean.*

From definition 2.1,and example 2.2, it is clear that every strongly r-clean is clean but converse need not be true. Now, we prove that converse of it is true if R is abelian.

Theorem 3.5. *Every Abelian r-clean ring is strongly r-clean.*

Proof. For every $x \in R$, we write $x = e+r$, where $e \in Id(R)$ and $r \in Reg(R)$. As $r \in Reg(R)$, $\exists\ y \in R$ such that $ryr = r$. Since R is an abelian ring so $er = re$ for every $r \in R$. Therefore R is strongly r-clean. □

In [2, Theorem 2.12] the authors have proved that if R is a commutative ring, then $R[x]$ is not r-clean. We show the above result for strongly r-clean.

Theorem 3.6. *If R is commutative ring, then $R[x]$ is not strongly r-clean.*

Proof. By [2, theorem 2.12] we know that if R is commutative ring then $R[x]$ is not r-clean. which further implies that $R[x]$ is not strongly r-clean. □

In [3, Proposition 2.5] it has already been proved that if R be an abelian ring and α an endomorphism of R. Then the following statements are equivalent:

(1) R is an r-clean ring.

(2) The formal power series ring $R[[x]]$ of R is an r-clean ring.

(3) The skew power series ring $R[[x;\alpha]]$ of R is an r-clean ring.

Also in [3, Theorem 2.2] the authors have proved that if R be an abelian ring. Then R is r-clean if and only if R is clean. □

Now in the following result we establish the relation of equivalence between clean, strongly clean, r-clean, Strongly r-clean and $R[[x]]$ being strongly r-clean ring.

Theorem 3.7. *If R is Abelian ring then the following structures are equivalent:*

(i) *R is a clean ring*

(ii) *R is a strongly clean ring*

(iii) *R is a r-clean ring*

(iv) *R is a strongly r-clean ring*

(v) *$R[[x]]$ is strongly r-clean ring*

Proof. (i) being equivalent to (iii) by [3, Theorem 2.2].
(ii) ⇔ (iv) from Theorem 3.2.
(iii) ⇔ (iv) from Theorem 3.5.
(iv) implies (v) since R is strongly r-clean, so R is r-clean. Now from [3, Proposition 2.5] $R[[x]]$ is r-clean. It follows that $R[[x]]$ is strongly r-clean from (iii) being equivalent to (iv). Conversely, from Theorem 2.4, homomorphic image of strongly r-clean is strongly r-clean therefore R is also strongly r-clean being homomorphic image of R.

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