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On Some Characterizations of o-fuzzy subgroups

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Abstract

In this paper, we define the notion of o-fuzzy subgroup and investigate the condition under which a fuzzy subgroup is o-fuzzy subgroup. We introduce the notion of o-fuzzy cosets and establish their algebraic properties. We also initiate the study of o-fuzzy normal subgroups and quotient group with respect to o-fuzzy normal subgroup and prove some of their various group theoretic properties.

1 Introduction

Zadeh initiated the study of fuzzy set in [12] and since then there has been a fabulous concentration in this particular branch of mathematics due to its various applications ranging from computer science and engineering to the study of social and economic behaviors. Rosenfeld initiated the idea of fuzzy groups on fuzzy set and established many basic results of fuzzy groups in

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[8]. Indeed, fuzzy subgroups admit many algebraic properties of groups. For more details about the developments on fuzzy groups, we refer the reader to [9] and [10]. The fuzzy subgroups were redefined in [2]. Later on, Das modified the work of Zadeh and Rosenfeld by defining the level subgroups of a given group in [4]. Chakrabatty and Khare defined the concept of fuzzy homomorphism between two groups and studied its effect to the fuzzy subgroups in [3]. Moreover, the idea of containment of an ordinary kernel of a group homomorphism in fuzzy subgroups was proposed by Ajmal in [1]. The recent developments about the applications of fuzzy sets in different algebraic structures may be viewed in [11], [13] and [14]. Gupta established the theory of t-operators on fuzzy sets in [5].
In this paper, we define a fuzzy set with respect to a t-operator. We use this fuzzy subset to define o-fuzzy subgroup and investigate further theory of this fuzzy subgroup and establish o-fuzzy versions of some basic results of group theory. We prove the homomorphic image (preimage) of an o-fuzzy subgroup is an o-fuzzy subgroup by using the classical homomorphism. We also introduce the concepts of o-fuzzy cosets and o-fuzzy normal subgroups and establish the isomorphism between the quotient group with respect to o-fuzzy normal subgroup and quotient group with respect to the normal subgroup G_{A_o}.

2 Preliminaries

In this section, we study some fundamental characterizations of a fuzzy subgroup which play a key role in obtaining the basic group theoretic results in terms of their respective fuzzy versions. Some details of these concepts are given below which are very essential for our further discussion.
Definition (2.1) [6]: Let X be a nonempty set. A mapping

$$A : X \longrightarrow [0,\ 1]$$

is called a fuzzy subset of X. Let A and B be two fuzzy subsets of a set X. Then the following characterizations of these fuzzy sets have been discussed in [6].

1. $A{\leq}B$ if and only if $A(x){\leq}B(x)$, for all $x{\in}X$.

2. $A = B$ if and only if $A{\leq}B$ and $B{\leq}A$.

3. The complement of the fuzzy set A is A^c and is defined as $A^c(x) = 1 - A(x)$
4. $(A \cap B)(x) = min\{A(x), B(x)\}$, for all $x \in X$.
5. $(A \cup B)(x) = max\{A(x), B(x)\}$, for all $x \in X$.

Definition (2.2) [6]: Let A be a fuzzy subset of a set X and $\delta \in [0, 1]$. The set $A^\delta = \{x \in X : A(x) \geq \delta\}$ is called a level subset of a fuzzy set A.
Definition (2.3) [6]: Let A be a fuzzy subset of a group G. Then A is called a fuzzy subgroup if

1. $A(xy) \geq min\{A(x), A(y)\}$
2. $A(x^{-1}) \geq A(x)$, for all $x, y \in G$

It is easy to show that a fuzzy subgroup of a group G satisfies $A(x) \leq A(e)$ and $A(x^{-1}) = A(x)$, for all $x \in G$ where e is the identity element of G.
Proposition (2.4) [6]: A function $A{:}X \longrightarrow [0, 1]$ is a fuzzy subgroup of a group G if and only if $A(xy^{-1}) \geq min\{A(x), A(y)\}$, for all $x, y \in G$
Proposition (2.5) [6]: If $A{:}G \longrightarrow$ [0, 1] is a fuzzy subgroup of a group G, then

1. $A(x) \leq A(e)$, for all $x \in G$, where e is the identity element of G.
2. $A(xy^{-1}) = A(e)$, which implies that $A(x) = A(y)$, for all $x, y \in G$.

Theorem (2.6) [4]: Let G be a group and A be a fuzzy subset of G. Then A is fuzzy subgroup if and only if the level subset A^δ for $\delta \in$ [0, 1], $A(e) \geq \delta$, is a subgroup of G, where e is an identity of G.
Definition (2.7) [7]: A fuzzy subgroup A of a group G is called a fuzzy normal subgroup if $A(xy) = A(yx)$, for all $x, y \in G$.
Definition (2.8) [7]: Let $A : G \longrightarrow$ [0, 1] be a fuzzy normal subgroup of a group G. For any $x \in G$, the fuzzy set $xA : G \longrightarrow$ [0, 1] defined by $(xA)(y) = A(x^{-1}y)$, for all $y \in G$ is called a left fuzzy coset of A. The right fuzzy coset of A may be defined in a similar way.
Definition (2.9) [1]: Let $f : G_1 \longrightarrow G_2$ be a homomorphism from a group G_1 into a group G_2. Let A and B be fuzzy subsets of G_1 and G_2 respectively. Then $f(A)$ and $f^{-1}(B)$ are respectively the image of fuzzy set A and the inverse image of fuzzy set B, for every $y \in G_2$ defined as

$$f(A)(y) = \begin{cases} sup(A(x) : x \in f^{-1}(y)), & \text{if } f^{-1}(y) \neq \phi \\ 1, & \text{if } f^{-1}(y) = \phi \end{cases}$$

for every $x \in G_1, f^{-1}(B)(x) = Bf(x)$.

Remark (2.10) [6]: It is quite evident that a group homomorphism f admits the following characteristics.

1. $f(A) f(x) \geq A(x)$, for every element $x \in G_1$
2. When f is bijective map, $f(A) f(x) = A(x)$, for all $x \in G_1$.

Definition (2.11) [5]: A function $t : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a t norm if and only if t admits following properties for all a, b, c, d in [0,1]

1. $t(a, b) = t(b, a)$
2. $t(a, t(b, c)) = t(t(a, b), c)$
3. $t(a, 1) = t(1, a) = 1$
4. If $a \leq c$ and $b \leq d$, then $t(a, b) \leq t(c, d)$

Definition (2.12) [5]: Let t_b: [0,1] $\times$ [0,1] $\rightarrow$ [0,1] be the bounded difference norm defined by

$$t_b(a, b) = \max(a + b - 1, 0) \quad , \quad 0 \leq a \leq 1 \ , \ 0 \leq b \leq 1$$

Clearly the bounded difference norm satisfies all the axioms of t-norm.

3 o-fuzzy subsets and their properties

Definition (3.1) Let A be a fuzzy subset of a set X and $\delta \in$ [0,1]. The fuzzy set A_o of X is called the o-fuzzy subset of X (w.r.t fuzzy set A) and is defined as $A_o(x) = t_b(A(x), \delta)$, for all $x \in X$.

Remark (3.2): It is important to note that one can obtain the classical fuzzy subset $A(x)$ by choosing the value of $\delta = 1$ in the above definition whereas the case become crisp for the choice of $\delta = 0$. These algebraic facts lead to note that the case illustrates the o-fuzzy version with respect to any fuzzy subset for the value of δ, when $\delta \in$ (0, 1).

Theorem (3.3): Let A and B be any two fuzzy subsets of X. Then $(A \cap B)_o = A_o \cap B_o$.

Proof: In view of definition (3.1), we have

$$(A \cap B)_o(x) = t_b((A \cap B)(x), \delta) = t_b(min(A(x), B(x)), \delta) = min(t_b(A(x), B(x)), \delta) =$$
$$min(t_b(A(x), \delta), t_b(B(x), \delta) = min(A_o(x), B_o(x))$$

This implies that $(A \cap B)_o = A_o \cap B_o$.

4 o-fuzzy subgroups

In this section, we define the notion of o-fuzzy subgroup and o-fuzzy normal subgroup. We prove that every fuzzy subgroup (normal subgroup) is also o-fuzzy subgroup (normal subgroup) but the converse need not be true. The notion of o-fuzzy coset is defined and discussed well in this section. Moreover, in view of the idea of an o-fuzzy normal subgroup, we introduce the concept of quotient group with respect to this particular fuzzy normal subgroup. This leads us to establish a natural homomorphism from a group G to its quotient group with respect to o-fuzzy normal subgroup. We also obtain the homomorphic image and pre-image of o-fuzzy subgroup (normal subgroup). We conclude this section by establishing an isomorphism between the quotient group with respect to o-fuzzy normal subgroup and quotient group with respect to the normal subgroup G_{A_o}.

Definition (4.1): Let A be a fuzzy subset of a group G and $\delta \in [0,1]$. Then A is called o-fuzzy subgroup of G. In other words, A is o-fuzzy subgroup if A_o satisfies the following

1. $A_o(xy) \geq min\{A_o(x), A_o(y)\}$
2. $A_o(x^{-1}) \geq A_o(x)$, for all $x, y \in G$.

Proposition (4.2): If A:$G \longrightarrow$ [0, 1] is an o-fuzzy subgroup of a group G, then

1. $A_o(x) \leq A_o(e)$, for all $x \in G$, where e is the identity element of G.
2. $A_o(xy^{-1}) = A_o(e)$, which implies that $A_o(x) = A_o(y)$, for all $x, y \in G$.

Proof: (i) $A_o(e) = A_o(xx^{-1}) \geq min(A_o(x), A_o(x^{-1})) = min(A_o(x), A_o(x)) = A_o(x)$
Hence $A_o(e) \geq A_o(x)$, for all $x \in G$.
(ii) $A_o(x) = A_o(xy^{-1}y) \geq min(A_o(xy^{-1}), A_o(y)) = min(A_o(e), A_o(y)) = A_o(y)$
Hence $A_o(x) \geq A_o(y)$.
Similarly $A_o(y) \geq A_o(x)$.
This implies that $A_o(x) = A_o(y)$, for all $x, y \in G$.
In the following result, we establish a condition under which an o-fuzzy subset of a group G is an o-fuzzy subgroup.

Theorem (4.3): Let A_o be an o-fuzzy subset of a group G. Then A_o is o-fuzzy subgroup of G if and only if $A_{\mathrm{o}}^{\mathrm{t}}$ is subgroup of G for all $t \leq A_o(e)$.

Proof: It is quite obvious that A_o is non-empty. Since A_o be o-fuzzy subgroup of a group G, $A_o(x) \leq A_o(e)$, for all $x \in G$

Let $x, y \in A_{\mathrm{O}}^{\mathrm{t}}$, then $A_o(x) \geq t$ and $A_o(y) \geq t$
Now $A_o(xy^{-1}) \geq \min(A_o(x), A_(y^{-1})) = \min(A_o(x), A_o(y)) \geq \min(t, t) = t$
This implies that $xy^{-1} \in A_{\mathrm{O}}^{\mathrm{t}}$
Hence $A_{\mathrm{O}}^{\mathrm{t}}$ is subgroup of G.
Conversely, suppose $A_{\mathrm{O}}^{\mathrm{t}}$ is subgroup of G, for all $t \leq A_o(e)$.
Let $x, y \in G$ and let $A_o(x) = a, A_o(y) = b$, where $a, b \in$ [0,1]
Let $c = min(a, b)$. Then $x, y \in A_{\mathrm{O}}^{\mathrm{c}}$, where $c \leq A_o(e)$.
So, by the assumption $A_{\mathrm{O}}^{\mathrm{c}}$ is subgroup of G.
This implies that $xy^{-1} \in A_{\mathrm{O}}^{\mathrm{c}}$ and hence $A_o(xy^{-1}) \geq \min(A_o(x), A_o(y))$.
Consequently, A_o is fuzzy subgroup of G.
The following result leads to note that every fuzzy subgroup of a group G is an o-fuzzy Subgroup of G.

Proposition (4.4): Every fuzzy subgroup of a group G is an o-fuzzy Subgroup of G.

Proof: Let A be a fuzzy subgroup of a group G and let x, y be any two elements in G.
Then

$$A_o(xy) = t_b(A(xy), \delta) \geq t_b(\min(A(x), A(y)), \delta) =$$

$$\min(t_b(A(x), A(y)), \delta) = \min(t_b(A(x), \delta), t_b(A(y), \delta))$$

and so

$$A_o(xy) \geq \min(A_o(x), A_o(y))$$

Moreover, $A_o(x^{-1}) = t_b(A(x^{-1}), \delta) = t_b(A(x), \delta) = A_o(x)$.
Consequently, A is o-fuzzy subgroup of G.

Remark (4.5): The converse of above proposition need not to be true

Example (4.6): Let $G = \{e, a, b, ab\}$, where $a^2 = b^2 = e$ and $ab = ba$ be the Klein four group. Let the fuzzy set A of G be defined as
$A = \{< e, 0.4 >, < a, 0.2 >, < b, 0.2 >, < ab, 0.1 >\}$
Taking $\delta = 0.19$, $A_o(x) = t_b(A(x), \delta) = \max(A(x) + \delta - 1, 0) = \max(A(x) + 0.19 - 1, 0)$
$A_o(x) = 0$, for all $x \in G$.
This implies that
$A_o(xy) \geq \min(A_o(x), A_o(y))$.
Moreover, we have $a^{-1} = a, b^{-1} = b$ and $(ab)^{-1} = ab$.
Hence we have $A_o(x^{-1}) = A_o(x)$, for all $x \in G$.
This implies that A is o-fuzzy subgroup of G. But clearly A is not fuzzy subgroup of G.

Proposition (4.7): Intersection of two o-fuzzy subgroups of a group G is also o-fuzzy Subgroup.

Proof: Let A and B be two o-fuzzy subgroups of a group G.
Consider, for all $x, y \in G$
$(A\cap B)_o(xy) = (A_o \cap B_o)(xy) = \min(A_o(xy), B_o(xy)) \geq \min(min(A_o(x), A_o(y)), min(B_o(x), B_o(y))) = min(min(A_o(x), B_o(x)), min(A_o(y), B_o(y))) = min((A\cap B)_o(x), (A\cap B)_o(y))$.
Thus $(A\cap B)_o(xy) \geq min((A\cap B)_o(x), (A\cap B)_o(y))$.
Moreover, $(A\cap B)_o(x^{-1}) = (A_o \cap B_o)(x^{-1}) = \min(A_o(x^{-1}), B_o(x^{-1})) = \min(A_o(x), B_o(x))$
$(A\cap B)_o(x^{-1}) = (A\cap B)_o(x)$.
Consequently, $(A\cap B)$ is o-fuzzy subgroup of G.
Corollary (4.8): The intersection of any finite number of o-fuzzy subgroups of a group G is also an o-fuzzy subgroup of G.
Remark (4.9): The union of two o-fuzzy subgroups of a group G need not be o-fuzzy subgroup of G:
Example (4.10): Consider the group of integers Z. Define the two Fuzzy subsets A and B of Z as follows

$$A(x) = \begin{cases} 0.5, & \text{if } x = 3Z \\ 0, & \text{otherwise} \end{cases}$$

and

$$B(x) = \begin{cases} 0.19, & \text{if } x = 2Z \\ 0.06, & \text{otherwise} \end{cases}$$

It can be easily verified that A and B are o-fuzzy subgroups of Z.
Now, $(A\cap B)(x) = \max(A(x), B(x))$

Therefore, $(A \cup B)(x) = \begin{cases} 0.5 & \text{if } x \in 3z \\ 0.19 & \text{if } x \in 2z - 3z \\ 0.06 & \text{otherwise} \end{cases}$

Take $x = 15$ and $y = 4$. Then, $(A\cup B)(x) = 0.5$ and $(A\cup B)(y) = 0.19$. But $(A\cup B)\ (x-y) = (A\cup B)\ (15\text{-}4) = (A\cup B)\ (11) = 0.06$ and $\min\ ((A\cup B)(x), (A\cup B)(y))$ $= \min\ (0.5, 0.19) = 0.19$.Clearly, $(A\cup B)\ (x-y) < \min\ ((A\cup B)(x), (A\cup B)(y))$.
Consequently, $A\cup B$ is not o-fuzzy subgroup of G.
Hence, we see that, the union of two o-fuzzy subgroups of G need not be o-fuzzy subgroup of G.
Definition (4.11): Let A be an o-fuzzy subgroup of a group G and $\delta \in$ [0,1]. The right o-fuzzy coset of A in G is denoted by $A_o x$ and is defined as
$A_o x(g) = t_b(A(gx^{-1}), \delta)$, for all $x, y \in G$
Similarly, we define the o-fuzzy left coset xA_o of G as follows
$xA_o(g) = t_b(A(x^{-1}g), \delta)$, for all $x, y \in G$.
Definition (4.12): Let A be an o-fuzzy subgroup of a group G and $\delta \in$ [0,1]. Then A is called o-fuzzy normal subgroup of G if and only if $xA_o = A_o x$, for all $x \in G$.

The following result leads to note that every fuzzy normal subgroup of a group G is an o-fuzzy normal subgroup of G.

Proposition (4.13): Every fuzzy normal subgroup of a group G is an o-fuzzy normal subgroup of G.

Proof: Let A be a fuzzy normal subgroup of a group G. Then for any $x \in G$, we have $xA = Ax$, which implies that $xA(g) = Ax(g)$, for any $g \in G$. Then we have $A(x^{-1}g) = A(gx^{-1})$, which implies that $t_b(A(x^{-1}g), \delta) = t_b(A(gx^{-1}), \delta)$. Hence, $xA_o = A_o x$, for all $x \in G$.

Consequently, A is o-fuzzy normal subgroup of G. The converse of the above result need not to be true:

Example (4.14): Consider the dihedral group of degree 3 with finite presentation

$G = D_3 = < a, b : a^3 = b^2 = e, ba = a^2 b >$. Define the fuzzy subgroup of D_3 by

$$A(x) = \begin{cases} 0.1, & \text{if } x \in < b > \\ 0.05, & \text{otherwise} \end{cases}$$

Taking $\delta = 0.3$, we have

$xA_o(g) = t_b(A(x^{-1}g), \delta) = t_b(A(x^{-1}g), 0.3) = 0 = A_o x$.

This shows that A is o-fuzzy normal subgroup of G.

$A(a^2(ab)) = A(a^3 b) = A(b) = 0.1$

$A((ab)a^2) = A(a(ba)a) = A(a(a^2 b)a) = A(a^3 ba) = A(ba) = 0.05$

This implies that A is not fuzzy normal subgroup of G.

Proposition (4.15): Let A be an o-fuzzy normal subgroup of a group G. Then $A_o(y^{-1}xy) = A_o(x)$ or equivalently, $A_o(xy) = A_o(yx)$, hold for all $x, y \in G$.

Proof: Since A is an o-fuzzy normal subgroup of a group G, $xA_o = A_o x$, holds for all $x \in G$.

This implies that

$$xA_o(y^{-1}) = A_o x(y^{-1}), y \in G$$

.

In view of definition (4.10), the above relation becomes

$t_b(A(x^{-1}y^{-1}), \delta) = t_b(A(y^{-1}x^{-1}), \delta)$, which implies that, $A_o((yx)^{-1}) = A_o((xy)^{-1})$

Consequently, $A_o(xy) = A_o(yx)$.

Definition (4.16): Let A be an o-fuzzy normal subgroup of a group G. We define a set $G_{A_o} = \{x \in G : A_o(x) = A_o(e)\}$, where e is the identity element of G

The following result illustrates that the set G_{A_o} is in fact a normal subgroup of G.

Proposition (4.17): Let A be an o-fuzzy normal subgroup of a group G. Then G_{A_o} is a normal subgroup of G.
Proof: Obviously, $G_{A_o}\neq\phi$, for $e\in G_{A_o}$
Let $x,y\in G_{A_o}$ be any element. Then we have
$A_o(xy^{-1})\geq\min(A_o(x),A_o(y))=\min(A_o(e),A_o(e))=A_o(e)$.
This implies that $A_o(xy^{-1})\geq A_o(e)$, but $A_o(xy^{-1})\leq A_o(e)$
Therefore $A_o(xy^{-1})=A_o(e)$, which implies that $xy^{-1}\in G_{A_o}$
Hence G_{A_o} is a subgroup of G. Further,, let $x\in G_{A_o}$ and $y\in G$, we have
$A_o(y^{-1}xy)=A_o(x)=A_o(e)$
This implies that $y^{-1}xy\in G_{A_o}$
Consequently, G_{A_o} is normal subgroup of G.
Proposition (4.18): Let A be an o-fuzzy normal subgroup of G. Then

1. $xA_o=yA_o$ if and only if $x^{-1}y\in G_{A_o}$
2. $A_ox=A_oy$ if and only if $xy^{-1}\in G_{A_o}$

Proof: (i) Suppose that $xA_o=yA_o$, for $x,y\in G$. In view of definition (3.1), the above relation yields
$A_o(x^{-1}y)=t_b(A(x^{-1}y),\delta)=(xA_o)(y)=(yA_o)(y)=t_b(A(y^{-1}y),\delta)=t_b(A(e),\delta)=A_o(e)$.
This implies that $x^{-1}y\in G_{A_o}$.
Conversely, let $x^{-1}y\in G_{A_o}$. Then $A_o(x^{-1}y)=A_o(e)$. For any element $z\in G_{A_o}$,$(xA_o)(z)=t_b(A(x^{-1}z),\delta)=A_o(x^{-1}z)=A_o((x^{-1}y)(y^{-1}z))\geq\min(A_o((x^{-1}y)),A_o(y^{-1}z))=\min(A_o(e),A_o(y^{-1}z))=A_o(y^{-1}z)=(yA_o)(z)$
Interchanging the roles of x and y, we get $(xA_o)(z)=(yA_o)(z)$, for all $z\in G$.
Consequently, $(xA_o)=(yA_o)$.
(ii) One can prove this part analogous to (i).
Proposition (4.19): Let A be an o-fuzzy normal subgroup of a group G and x,y,u,v be any element in G. If $xA_o=uA_o$ and $yA_o=vA_o$, then $xyA_o=uvA_o$.
Proof: Given $xA_o=uA_o$ and $yA_o=vA_o$, we have $x^{-1}u$ and $y^{-1}v\in G_{A_o}$.
Consider, $(xy)^{-1}uv=y^{-1}(x^{-1}u)(yy^{-1})v=[y^{-1}(x^{-1}u)y](y^{-1})v)\in G_{A_o}$.
This implies that $(xy)^{-1}uv\in G_{A_o}$
Consequently, $xyA_o=uvA_o$.
Definition (4.20): Let A be an o-fuzzy normal subgroup of a group G. The set of all o-fuzzy cosets of A denoted by G/A_o forms a group under the binary operation * defined as follows:
Let $xA_o,yA_o\in G/A_o,xA_o*yA_o=(x*y)A_o)$, for $x,y\in G$. This group is called the factor group or the quotient group of G with respect to o-fuzzy normal

subgroup A_o.
Theorem (4.21): The set G/A_o in definition (4.18) forms a group under the above stated binary operation *.
Proof: Let $A_o x_1 = A_o x_2$ and $A_o y_1 = A_o y_2$, for $x_1, x_2, y_1, y_2 \in G$.
Let $g \in G$ be any element of G
$[A_o x_1 * A_o y_1](g) = (A_o x_1 y_1)(g) = t_b(A(g(x_1 y_1)^{-1}), \delta) = t_b(A(g y_1^{-1} x_1^{-1}), \delta)$
$= t_b(A(g y_1^{-1}) x_1^{-1}, \delta) = A_o x_1 (g y_1^{-1}) = A_o x_2 (g y_1^{-1}) = t_b(A(g y_1^{-1}) x_2^{-1}, \delta)$
$= t_b(A(x_2^{-1} g) y_1^{-1}, \delta) = A_o y_1 (x_2^{-1} g) = A_o y_2 (x_2^{-1} g) = t_b(A(x_2^{-1} g) y_2^{-1}, \delta)$
$= t_b(A(y_2^{-1} x_2^{-1}) g, \delta) = t_b(A(x_2 y_2)^{-1} g, \delta) = t_b(A g (x_2 y_2)^{-1}, \delta) = (A_o x_2 y_2)(g)$.
This implies that * is well defined.
Obviously, the set G/A_o admits closure and associative properties with respect to the binary operation $*$.
Moreover, $A_o * x A_o = e A_o * x A_o = (e * x) A_o = x A_o$, which implies that A_o is the identity of G/A_o.
It is easy to note that the inverse of each element of G/A_o exists as if for $x A_o \in G/A_o$, there exist $x^{-1} A_o \in G/A_o$ such that
$(x^{-1} A_o) * (x A_o) = (x^{-1} * x) A_o = A_o$.
Consequently, (G/A_o) is a group under *.
Theorem (4.22): Let A_o be an o-fuzzy normal subgroup of a group G. Then there exists a natural epimorphism between G and G/A_o which may be defined as $x \longmapsto A_o x, x \in G$ where G_{A_o} is the kernel of this homomorphism.
Proof: f is homomorphism as if for $x, y \in G$, we have
$f(xy) = A_o xy = A_o x A_o y = f(x) f(y)$.
Obviously f is surjective as well.
Consequently, f is an epimorphism from G to G/A_o.
Moreover, $ker f = \{x \in G : f(x) = A_o e\} = \{x \in G : A_o x = A_o e\} = \{x \in G : x e^{-1} \in G_{A_o}\} = \{x \in G : x \in G_{A_o}\} = G_{A_o}$.
Theorem (4.23): Let A_o be an o-fuzzy normal subgroup of a group G. Then $G/A_o \cong G/G_{A_o}$.
Proof: In view of definition (4.15), G/G_{A_o} is well defined.
Define a map $f : G/A_o \to G/G_{A_o}$ by the rule

$$f(x A_o) = x G_{A_o}, x \in G$$

f is well defined because if $x A_o = y A_o$, $x G_{A_o} = y G_{A_o}$.
This implies that $f(x A_o) = f(y A_o)$
f is injective as if $f(x A_o) = f(y A_o)$, which implies that $x G_{A_o} = y G_{A_o}$. Hence, $x A_o = y A_o$.f is surjective as for each $x G_{A_o} \in G/G_{A_o}$, there exists $x A_o \in G/A_o$ such that $f(x A_o) = x G_{A_o}$

f is homomorphism as for each $xA_o, yA_o \in G/A_o$
$f(xA_oyA_o) = f((xy)A_o) = xyG_{A_o} = xG_{A_o}yG_{A_o} = f(xA_o)f(yA_o)$.
Consequently, there is an isomorphism between G/A_o and G/G_{A_o}.

5 Homomorphism of o-fuzzy subgroups

Theorem (5.1): Let $f{:}G_1 \longrightarrow G_2$ be a bijective homomorphism from a group G_1 to a group G_2 and let A be an o-fuzzy subgroup of group G_1. Then $f(A)$ is an o-fuzzy subgroup of group G_2.
Proof: Let A be an o-fuzzy subgroup of group G_1. Let $y_1, y_2 \in G_2$. Then there exist $x_1, x_2 \in G_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.
Consider
$(f(A))_o(y_1y_2) = t_b(f(A)(y_1y_2), \delta) = t_b(f(A)f(x_1)f(x_2), \delta) = t_b(f(A)f(x_1x_2), \delta) = t_b(A(x_1x_2), \delta) = A_o(x_1x_2) \geq \min(A_o(x_1, A_o(x_2)$, for all $x_1, x_2 \in G_1 \geq \min(max\{A_o(x_1) : f(x_1) = y_1\}, max\{A_o(x_2) : f(x_2) = y_2\})$ $=\min(f(A_o)(y_1), f(A_o)(y_2))$ $=\min((f(A))_o(y_1), (f(A))_o(y_2))$.
Moreover,
$(f(A))_o(y^{-1}) = f(A_o)(y^{-1})$ $=\max\{A_o(x^{-1}) : f(x^{-1}) = y^{-1}\}$ $=\max\{A_o(x) : f(x) = y\} = (f(A))_o(y)$.
Consequently, $f(A)$ is o-fuzzy subgroup of G_2.
Theorem (5.2): Let $f{:}G_1 \longrightarrow G_2$ be a bijective homomorphism from a group G_1 to a group G_2 and let A be an o-fuzzy normal subgroup of group G_1. Then $f(A)$ is an o-fuzzy normal subgroup of group G_2.
Proof: In view of theorem (5.1), it is sufficient to show that $f(A_o)$ is fuzzy normal in G_2. Let A be an o-fuzzy normal subgroup of group G_1. Let $y_1, y_2 \in G_2$.
Then there exist $x_1, x_2 \in G_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.
Consider,
$(f(A))_o(y_1y_2) = t_b(f(A)(y_1y_2), \delta) = t_b(f(A)f(x_1)f(x_2), \delta) = t_b(f(A)f(x_1x_2), \delta) = t_b(A(x_1x_2), \delta) = A_o(x_1x_2) = A_o(x_2x_1) = t_b(A(x_2x_1), \delta) = t_b(f(A)f(x_2x_1), \delta) = t_b(f(A)f(x_2)f(x_1), \delta) = t_b(f(A)(y_2y_1), \delta) = (f(A))_o(y_2y_1)$.
Consequently, $f(A)$ is o-fuzzy normal subgroup of G_2.
Theorem (5.3): Let $f{:}G_1 \longrightarrow G_2$ be a bijective homomorphism from a group G_1 to a group G_2 and let B be an o-fuzzy subgroup of group G_2. Then $f^{-1}(B)$ is an o-fuzzy subgroup of group G_1.
Proof: Let B be an o-fuzzy subgroup of group G_2. Let $x_1, x_2 \in G_1$.
Then
$(f^{-1}(B))_o(x_1x_2) = B_o(f(x_1x_2)) = B_o((f(x_1)f(x_2)) \geq \min(B_o(f(x_1), B_o(f(x_2))$= $\min(f^{-1}(B_o)(x_1), f^{-1}(B_o)(x_2))$ $=\min((f^{-1}(B))_o(x_1), (f^{-1}(B))_o(x_2))$

Further, $(f^{-1}(B))_o(x^{-1}) = f^{-1}(B_o)(x^{-1}) = B_o(f(x^{-1})) = B_o(f(x))^{-1} = B_o(f(x)) = (f^{-1}(B))_o(x)$.
Consequently, $f^{-1}(B)$ is o-fuzzy subgroup of a group G_1.
Theorem (5.4): Let $f{:}G_1 \longrightarrow G_2$ be a bijective homomorphism from a group G_1 to a group G_2 and let B be an o-fuzzy normal subgroup of group G_2. Then $f^{-1}(B)$ is an o-fuzzy normal subgroup of group G_1.
Proof: In view of theorem (5.3), it is sufficient to show that $f^{-1}(B_o)$ is fuzzy normal in G_1.
Let B be an o-fuzzy normal subgroup of group G_2. Let $x_1, x_2 \in G_1$. Then we have
$(f^{-1}(B))_o(x_1x_2) = B_o(f(x_1x_2)) = B_o(f(x_1)(x_2)) = B_o(f(x_2)(x_1)) = B_o(f(x_2)(x_1)) = (f^{-1}(B))_o(x_2x_1)$.
Consequently, $f^{-1}(B)$ is an o-fuzzy normal subgroup of G_1.

6 Conclusion

In this paper, we have introduced the concept of an o-fuzzy subgroup and an o-fuzzy coset of a given group and have used them to introduce the concept of an o-fuzzy normal subgroup and have discussed various related properties. We have also studied the effect on the image and inverse image of an o-fuzzy subgroup (normal subgroup) under group homomorphism.
In later studies, we shall extend this idea to intuitionistic fuzzy sets and will investigate its various algebraic properties.

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