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New Theorems for Hyers-Ulam Stability of Lienard Equation with Variable Time Lags

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Abstract

We investigate a modified Lienard equation (LE) with multiple variable time lags and find assumptions for the Hyers-Ulam stability (HUS) and Hyers-Ulam -Rassias stability (HURS) of the modified (LE) considered. We obtain two results on (HUS) and (HURS) of the equation considered where we use Banach's contraction principle (BCP).

1 Introduction

In 2009, Li and Shen [11] proved the (HUS) of the second order non-homogeneous linear differential equation (LDE)

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = -r(x),\ x \in I, I = (a, b),$$

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by establishing some special conditions. Two theorems on the (HUS) were given by thoe authors. Later, in 2010, Li and Shen [12] focused on the (HUS) of the (LDEs) of second order, respectively:

$$\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y = 0$$

and

$$\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y = f(x), \alpha, \beta \in \mathbb{R}, -\infty < a < b < \infty.$$

They proved two theorems on the (HUS) of these equations.
After that, Javadian et al[8] discussed the (HURS) of a certain (LDE) of second order.
Later, Ghaemi et al [6] also studied the (HUS) of a different (LDE) of second order. As a consequence, they showed the (HUS) and (HURS) of a (LDE) of second-order with constant coefficients and different forms of (LDEs) called such as Euler, Hermite, Cheybyshev and Legendre equations.
Motivated by the papers, Abdollahpour and Park [1], Alsina and Ger [2], Biçer and Tunç [3, 4], Cimpean and Popa [5], Ghaemi et al [6], Hyers [7], Javadian et al [8], Jung [9, 10], Li and Shen [11, 12], Rassias [13], Tunç [14, 15, 16, 17, 18, 19], Tunç and Biçer [20, 21], Ulam [22] and the references therein, we consider the following modified (LE) with multiple variable time lags:

$$\frac{d^2y}{dt^2} + F(t,y)\frac{dy}{dt} + H(t,y) = 0, \tag{1}$$

throughout the whole paper we suppose that $F(t,Y) = f(t,y,y(t-\tau_1(t)),y(t-\tau_2(t))$, $H(t,Y) = h(t,y,y(t-\tau_1(t)),y(t-\tau_2(t))$, $f \in C^1(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $h \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, where $\mathbb{R}^+ = (0,\infty), \mathbb{R} = (-\infty,\infty)$. We also assume that $\tau_1, \tau_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ and

$$m_j(t_0) = \inf\{t - \tau_j(t), t \geq t_0\}, m(t_0) = \min\{m_j(t_0), 1 \leq j \leq 2\}.$$

Whenever needed, we take $y = y(t)$, $y_0 = y_0(t), y' = \frac{dy}{dt}, y'' = \frac{d^2y}{dt^2}, \zeta = \zeta(t)$ and $\eta = \eta(t)$, respectively.
Meanwhile, when we look at the relevant literature on the (HUS) and (HURS) of solutions, we see that both of these topics attract huge research interest. In fact, in recent years, especially, the (HUS) and (HURS) of first and second order ordinary (LDEs), partially, (LDEs) with constant or variable time lags were investigated many interesting and novel results on that topics were obtained. We investigate the (HUS) and (HURS) of modified (LE) (1) with two variable time lags.

The main idea and aim this paper is to discuss the (HUS) and (HURS) of modified (LE) (1). To the best of our knowledge, the (HUS) and (HURS) for (LE) (1) were not discussed in the literature. Our results might be beneficial to researchers working on the (HUS) and (HURS).

2 (HUS) and (HURS) Results

Definition 2.1. Let $I = [m(t_0), T]$. For some $\varepsilon \geq 0, \psi \in C([m(t_0), t_0], \mathbb{R})$ and $t_0, T \in \mathbb{R}$ with $T > t_0$, we suppose that for any continuous function $y : I \to \mathbb{R}$ with $y \in C(I, \mathbb{R})$ satisfying

$$\begin{cases} \mid \frac{d^2y}{dt^2} + F(t,Y)\frac{dy}{dt} + H(t,Y) \mid \leq \varepsilon, \quad t \in [t_0, T] \\ \\ \mid y - \psi(t) \mid \leq \varepsilon, t \in [m(t_0), t_0], \end{cases}$$

there exists a continuous function $y_0 : I \to \mathbb{R}$ with $y_0 \in C(I, \mathbb{R})$ satisfying:

$$\begin{cases} \frac{d^2y_0}{dt^2} = -F(t,Y_0)\frac{dy_0}{dt} - H(t,Y_0), \quad t \in [t_0, T] \\ \\ y_0 = \psi(t), t \in [m(t_0), t_0], \end{cases}$$

and

$$\mid y - y_0 \mid \leq K(\varepsilon),$$

where

$$F(t, Y_0) = f(t, y_0, y_0(t - \tau_1(t)), y_0(t - \tau_2(t)))$$

and

$$H(t, Y_0) = h(t, y_0, y_0(t - \tau_1(t)), y_0(t - \tau_2(t))).$$

Then we call that (LE) (1) has the (HUS). If the information given above is also true, when we substitute ε and $K(\varepsilon)$ by $\phi(t)$ and $\varphi(t)$, respectively, where $\phi, \varphi \in C[m(t_0, T]$ are functions not depending on y and y_0, explicitly, then we call that the corresponding (DE) has the (HUS) (or the (HURS)).

Theorem 2.2. Let $m(t_0) > 0, I = \{t \in \mathbb{R} : t_0 - m(t_0) \leq t \leq t_0 + m(t_0)\}$ and $K : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\mid K(t, x_1, y_1, z_1) - K(t, x_2, y_2, z_2) \mid \leq k_1 \mid x_1 - x_2 \mid + k_2 \mid y_1 - y_2 \mid + k_3 \mid z_1 - z_2 \mid \tag{2}$$

for all $t \in I$ and $x_i \in \mathbb{R}, y_i \in \mathbb{R}, z_i \in \mathbb{R}, (i = 1, 2)$, where $k_i > 0, k_i \in \mathbb{R}, (i = 1, 2, 3)$, with $0 < \sum\limits_{i=1}^{3} m(t_0)k_i < 1$. For $\psi \in C([m(t_0), t_0], \mathbb{R})$,if a function $y : I \to \mathbb{R}$ with $y \in C(I, \mathbb{R})$ satisfies

$$\begin{cases} \mid \frac{d^2y}{dt^2} + F(t, Y)\frac{dy}{dt} + H(t, Y) \mid \leq \varepsilon, \quad t \in [t_0, T] \\ \\ \mid y - \psi(t) \mid \leq \varepsilon, t \in [m(t_0), t_0], \end{cases} \tag{3}$$

then we can find a unique function $y_0 : I \to \mathbb{R}$ with $y_0 \in C(I, \mathbb{R})$ such that

$$\begin{cases} \frac{d^2y_0}{dt^2} = -F(t, Y_0)\frac{dy_0}{dt} - H(t, Y_0), \quad t \in [t_0, T] \\ \\ y_0 = \psi(t), t \in [m(t_0), t_0], \end{cases}$$

and

$$\mid y - y_0 \mid \leq \frac{m(t_0)}{1 - m(t_0)\sum\limits_{i=1}^{3} k_i}\varepsilon, t \in I.$$

Proof. Consider the functional (DE) of the first order

$$\frac{dy}{dt} + K(t, Y) = 0, \tag{4}$$

where

$$K(t, Y) = K(t, y, y(t - \tau_1(t)), y(t - \tau_2(t))).$$

Let

$$\begin{cases} \mid \frac{dy}{dt} + K(t, Y) \mid < \varepsilon, \quad t \in [t_0, T] \\ \\ \mid y - \psi(t) \mid \leq \varepsilon, t \in [m(t_0), t_0]. \end{cases} \tag{5}$$

Assume that C is the space of all continuous functions from $I \to \mathbb{R}$. Let d be a metric and P an operator on C, respectively:

$$d(\zeta, \eta) = \sup_{t \in I} \mid \zeta(t) - \eta(t) \mid$$

and

$$\begin{cases} (P\zeta)(t) = \psi(t), t \in [m(t_0), t_0], \\ \\ (P\zeta(t) = \psi(t_0) - \int\limits_{t_0}^{t} K(\zeta)ds, t \in [t_0, T] \end{cases}$$

for all $\zeta \in C$, where

$$K(\zeta) = K(s, \zeta(s), \zeta(s - \tau_1(s)), \zeta(s - \tau_2(s))).$$

By means of inequality (2), we have

$$\begin{aligned} d(P\zeta, P\eta) &= \sup_{t\in I} \mid \int_{t_0}^{t} [K(\zeta) - K(\eta)] ds \mid \\ &\leq \sup_{t\in I} \int_{t_0}^{t} \sum_{i=1}^{3} k_i \mid \zeta(s) - \eta(s) \mid ds \\ &\leq \sum_{i=1}^{3} k_i m(t_0) d(\zeta, \eta), \quad t \in [t_0, T], \end{aligned}$$

and

$$d(P\zeta, P\eta) = \psi(t) - \psi(t) = 0, \qquad t \in [m(t_0), t_0],$$

which implies that

$$d(P\zeta, P\eta) \leq (\sum_{i=1}^{3} k_i) m(t_0) d(\zeta, \eta).$$

From (BCP), there exists a unique $y_0 \in C$ such that $Py_0 = y_0$, thus y_0 satisfies

$$\begin{cases} \frac{dy_0}{dt} = K(t, Y_0), \quad t \in [t_0, T] \\ y_0 = \psi(t), t \in [m(t_0), t_0], \end{cases}$$

where

$$K(t, Y_0) = K(t, y_0(t), y_0(t - \tau_1(t)), y_0(t - \tau_2(t))).$$

It is also clear that

$$d(y_0, y) \leq \frac{1}{1 - (k_1 + k_2 + k_3) m(t_0)} d(y, Py), \quad y \in C. \tag{6}$$

Furthermore, from (5), we can write

$$-\varepsilon \leq \frac{dy}{dt} + K(t, Y) \leq \varepsilon, \qquad t \in [t_0, T].$$

By integrating of each term in previous inequality, we find

$$\mid y - \psi(t_0) - \int_{t_0}^{t} K(s, Y(s)) ds \mid \leq \int_{t_0}^{t} \varepsilon ds \leq \varepsilon m(t_0), \quad t \in [t_0, T].$$

Hence, we have

$$| y - (Py)(t) | \leq \varepsilon m(t_0)$$

so that

$$\sup_{t\in I} | y(t) - (Py)(t) | \leq \varepsilon m(t_0).$$

Thus, we find

$$d(y, Py) \leq \varepsilon m(t_0). \tag{7}$$

In view of (6) and (7), we reach that

$$| y - y_0 | \leq \frac{m(t_0)}{1-(k_1+k_2+k_3)m(t_0)}\varepsilon,\ t \in I.$$

Then, we can arrive at functional (DE) (4) has the (HUS).
One can naturally obtain from (3) that

$$| \frac{d^2y}{dt^2} + F(t, Y)\frac{dy}{dt} + H(t, Y) | \leq \varepsilon.$$

Hence,

$$-\varepsilon \leq \frac{d^2y}{dt^2} + F(t, Y)\frac{dy}{dt} + H(t, Y) \leq \varepsilon.$$

If we integrate the former estimate, then we get

$$| \frac{dy}{dt} - y'(t_0) + \int_{t_0}^{t} F(s, Y(s))\frac{dy}{ds}ds + \int_{t_0}^{t} H(s, Y(s))ds | \leq \int_{t_0}^{t} \varepsilon ds,$$

where

$$F(s, Y(s)) = f(s, y(s), y(s - \tau_1(s)), y(s - \tau_2(s))).$$

An application of the integration by parts implies that

$$| \frac{dy}{dt} + F(t, Y)y - \int_{t_0}^{t} F_s(s, Y(s))y(s)ds + \int_{t_0}^{t} H(s, Y(s))ds + r(t_0) | \leq \int_{t_0}^{t} \varepsilon ds,$$

where

$$r(t_0) = -f(t_0, y(t_0), y(t_0 - \tau_1(t)), y(t_0 - \tau_2(t)))y(t_0) - y'(t_0)$$

and

$$H(s, Y(s)) = h(s, y(s), y(s - \tau_1(s)), y(s - \tau_2(s))).$$

Choose

$$K(t,y(t),y(t-\tau_1(t)),y(t-\tau_2(t))) = -F(t,Y)y + \int\limits_{t_0}^{t} F_s(s,Y(s))y(s)ds - \int\limits_{t_0}^{t} H(s,Y(s))ds - r(t_0).$$

As a consequence, we can reach that

$$\mid \frac{dy}{dt} - K(t,y(t),y(t-\tau_1(t)),y(t-\tau_2(t)) \mid \leq \int\limits_{t_0}^{t} \varepsilon ds \leq \varepsilon m(t_0).$$

This is the end of the proof of Theorem 2.2.

Theorem 2.3. For $I = [m(t_0), T]$, let $\phi : I \to (0,\infty)$ be a continuous function. In addition, we suppose that $\psi \in (C[m,t_0],\mathbb{R})$ and there exists a L, L is a real constant with $0 < L < 1$ such that

$$\int\limits_{t_0}^{t} \phi(s)ds < L\phi(t) \tag{8}$$

with $0 < \sum\limits_{i=1}^{3} Lk_i < 1$. If a function $y : I \to \mathbb{R}$ satisfies

$$\begin{cases} \mid \frac{d^2y}{dt^2} + F(t,Y)\frac{dy}{dt} + H(t,Y) \mid \leq \phi(t), & t \in [t_0,T] \\ \mid y - \psi(t) \mid \leq \phi(t), t \in [m(t_0),t_0], \end{cases} \tag{9}$$

then there exists a unique function $y_0 : I \to \mathbb{R}$ with $y_0 \in C(I,\mathbb{R})$ such that

$$\begin{cases} \frac{d^2y_0}{dt^2} = -F(t,Y_0)\frac{dy_0}{dt} - H(t,Y_0), & t \in [t_0,T] \\ y_0 = \psi(t), t \in [m(t_0),t_0], \end{cases}$$

and

$$\mid y - y_0 \mid \leq \frac{L}{1 - \sum\limits_{i=1}^{3} Lk_i}\phi(t), t \in I.$$

Proof. We re-consider functional (DE) (4) and let

$$\begin{cases} \mid \frac{dy}{dt} + K(t,Y) \mid < \phi(t), & t \in [t_0,T] \\ \mid y - \psi(t) \mid \leq \phi(t), t \in [m(t_0),t_0]. \end{cases} \tag{10}$$

Further, let C be the space of all continuous functions from $I \to \mathbb{R}$. We define a metric d and an operator P on C as given below, respectively:

$$d(\zeta,\eta)=\sup_{t\in I}\frac{|\ \zeta(t)-\eta(t)\ |}{\phi(t)}$$

and

$$\begin{cases} (P\zeta)(t)=\psi(t), t\in[m(t_0),t_0], \\ \\ (P\zeta(t)=\psi(t_0)-\int\limits_{t_0}^{t}K(s,\zeta(s),\zeta(s-\tau_1(s)),\zeta(s-\tau_2(s)))ds, t\in[t_0,T] \end{cases}$$

for all $\zeta\in C$. Using the assumptions of Theorem 2.3. and (8), we find

$$\begin{aligned} d(P\zeta,P\eta) &\le \sup_{t\in I}\frac{\int\limits_{t_0}^{t}\sum\limits_{i=1}^{3}k_i\ |\ \zeta(s)-\eta(s)\ |\ ds}{\phi(t)} \\ &\le \sup_{t\in I}\frac{\int\limits_{t_0}^{t}\sum\limits_{i=1}^{3}k_i\phi(s)\frac{|\zeta(s)-\eta(s)|}{\phi(s)}ds}{\phi(t)} \\ &\le \sup_{t\in I}\frac{\int\limits_{t_0}^{t}\sum\limits_{i=1}^{3}k_i\phi(s)\sup\limits_{s\in I}\frac{|\zeta(s)-\eta(s)|}{\phi(s)}ds}{\phi(t)} \\ &= d(\zeta,\eta)\sup_{t\in I}\frac{\int\limits_{t_0}^{t}\sum\limits_{i=1}^{3}k_i\phi(s)ds}{\phi(t)} \\ &\le (\sum_{i=1}^{3}k_i)Ld(\zeta,\eta),\ t\in[t_0,T], \end{aligned}$$

and

$$d(P\zeta,P\eta)=0,\ t\in[m(t_0),t_0],$$

which implies that

$$d(P\zeta,P\eta)\le(\sum_{i=1}^{3}Lk_i)d(\zeta,\eta).$$

From (BCP), there exists a unique $y_0\in C$ such that $Py_0=y_0$ and y_0 satisfies

$$\begin{cases} \frac{dy_0}{dt}=-K(t,y_0(t),y_0(t-\tau_1(t)),y_0(t-\tau_2(t))), \quad t\in[t_0,T] \\ \\ y_0=\psi(t), t\in[m(t_0),t_0]. \end{cases}$$

Furthermore, by (BCP), we have

$$d(y_0, y) \leq \frac{1}{1 - (\sum_{i=1}^{3} Lk_i)} d(y, Py) \tag{11}$$

for all $y \in C$. In addition, from (10), we can write

$$-\phi(t) \leq \frac{dy}{dt} + K(t, y(t), y(t - \tau_1(t)), y(t - \tau_2(t))) \leq \phi(t),\ t \in [t_0, T].$$

Obviously

$$| y - \psi(t_0) - \int_{t_0}^{t} K(s, Y(s))ds | \leq \int_{t_0}^{t} \phi(s)ds \leq L\phi(t), \quad t \in [t_0, T].$$

Hence,

$$\frac{| y(t) - (Py)(t) |}{\phi(t)} \leq L$$

so that

$$\sup_{t \in I} \frac{| y(t) - (Py)(t) |}{\phi(t)} \leq L.$$

It follows that

$$d(y, Py) \leq L. \tag{12}$$

Therefore, by (11) and (12), we obtain

$$| y(t) - y_0(t) | \leq \frac{L}{1 - (\sum_{i=1}^{3} Lk_i)} \phi(t),\ t \in I.$$

Then, we can conclude that functional (DE) (4) has the (HURS). It follows from (10) that

$$| \frac{d^2y}{dt^2} + F(t, Y)\frac{dy}{dt} + H(t, Y) | \leq \phi(t).$$

If we integrate the above inequality, then we get

$$| \frac{dy}{dt} - y'(t_0) + \int_{t_0}^{t} F(s, Y(s))\frac{dy}{ds}ds + \int_{t_0}^{t} H(s, Y(s))ds | \leq \int_{t_0}^{t} \phi(s)ds.$$

Applying the integration by parts, we have

$$|\frac{dy}{dt}+F(t,Y)y-\int_{t_0}^{t}F_s(s,Y(s))y(s)ds+\int_{t_0}^{t}H(s,Y(s))ds+r(t_0)|\leq\int_{t_0}^{t}\phi(s)ds,$$

where $r(t_0)=-f(t_0,y(t_0),y(t_0-\tau_1(t)),y(t_0-\tau_2(t)))y(t_0)-y'(t_0)$.
Choose

$$\begin{aligned}K(t,y(t),y(t-\tau_1(t)),y(t-\tau_2(t)))=&-F(t,Y)y+\int_{t_0}^{t}F_s(s,Y(s))y(s)ds\\&-\int_{t_0}^{t}H(s,Y(s))ds-r(t_0).\end{aligned}$$

As a consequence of this fact, we can reach that

$$|\frac{dy}{dt}-K(t,y(t),y(t-\tau_1(t)),y(t-\tau_2(t)))|\leq\int_{t_0}^{t}\phi(s)ds\leq L\phi(t)\leq\phi(t).$$

This finishes the proof of Theorem 2.3.

3 Conclusion

A modified (LE) with multiple variable time lags was considered. The (HUS) and (HURS) of this modified (LE) were investigated. We benefited from (BCP) in proving our main results thus extending and improving some recent results found in the literature from the cases of without time lags to the case with multiple variable time lags.

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