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Left and right magnifying elements in semigroups of linear transformations with restricted range

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Abstract

An element a of a semigroup S is called left [right] magnifying if there exists a proper subset M of S such that $S = aM$ [$S = Ma$]. Let $L(V)$ be the linear transformation semigroup on a vector space V. It is well-known that $L(V)$ contains left [right] magnifying elements if and only if the dimension of V is infinite. In case its dimension is infinite, α is left magnifying if and only if α is surjective and not injective and α is right magnifying if and only if α is injective and not

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surjective. To generalize this results, let W be a subspace of V and $L(V,W) = \{\alpha \in L(V) \mid \operatorname{im}\alpha \subseteq W\}$. Then $L(V,W)$ is a subsemigroup of $L(V)$ and if $W = V$, then $L(V,W) = L(V)$. Our purpose in this paper is to give necessary and sufficient conditions for elements in $L(V,W)$ to be left or right magnifying.

1 Introduction and Preliminaries

The notions of left and right magnifying elements of a semigroup were introduced by Ljapin [6]. An element a of a semigroup S is called left [right] magnifying if there exists a proper subset M of S such that $S = aM$ [$S = Ma$]. Some author determined several properties of left and right magnifying elements in semigroups. Migliorini [8] gave some remarkable properties of left and right magnifying elements in semigroups. Minimal subsets associated with the left [right] magnifying were introduced and studied by Migliorini [9]. Catino and Migliorini [1] gave necessary and sufficient conditions for any semigroup to contain left and right magnifying elements. Gutan [4] studied semigroups with strong and non strong magnifying elements. Gutan [5] showed that every semigroup containing magnifying elements is factorizable. Recently, Chinram and Baupradist gave necessary and sufficient conditions for elements in some generalized transformation semigroups in [2] and [3]. Let V be a vector space over a field F and let $L(V)$ denote the set of all linear transformations from V into itself, that is, $L(V) = \{\alpha : V \to V \mid \alpha$ is a linear transformation$\}$. It is well-known that $L(V)$ is a semigroup under the composition of maps and the semigroup $L(V)$ is called the linear transformation semigroup on V. Magill, Jr. [7] studied left magnifying elements and right magnifying elements in transformation semigroups and applied to linear transformation semigroups over a vector space and semigroups of all continuous selfmaps of a topological space. Moreover, he gave necessary and sufficient conditions for elements in $L(V)$ to be left or right magnifying.

Theorem 1.1. ([7]) *$L(V)$ contains left [right] magnifying elements if and only if the dimension of V is infinite. In case its dimension is infinite, $\alpha \in L(V)$ is a left magnifying element if and only if α is surjective and not injective and α is a right magnifying element if and only if α is injective and not surjective.*

To generalize a semigroup $L(V)$ and Theorem 1.1, let W be a subspace of V and $L(V,W) = \{\alpha \in L(V) \mid \operatorname{im}\alpha \subseteq W\}$. Then $L(V,W)$ is a subsemigroup of $L(V)$ and $L(V,V) = L(V)$. Sulivan gave some remarkable properties of

$L(V, W)$ in [11]. Recently, Sommanee and Sanghanan [10] studied the regular part of $L(V, W)$. Our purpose in this paper is to give necessary and sufficient conditions for elements in $L(V, W)$ to be left or right magnifying.

2 Main Results

We will write functions from the right, $(v)\alpha$ rather than $\alpha(v)$ and compose from the left to the right, $(v)(\alpha\beta)$ rather than $(\beta \circ \alpha)(v)$, for $\alpha, \beta \in L(V)$ and $v \in V$.

2.1 Left magnifying elements

Lemma 2.1. *If* $\dim W < \dim V$*, then* $L(V, W)$ *has no left magnifying element.*

Proof. If $\dim W = 0$, then $W = \{0\}$ and $|L(V, W)| = 1$. This implies that $L(V, W)$ has no left magnifying element. Assume that $\dim W > 0$. Let α be a left magnifying element in $L(V, W)$. So there exists a proper subset M of $L(V, W)$ such that $\alpha M = L(V, W)$. Since $\dim W < \dim V$, α is not injective. So there exist $w \in W$ and $v_1, v_2 \in V$ such that $\{v_1, v_2\}$ is linearly independent and $(v_1)\alpha = (v_2)\alpha = w$. Let $w' \in W$ be such that $w' \neq w$ and B be a basis of V containing v_1 and v_2. Define $\beta \in L(V, W)$ on B by for $b \in B$,

$$(b)\beta = \begin{cases} w & \text{if } v = v_1, \\ w' & \text{if } v \neq v_1. \end{cases}$$

Then there is no $\gamma \in L(V, W)$ such that $\alpha\gamma = \beta$, a contradiction. Hence $L(V, W)$ has no left magnifying element. □

Lemma 2.2. *Assume that* $\dim W = \dim V$*. If* α *is a left magnifying element in* $L(V, W)$*, then* α *is injective.*

Proof. Assume that α is a left magnifying element in $L(V, W)$. Then there exists a proper subset M of $L(V, W)$ such that $\alpha M = L(V, W)$. Since $\dim W = \dim V$, there exists an injective linear transformation β in $L(V, W)$. Therefore there exists $\gamma \in M$ such that $\alpha\gamma = \beta$. This implies α is injective. □

Lemma 2.3. *Assume that* $W \neq V$*. Let* $\alpha \in L(V, W)$*. If* α *is injective, then* α *is a left magnifying element in* $L(V, W)$*.*

Proof. Assume that $W \neq V$ and α is injective. Let $M = \{\gamma \in L(V,W) \mid (v)\gamma = 0 \text{ for all } v \notin \operatorname{im}\alpha\}$. We claim that $\alpha M = L(V,W)$. Let $\beta \in L(V,W)$. Let B' be a basis of V. Since α is injective, $A = \{(b)\alpha \mid b \in B'\}$ is linearly independent and $< A >= \operatorname{im}\alpha$. Let B be a basis of V containing A. Define $\gamma \in L(V,W)$ on B by for $x \in B$

$$(x)\gamma = \begin{cases} (b)\beta & \text{if } x \in A \text{ and } x = (b)\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(v)\gamma = 0$ for all $v \notin \operatorname{im}\alpha$, and so $\gamma \in M$. For $b \in B'$, we have $(b)\alpha\gamma = ((b)\alpha)\gamma = (b)\beta$. Then $\alpha\gamma = \beta$, this implies that $\alpha M = L(V,W)$. Hence α is a left magnifying element in $L(V,W)$. □

Example 2.1. Let V be a vector space over a field $\mathbb{R}$ such that $\dim V = \aleph_0$ and $B = \{b_n \mid n \in \mathbb{N}\}$ is a basis of V. Let $W =< \{b_n \mid n \in 2\mathbb{N}\} >$. Define $\alpha \in L(V,W)$ on B by $(b_n)\alpha = b_{2n}$ for all positive integers n. Then α is injective. Let $M = \{\gamma \in L(V,W) \mid (b_{2n-1})\gamma = 0 \text{ for all } n \in \mathbb{N}\}$. Let $\beta \in L(V,W)$. By Lemma 2.3, we define $\gamma \in L(V,W)$ by for all $n \in \mathbb{N}, (b_{2n})\gamma = (b_n)\beta$ and $(b_{2n-1})\gamma = 0$. So $\gamma \in M$ and $\alpha\gamma = \beta$.

For example, if $\beta \in L(V,W)$ such that $(b_n)\beta = b_{4n}$ for all $n \in \mathbb{N}$. Define $\gamma \in L(V,W)$ on B by $(b_{2n})\gamma = b_{4n}$ and $(b_{2n-1})\gamma = 0$ for all $n \in \mathbb{N}$. So $\gamma \in M$ and if $n \in \mathbb{N}$, we have $(b_n)\alpha\gamma = ((b_n)\alpha)\gamma = (b_{2n})\gamma = b_{4n} = (b_n)\beta$.

Theorem 2.4. *Assume that* $\dim W = \dim V$ *and* $W \neq V$. *Then* α *is left magnifying of* $L(V,W)$ *if and only if* α *is injective.*

Proof. This follows from Lemma 2.2 and Lemma 2.3. □

Corollary 2.5. *Let* $\alpha \in L(V)$. α *is a left magnifying element in* $L(V)$ *if and only if* α *is injective but not surjective.*

Proof. Assume that α is injective but not surjective. Let $M = \{\gamma \in L(V) \mid (v)\gamma = 0 \text{ for all } v \notin \operatorname{im}\alpha\}$. We claim that $\alpha M = L(V)$. Let $\beta \in L(V)$. Let B' be a basis of V. Clearly, $A = \{(b)\alpha \mid b \in B'\}$ is linearly independent and so A is a basis of $\operatorname{im}\alpha$. Let B be a basis of V containing A. Define $\gamma \in L(V)$ on B by for $x \in B$

$$(x)\gamma = \begin{cases} (b)\beta & \text{if } x \in A \text{ and } x = (b)\alpha, \\ 0 & \text{if } x \notin A. \end{cases}$$

Then $\gamma \in M$ and for $b \in B'$, we have $(b)\alpha\gamma = ((b)\alpha)\gamma = (b)\beta$. Thus $\alpha\gamma = \beta$, this implies that $\alpha M = L(V)$. Therefore α is a left magnifying element in $L(V)$. Conversely, assume that α is a left magnifying element in $L(V)$. By Lemma 2.2, we have α is injective. Suppose α is surjective. Since α is bijective, α^{-1} is defined and $\alpha^{-1} \in L(V)$. Since α is a left magnifying element in $L(V)$, there exists a proper subset M of $L(V)$ such that $\alpha M = L(V)$. We have $\alpha M = \alpha L(V)$ and so $M = \alpha^{-1}\alpha M = \alpha^{-1}\alpha L(V) = L(V)$, a contradiction, this implies that α is injective but not surjective. □

2.2 Right magnifying elements

Lemma 2.6. *If α is a right magnifying element in $L(V, W)$, then α is surjective.*

Proof. Assume that α is a right magnifying element in $L(V, W)$. Therefore there exists a proper subset M of $L(V, W)$ such that $M\alpha = L(V, W)$. Since $W \subseteq V$, there exists a surjective linear transformation β in $L(V, W)$. Then there exists $\gamma \in M$ such that $\gamma\alpha = \beta$. This implies that α is surjective. □

Lemma 2.7. *Let $\alpha \in L(V, W)$ be surjective but not injective.*

1. *If $(w)\alpha^{-1} \cap W = \emptyset$ for some $w \in W$, then α is not right magnifying.*
2. *If $|(w)\alpha^{-1} \cap W| = 1$ for all $w \in W$, then α is not right magnifying.*
3. *If $(w)\alpha^{-1} \cap W \neq \emptyset$ for all $w \in W$ and $|(w)\alpha^{-1} \cap W| > 1$ for some $w \in W$, then α is right magnifying.*

Proof. Let $\alpha \in L(V, W)$ be surjective but not injective.

(1) Assume that $(w)\alpha^{-1} \cap W = \emptyset$ for some $w \in W$. Let $w_0 \in W$ be such that $(w_0)\alpha^{-1} \cap W = \emptyset$. Let B be a basis of V and define $\beta \in L(V, W)$ on B by $(b)\beta = w_0$ for all $b \in B$. Then there is no $\gamma \in L(V, W)$ such that $\gamma\alpha = \beta$. Then α is not right magnifying.

(2) Assume that $|(w)\alpha^{-1} \cap W| = 1$ for all $w \in W$. Then $\alpha|_W$ is bijective. Suppose α is right magnifying. Then there exists a proper subset M of $L(V, W)$ such that $M\alpha = L(V, W)$. Hence $M\alpha = L(V, W)\alpha$. Since $\alpha|_W$ is bijective, $M = L(V, W)$, a contradiction. Therefore α is not right magnifying.

(3) Assume that $(w)\alpha^{-1} \cap W \neq \emptyset$ for all $w \in W$ and $|(w)\alpha^{-1} \cap W| > 1$ for some $w \in W$. Let $M = \{\gamma : V \to W \mid \gamma \text{ is not surjective}\}$. Then $M \neq L(V, W)$. Let β be any linear transformation in $L(V, W)$. Let B be a basis of V. Since α is surjective and $(w)\alpha^{-1} \cap W \neq \emptyset$ for all $w \in W$, we

have for all $b \in B$, there exists $w_b \in W$ such that $(w_b)\alpha = (b)\beta$. Define $\gamma \in L(V,W)$ on a basis B of V by $(b)\gamma = w_b$ for all $b \in B$. Since α is not injective and $|(w)\alpha^{-1} \cap W| > 1$ for some $w \in W$, γ is not surjective. Then $\gamma \in M$ and for all $b \in B$, we have $(b)\gamma\alpha = ((b)\gamma)\alpha = (w_b)\alpha = (b)\beta$. Thus $\gamma\alpha = \beta$, hence $M\alpha = L(V,W)$. Therefore α is right magnifying. □

Example 2.2. Let V be a vector space over a field $\mathbb{R}$ such that $\dim V = \aleph_0$ and $B = \{b_n \mid n \in \mathbb{N}\}$ is a basis of V. Let $W =< \{b_n \mid n \in 2\mathbb{N}\} >$. Let $\alpha \in L(V,W)$ by $(b_1)\alpha = (b_2)\alpha = b_2$ and $(b_{2n})\alpha = (b_{2n-1})\alpha = b_{2n-2}$ for all positive integer $n > 1$. Then α is surjective but not injective such that $(w)\alpha^{-1} \cap W \neq \emptyset$ for all $w \in W$ and $|(w)\alpha^{-1} \cap W| > 1$ for some $w \in W$. Let $M = \{\gamma \in L(V,W) \mid \gamma \text{ is not surjective}\}$. Let $\beta \in L(V,W)$ be any linear transformation. By Lemma 2.7(3), we can define $\gamma \in L(V,W)$ such that $\gamma \in M$ and $\gamma\alpha = \beta$.

For example, if β is an element in $L(V,W)$ such that $(b_n)\beta = b_{2n}$ for all $b_n \in B$. Define a linear transformation $\gamma \in L(V,W)$ by $(b_n)\gamma = b_{2n+2}$ for all $n \in \mathbb{N}$. So $\gamma \in M$ and if $n \in \mathbb{N}$, we have $(b_n)\gamma\alpha = ((b_n)\gamma)\alpha = (b_{2n+2})\alpha = b_{2n} = (b_n)\beta$.

Theorem 2.8. *α is right magnifying in $L(V,W)$ if and only if α is surjective but not injective such that $(w)\alpha^{-1} \cap W \neq \emptyset$ for all $w \in W$ and $|(w)\alpha^{-1} \cap W| > 1$ for some $w \in W$.*

Proof. Assume that α is right magnifying. By Lemma 2.6, α is surjective. Suppose α is injective. Since α is right magnifying, there exists a proper subset M of $L(V,W)$ such that $M\alpha = L(V,W)$. This implies that $M\alpha = L(V,W)\alpha$. Since α is injective, $M = L(V,W)$, a contradiction. Hence α is not injective. By Lemma 2.7, we have α is surjective but not injective such that $(w)\alpha^{-1} \cap W \neq \emptyset$ for all $w \in W$ and $|(w)\alpha^{-1} \cap W| > 1$ for some $w \in W$. Conversely, assume that α is surjective but not injective such that $(w)\alpha^{-1} \cap W \neq \emptyset$ for all $w \in W$ and $|(w)\alpha^{-1} \cap W| > 1$ for some $w \in W$. By Lemma 2.7, we have α is right magnifying. □

Corollary 2.9. *Let $\alpha \in L(V)$. α is right magnifying in $L(V)$ if and only if α is surjective but not injective.*

Proof. This follows by Theorem 2.8 and the fact that if α is surjective but not injective, then $(v)\alpha^{-1} \cap V \neq \emptyset$ for all $v \in V$ and $|(v)\alpha^{-1} \cap V| > 1$ for some $v \in V$. □

3 Conclusion

We give necessary and sufficient conditions for elements in $L(V, W)$ to be left or right magnifying.

1. If $\dim W < \dim V$, then $L(V, W)$ has no left magnifying element.

2. If $\dim W = \dim V$ and $W \neq V$, then α is left magnifying in $L(V, W)$ if and only if α is injective.

3. α is left magnifying in $L(V)$ if and only if α is injective but not surjective.

4. α is right magnifying in $L(V, W)$ if and only if α is surjective but not injective such that $(w)\alpha^{-1} \cap W \neq \emptyset$ for all $w \in W$ and $|(w)\alpha^{-1} \cap W| > 1$ for some $w \in W$.

5. α is right magnifying in $L(V)$ if and only if α is surjective but not injective.

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