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Transverse Levi civita connection of a Riemannian foliation having dense leaves and admitting a flag of extension on a compact manifold having a finite fundamental group

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Abstract

In this paper we show that the transverse Levi civita connection of a Riemannian foliation having dense leaves and admitting a flag of extension on a compact manifold M is integrable and if additionally the fundamental group of M is finished then the foliation of the closure of leaf $F^{\natural}$ of lifted foliation $\mathcal{F}^{\natural}$ of $\mathcal{F}$ on the orthonormal transverse frame bundle $M^{\natural}$ is defined by the connection of transverse Levi civita $\omega_T\cdot$ In summary, the horizontal spaces of the transverse Levi civita connection ω_T of a Riemannian foliation having dense leaves and admitting a flag of extension on a compact manifold M having a finite fundamental group is the leaves of the closure $\overline{\mathcal{F}^{\natural}}$ of lifted foliation $\mathcal{F}^{\natural}$ of $\mathcal{F}$ on the orthonormal transverse frame bundle $M^{\natural}\cdot$ We also establish that the structural Lie algebra of such Riemannian foliation is an abelian Lie algebra·

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1 Introduction

Here $(U_i, f_i, T^q, \gamma_{ij})_{i\in I}$ is a foliated cocycle defining a Riemannian foliation $\mathcal{F}_q$ having dense leaves and admitting a complete flag of extension

$$\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_q, \mathcal{F}_{q-1},\ \mathcal{F}_{q-2},\ ...,\ \mathcal{F}_1)$$

on a compact Riemannian manifold M having finite fundamental group·
The foliation $\mathcal{F}_q$ is said Riemannian transversaly diagonal if only and only the foliated cocycle $(U_i, f_i, T^q, \gamma_{ij})_{i\in I}$ defining $\mathcal{F}_q$ satisfy the following conditions:
$i)$ the open sets U_i are $\mathcal{F}$ -distinguished,
$ii)$ there exists a metric on the transverse manifold T^q for which the γ_{ij} are local isometries,
$iii)$ for every $(i, j) \in I \times I$, in coordinate systems $\mathcal{F}$-local transversals on the open $\mathcal{F}$-distinguished U_i, the Jacobian matrix $J_{\gamma_{ij}}$ of γ_{ij} is diagonal.
Let $\mathcal{F}_q^\natural$ be the lifted foliation of $\mathcal{F}$ on the orthonormal $\mathcal{F}$-transverse frame bundle $M^\natural$, let $\overline{\mathcal{F}^\natural}$ be the foliation of closure of the leaves of $\mathcal{F}^\natural$ and let ω_T be the transverse Levi civita connection of $M^\natural$. Then ω_T is integrable and :
$i)$ the foliation defined by ω_T is the foliation $\overline{\mathcal{F}^\natural}$,
$ii)$ the structural Lie algebra $\mathcal{G}_{\mathcal{F}_q}$ of $\mathcal{F}_q$ is abelian of dimension equal to q,
In all that follows, the manifolds considered are supposed to be related and the differentiability C^∞.

2 Definitions

We reformulate certain definitions and theorems that are found in the literature (For instance [1], [2]).

Definition 2.1. *Let $P^\natural$ be a main bundle of a structural group G and $\omega_{P^\natural}$ a connection on $P^\natural$· The holonomy group at $z \in P^\natural$ of the connection $\omega_{P^\natural}$ is the set*

$$Hol_z(\omega_{P^\natural}) = \{g \in G \ / \ z \sim \ z \cdot g\}$$

where $z \sim z'$ means that there is a piecewise horizontal curve with extremities z and z'. (The relation $\sim$ is an equivalence relation).

The restricted holonomy group $Hol_z^0(\omega_{P^\natural})$ is also defined as the subgroup of $Hol_z(\omega_{P^\natural})$, but adding the following condition:
if $g \in Hol_z^0(\omega_{P^\natural})$, then z is connected to $g \cdot z$ by a different piecewise horizontal curve whose projection on the basic manifold of fiber $P^\natural$ is zero homotopy·

We also note that if $P^{\natural}$ and its base are related, then the holonomy group $Hol_z\left(\omega_{P^{\natural}}\right)$ depends on the point z only at a close conjugation (in G).
More precisely, if z' is another point of $P^{\natural}$, there is a single g of G such that $z \sim z \cdot g$, and then:

$$Hol_{z'}\left(\omega_{P^{\natural}}\right) = g^{-1} \cdot Hol_z\left(\omega_{P^{\natural}}\right) \cdot g$$

Proposition 2.2. *$Hol_z^0\left(\omega_{P^{\natural}}\right)$ is a related Lie group that coincides with the connected component by arc of $Hol_z\left(\omega_{P^{\natural}}\right)$ which contains the neutral element. Moreover, there is a surjective morphism of group of $\pi_1(M)$ on $\frac{Hol_z\left(\omega_{P^{\natural}}\right)}{Hol_z^0\left(\omega_{P^{\natural}}\right)}$ where $\pi_1(M)$ is the basic group of basic manifold M of the main bundle $P^{\natural}\cdot$*

Proposition 2.3. *Let $P^{\natural}$ be a main bundle and let $K^{\natural}$ be a sub-bundle of $P^{\natural}$ having $\mathcal{K}$ for Lie structural algebra$\cdot$*
A connection $\omega_{P^{\natural}}$ on $P^{\natural}$ is said to be suitable for $K^{\natural}$ if and only if the restriction $\omega_{K^{\natural}}$ of $\omega_{P^{\natural}}$ at $K^{\natural}$ has its values in $\mathcal{K}$ $\cdot$
Note that this is similar to imposing on the infinitesimal connection $H_{P^{\natural}}$ having $\omega_{P^{\natural}}$ for form of connection to check the condition $H_{P^{\natural},z} \subset T_z K^{\natural}$ for every $z \in K^{\natural}$, where $H_{P^{\natural},z}$ is the horizontal space of $H_{P^{\natural}}$ at z; that is to say, $H_{P^{\natural},z} \subset Ker\omega_{P^{\natural},z}\cdot$

The following theorem is the Ambrose-Singer theorem.

Theorem 2.4. *Let $P^{\natural}$ be a main bundle, let $\omega_{P^{\natural}}$ be a connection on $P^{\natural}$, let R be the curvature shape of $\omega_{P^{\natural}}$ and let $z \in P^{\natural}\cdot$*
i) The subset $P^{\natural}\left(\omega_{P^{\natural}}\right)$ of $P^{\natural}$ of points of $P^{\natural}$ that can be attached to z by a piecewise differentiable horizontal curve is a main bundle of group of structure $Hol_z\left(\omega_{P^{\natural}}\right)\cdot$
ii) The connection $\omega_{P^{\natural}}$ is suitable at $P^{\natural}\left(\omega_{P^{\natural}}\right)\cdot$
iii) The connection obtains by the restriction of $\omega_{P^{\natural}}$ at $P_z^{\natural}\left(\omega_{P^{\natural}}\right)$ have in any point of $P_z^{\natural}\left(\omega_{P^{\natural}}\right)$ the same holnomy group as the connection $\omega_{P^{\natural}}\cdot$
iv) The Lie algebra $hol_z^0\left(\omega_{P^{\natural}}\right)$ of the Lie group $Hol_z^0\left(\omega_{P^{\natural}}\right)$ is generated by the elements $R(X_z\ ,\ Y_z)$ for $X_z\ ,\ Y_z \in H_{P^{\natural},z} = Ker\omega_{P^{\natural},z}$

Note that the curvature shape R of $\omega_{P^{\natural}}$ is null if and only if $Hol_z^0\left(\omega_{P^{\natural}}\right)$ is trivial.
Note that $P^{\natural}$ breaks down into a disjointed union of principal bundles of the type $P_z^{\natural}\left(\omega_{P^{\natural}}\right)\cdot$ In addition, the main fibers of the type $P_z^{\natural}\left(\omega_{P^{\natural}}\right)$ are isomorphic to each other.

Definition 2.5. *We say that a curve $(I;\gamma)$ of a differential manifold M is tangent to a foliation $\mathcal{F}$ of M if only if $\dot{\gamma}(t) \in T_{\gamma(t)}\mathcal{F}\cdot$*

Proposition 2.6. *If* $(I,\ \gamma)$ *is a curve of a different manifold* M *tangent to a foliation* $\mathcal{F}$ *defined on* M*, then his image* $\gamma(I)$ *is contained in a leaf of* $\mathcal{F}\cdot$

Definition 2.7. *The main bundle* $P_z^{\natural}(\omega_{P^\natural})$ *is called the holonomy bundle of* z *of the connection* $\omega_{P^\natural}\cdot$

Definition 2.8. *A* $q-$*codimensional foliation* $(M,\mathcal{F})$*on* M *is the data of an open recovering* $(U_i)_{i\in I}$ *of* M *of submersion* $f_i : U_i \to T$*, where* T *is a* $q-$*dimension q manifold and, for* $U_i \cap U_j \neq \emptyset$*, of a diffeomorphism*

$$\gamma_{ij} : f_j(U_i \cap U_j) \subset T \to f_i(U_i \cap U_j) \subset T$$

satisfying $f_i(x) = (\gamma_{ij} \circ f_j)(x)$ *for every* $x \in U_i \cap U_j$.
We say that $(U_i, f_i, T, \gamma_{ij})_{i\in I}$ *is a foliated cocycle and the manifold* T *is called transverse manifold of foliation* $(M,\mathcal{F})$.

We note that the manifold T can be considered as a disjoint intersection of $f_i(U_i)$.
The connected components of fibers of f_i, called plaques**,** form a basis for a topology for which the connected components are called leaves of foliation.
The manifold M provided with leaf topology is denoted by$M^{\mathcal{F}}$.
It should also be mentioned that any leaf of $\mathcal{F}$ is connected by arc.
We emphasize that for a foliation $\mathcal{F}$ on M and $f \in C^\infty(N,M)$ where N is a manifold, if the image $f(N)$ of f is contained in a countable infinity of leaves of $\mathcal{F}$, then $f \in C^\infty\left(N, M^{\mathcal{F}}\right)$.

Theorem 2.9. *Let* $\mathcal{F}$ *be a* $q-$*codimensional Riemannian foliation on a compact connected manifold* M*. Let* $\overline{F^\natural}$ *be the closure of a leaf* $F^\natural$ *of lifted foliation* $\mathcal{F}^\natural$ *of* $\mathcal{F}$ *on the orthonormal transverse frame bundle* $M^\natural$ *and let* $\phi : M^\natural \to M$ *be the projection which associates* x *to a frame.*
Them:
i) $\phi\left(F^\natural\right)$ *is a leaf of* F *and* $\overline{\phi\left(F^\natural\right)} = \phi\left(\overline{F^\natural}\right)$,
ii) the map $\phi : \overline{F^\natural} \longrightarrow \phi\left(\overline{F^\natural}\right)$ *is a locally trivial fibration.*

The structural Lie Algebra $\mathcal{H}_m$ of this fibration is the Lie algebra of fundamental fields $\widetilde{\lambda}$ of $M\natural$ whose value at z is tangent to $\overline{F\natural}$; that is,

$$\mathcal{H}_m = \left\{\lambda \in \widetilde{\mathcal{O}}(q,\mathbb{R}\ ) \,/\, \widetilde{\lambda}_z \in T_z\overline{F^\natural}\ \right\}$$

where $\widetilde{\mathcal{O}}(q,\mathbb{R}\ )$ is the Lie algebra of the orthogonal group $O(q,\mathbb{R}^q\ )\cdot$

Theorem 2.10. *Let* $\mathcal{F}$ *be a* $q-$*codimensional Riemannian foliation on a compact related manifold* M*, let* $\pi : M^\natural \to M$ *be the orthonormal transverse*

frame bundle and let $\mathcal{F}^\natural$ be lifted foliation of $\mathcal{F}$ on the orthonormal transverse frame bundle $M^\natural$·
Then the fundamental form θ_T and the Levi civita transverse connection ω_T of $M^\natural$ are $\mathcal{F}^\natural -$ basic·
Note that the twist δ_T of the Levi civita transverse connection ω_T is zero, where

$$\delta_T = d\theta_T + \omega_T \wedge \theta_T$$

We also have for every $z \in M^\natural$,

$$Y_z \in T_z\mathcal{F}^\natural \iff i_{Y_z}\theta_T = i_{Y_z}d\theta_T = 0\cdot$$

Finally, note that if u is a vector of $\mathbb{R}^q$ we will call the basic field associated with u (for the connection of Levi-civita transverse ω_T) the vector field $\widetilde{u}$ horizontal on $M^\natural$ defined by the conditions:

$$\omega_T(\widetilde{u}) = 0 \quad and \quad \theta_T(\widetilde{u}) = u\cdot$$

Definition 2.11. *A $q-$codimensional foliation $\mathcal{F}$ on a manifold M is said transversally diagonal relative to $\mathcal{F}$-transverse local coordinates systems $\left(y^i_{1,}y^i_{2,\ldots,}y^i_q\right)_{i\in I}$ if and only if it is defined by a foliated cocycle $(U_i, f_i, T, \gamma_{ij})_{i\in I}$ such that the open sets U_i are $\mathcal{F}$-distinguished and on each open $f_i(U_i)$ there exists a $\mathcal{F}$-transverse local coordinates system $\left(y^i_{1,}y^i_{2,\ldots,}y^i_q\right)$ such that relative to $\mathcal{F}$-transverse local coordinates systems $\left(y^i_{1,}y^i_{2,\ldots,}y^i_q\right)_{i\in I}$ on open $f_i(U_i)$, the Jacobian matrix $J_{\gamma_{ij}}$ of γ_{ij} is diagonal. In the case where there exists a metric λ_T on the transverse manifold T such that the γ_{ij} are local isometries for this transverse metric, we say that $\mathcal{F}$ is Riemannian transversally diagonal.*

Remark 2.12. [*18*] *Let $\mathcal{F}$ be a $q-$codimensional Riemannian transversally diagonal foliation admetting $\left(y^i_{1,}y^i_{2,\ldots,}y^i_q\right)_{i\in I}$ for $\mathcal{F}$-traverse diagonal local coordinates system on a manifold M relative to a transverse metric g_T defined by a foliated cocycle $(U_i, f_i, T, \gamma_{ij})_{i\in I}$ ·*
We have for all $x \in U_i$, $\left(\frac{\partial}{\partial y^i_1}(x), \frac{\partial}{\partial y^i_2}(x), \cdots, \frac{\partial}{\partial y^i_q}(x)\right)$ is a $\mathcal{F}$ $-$transverse orthonormal basis

The family of local transverse coordinates systems $\left(y^i_{1,}y^i_{2,\ldots,}y^i_q\right)_{i\in I}$ is called an associated family to $\mathcal{F}$.
We note that the fact that the Jacobian matrix $J_{\gamma_{ij}}$ of the isometry γ_{ij} is diagonal relative to $\mathcal{F}$-transverse local coordinates systems $\left(y^i_{1,}y^i_{2,\ldots,}y^i_q\right)_{i\in I}$ implies that relative to $\left(y^i_{1,}y^i_{2,\ldots,}y^i_q\right)_{i\in I}$ we have

$$J_{\gamma_{ij}} = \left(\varepsilon^{ij}_{rs}\right)_{rs},$$

where

$$\varepsilon_{rs}^{ij} = 1 \text{ if } r = s \text{ and } \varepsilon_{rs}^{ij} = 0 \text{ if } r \neq s\cdot$$

Proposition 2.13. *Let $\mathcal{F}$ be a $q-$codimensional Riemannian transversally diagonal foliation admitting $\left(y_{1,}^{i}y_{2,\dots,}^{i}y_{q}^{i}\right)_{i\in I}$ for $\mathcal{F}$-traverse diagonal local coordinates system on a manifold M relative to a transverse metric g_{T} then we have for all $x \in U_{i}$,*

$$g_{Tsr}^{i} = g_{T}\left(\frac{\partial}{\partial y_{k}^{i}}(x), \frac{\partial}{\partial y_{r}^{i}}(x)\right) = 1 \quad if \quad r = s$$

and

$$g_{Tsr}^{i} = g_{T}\left(\frac{\partial}{\partial y_{k}^{i}}(x), \frac{\partial}{\partial y_{r}^{i}}(x)\right) = 0 \quad if \quad r \neq s\cdot$$

Definition 2.14. *Let M be a manifold. An extension of a $q-$codimensional foliation $(M, \mathcal{F})$ is a $q'-$codimensional foliation $(M, \mathcal{F}')$ such that $0 < q' < q$ and $(M, \mathcal{F}')$ leaves are $(M, \mathcal{F})$ leaves meetings ($\mathcal{F} \subset \mathcal{F}'$).*

Definition 2.15. *Given a $q-$codimensional foliation $\mathcal{F}_q$ on a manifold M, a flag of extensions of $\mathcal{F}_q$ is a sequence $\mathcal{D}_{\mathcal{F}_q}^{k} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, \dots, \mathcal{F}_k)$ of foliations on M such as $\mathcal{F}_q \subset \mathcal{F}_{q-1} \subset \mathcal{F}_{q-2} \subset \dots \subset \mathcal{F}_k$ and each foliation $\mathcal{F}_s$ is a $s-$codimensional foliation.*
For $\dim \mathcal{F}_q = 1$, the flag of extensions $\mathcal{D}_{\mathcal{F}_q}^{k}$ will be called flag of foliations.
For $k = 1$, the flag of extensions $\mathcal{D}_{\mathcal{F}_q}^{k}$ will be called complete and will be denoted by $\mathcal{D}_{\mathcal{F}_q}$.
If each foliation $\mathcal{F}_s$ is Riemannian, the flag of extensions $\mathcal{D}_{\mathcal{F}_q}^{k}$ will be called flag of Riemannian extensions of $\mathcal{F}_q$.

Proposition 2.16. *Let $\mathcal{F}$ be a $q-$codimensional Riemannian foliation having dense leaves·*
Then $\mathcal{F}$ is transversally diagonal foliation if only and only $\mathcal{F}$ admit a complete flag of Riemannian extension $\mathcal{D}_{\mathcal{F}} = (\mathcal{F}_q, \mathcal{F}_{q-1}, \dots, \mathcal{F}_1)\cdot$

The following theorem is the structure theorem of complete Riemannian flags [8]:

Theorem 2.17. *Let $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, \dots, \mathcal{F}_1)$ be a complete flag of Riemannian foliations where $\mathcal{F}_q$ is a Riemannian foliation having T^q for transverse monifold and λ for bundle-like metric on a $(q+1)-$dimensional connected Riemannian monifold $(M; \lambda)$ not necessarily compact.*
If the metric λ is bundle-like for any foliation of $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, \dots, \mathcal{F}_1)$,

then:
1) The manifold M is completely parallelizable and foliations of $\mathcal{D}_{\mathcal{F}_q}$ are both transversally parallelizable and transversally integrable. In particular, there is a single parallelism $(Y_i)_{0\leq i\leq q}$ of M, called parallelism of $\mathcal{D}_{\mathcal{F}_q}$, such as:
- For every $i \geq 0$, Y_i is unitary tangent to $\mathcal{F}_i$ and orients the flow it defines
- For all $q \geq j > i \geq 0$, the vector fields Y_j and Y_i are orthogonal and $[Y_j, Y_i] = k_{ji}Y_j$ where functions k_{ji}, called structure functions of $\mathcal{D}_{\mathcal{F}_q}$, are, for $i \leq q-1$, basic for $\mathcal{F}_{i+1}$.
2) If the metric is complete, $\mathcal{D}_{\mathcal{F}_q}$ can be lifted on the universal covering on a simple flag of foliations.
When the manifold is compact, $\mathcal{D}_{\mathcal{F}_q}$ is a flag of Lie homogeneous foliatons if only if its structure functions are constant.

We note that the parallelism $(Y_i)_{0\leq i\leq q}$ of M will be called parallelism of Diallo of $\mathcal{D}_{\mathcal{F}_q}$.

3 Main Results

Proposition 3.1. *Let $\mathcal{F}$ be a $q-$codimensional Riemannian transversally diagonal foliation admitting $\left(y^i_{1,}y^i_{2,\dots,}y^i_q\right)_{i\in I}$ for $\mathcal{F}$-traverse diagonal local coordinates system on a compact manifold M relative to a transverse metric g_T defined by a foliated cocycle $(U_i, f_i, T, \gamma_{ij})_{i\in I}$ $\cdot$ Let $\pi : M^\natural \to M$ be the orthonormal transverse frame bundle. Let $\mathcal{F}^\natural$ be lifted foliation of $\mathcal{F}$ on the orthonormal transverse frame bundle $M^\natural$, let ω_T be the transverse Levi civita connection of $M^\natural$, and let $\overline{\mathcal{F}^\natural}$ be the foliation of adhesions of the leaves of $\mathcal{F}^\natural$.*
Then ω_T is integrable.

Proof. Consider the Christofel symbols Γ^j_{ik} of the metric $\mathcal{F}-$trasverse $g_T\cdot$
Using Einstein's summation, we have [16]:

$$\Gamma^j_{ik} = \frac{1}{2} g_{Tjs}\left(\partial_k g_{Tsi} + \partial_i g_{Tsk} + \partial_s g_{Tik}\right),$$

where the g_{Tjs} are the coefficients of the inverse matrix of $(g_{Tij})_{ij}\cdot$
From the previous equality and the consistency of each g_{Tij}

$$\Gamma^j_{ik} = 0, forall i, j, k\cdot$$

We recall [16] that the 2-form curvature of the connection $\omega_{_T}$ local coordinates has its components

$$R^j_{_Tiks} = \partial_k \Gamma^j_{si} - \partial_s \Gamma^j_{ki} + \Gamma^j_{ku}\Gamma^u_{si} - \Gamma^j_{su}\Gamma^u_{ki}\cdot$$

Hence the nullity of Γ^j_{ik} shows that $R_{_T} = 0\cdot$ But curvature $R_{_T}$ checks the equality:

$$R_{_T} = d\omega_{_T} + \frac{1}{2}\left[\omega_{_T}, \omega_{_T}\right]$$

so

$$d\omega_{_T} + \frac{1}{2}\left[\omega_{_T}, \omega_{_T}\right] = 0\cdot$$

Thus $\omega_{_T}$ defines a Lie foliation $\mathcal{F}^{\omega_T}\cdot$ □

Theorem 3.2. *Let $\mathcal{F}$ be a Riemannian foliation having dense leaves and admitting a flag of extension on a compact manifold M admitting a finite fundamental group $\pi_1(M)$, let $\omega_{_T}$ be the $\mathcal{F}-$transverse Levi civita connection of $M^\natural$, and let $\overline{\mathcal{F}^\natural}$ be the foliation defined by the closure of the leaves of $\mathcal{F}^\natural$ and let $\mathcal{F}^{\omega_T}$ the foliation defined by the transverse Levi-civita connection $\omega_{_T}$ of $M^\natural\cdot$*
Then $\mathcal{F}^{\omega_T} = \overline{\mathcal{F}^\natural}\cdot$

Proof. Note that the fact the foliation $\mathcal{F}$ is a Riemannian foliation having dense leaves and admitting a complete flag of extension

$$\mathcal{D}_{_\mathcal{F}} = (\mathcal{F}, \mathcal{F}_{q-1},\ \mathcal{F}_{q-2},\ ...,\ \mathcal{F}_1)$$

on a compact manifold implies according to ([18], [19]) that

$$\mathcal{D}_{_\mathcal{F}} = (\mathcal{F}, \mathcal{F}_{q-1},\ \mathcal{F}_{q-2},\ ...,\ \mathcal{F}_1)$$

is a complete flag of extensions where each foliation $\mathcal{F}_i$ is a Riemannian foliations where having dense leaves. Therefore the flag of extension

$$\mathcal{D}_{_\mathcal{F}} = (\mathcal{F}, \mathcal{F}_{q-1},\ \mathcal{F}_{q-2},\ ...,\ \mathcal{F}_1)$$

is a flag of extension where each foliation $\mathcal{F}_i$ is Riemannian transverselly diagonal$\cdot$
This being said, to establish that

$$\mathcal{F}^{\omega_T} = \overline{\mathcal{F}^\natural}$$

we begin to establish that the leaves of $\mathcal{F}^{\omega_T}$ are compact$\cdot$
Let $z \in M^\natural$ and let $F_z^{\omega_T}$ the leaft of $\mathcal{F}^{\omega_T}$ passing at $z\cdot$
By a previous proposition, any horizontal curve is contained in a leaft of $\mathcal{F}^{\omega_T}\cdot$ Therefore the holonomy bundle $M^\natural_{(z,\omega_T)}$ of the connection ω_T, on the main bundle $M^\natural$, at z is contained in the leaft $F_z^{\omega_T}$ $\cdot$
The connectivity by arc of $F_z^{\omega_T}$ and the fact that any vector is tangent to $F_z^{\omega_T}$ horizontal ensures that

$$F_z^{\omega_T} \subset M^\natural_{(z,\omega_T)}\cdot$$

It follows that

$$F_z^{\omega_T} = M^\natural_{(z,\omega_T)}\cdot$$

Since the connection ω_T is integrable, we have $Hol_z^0(\omega_{P^\natural})$ which is trivial$\cdot$
Thus

$$\frac{Hol_z(\omega_{P^\natural})}{Hol_z^0(\omega_{P^\natural})} = Hol_z(\omega_T)\cdot$$

We also observe that $\pi_1(M)$ being finished, that it results from the existence of the surjective morphism of $\pi_1(M)$ on $\frac{Hol_z(\omega_{P^\natural})}{Hol_z^0(\omega_{P^\natural})}$ and of equality

$$\frac{Hol_z(\omega_{P^\natural})}{Hol_z^0(\omega_{P^\natural})} = Hol_z(\omega_T)$$

that $Hol_z(\omega_T)$ is a Lie subgroup finite so compact of the structural group O $(q,\mathbb{R})$ of is a sub-bundle of$\cdot$
The equality

$$F_z^{\omega_T} = M^\natural_{(z,\omega_T)}$$

shows according to the Ambrose-Singer theorem [17] that $F_z^{\omega_T}$ is a sub-bundle of $M^\natural$ of structural group the group of compact Lie $Hol_z(\omega_T)\cdot$ So $F_z^{\omega_T}$ is a compact sub-bundle of $M^\natural\cdot$
We note, in passing that $F_z^{\omega_T}$ is a covering of M having a finite number of slips. Indeed, the structural group $Hol_z(\omega_T)$ of $F_z^{\omega_T}$ is finished$\cdot$ The transverse Levi civita connection is basic for the lifted foliation $\mathcal{F}^\natural$ of $\mathcal{F}$ on $M^\natural\cdot$ Therefore the lifted foliation $\mathcal{F}^\natural$ is horizontal$\cdot$ Thus for every leaves $F_z^{\omega_T}$ and $F_z^\natural$ we have

$$F_z^\natural \subset F_z^{\omega_T}$$

where $F_z^\natural$ is a leaft of $\mathcal{F}^\natural$ passing through $z\cdot$ The fact that $F_z^{\omega_T}$ is compac implies that

$$\overline{F_z^\natural} \subset \overline{F_z^{\omega_T}} = F_z^{\omega_T}$$

which implies again that

$$\dim \overline{F_z^\natural} \leq \dim \overline{F_z^{\omega_T}} = \dim M\cdot$$

According to Molino [12], $\overline{F_z^\natural}$ is a sub-bundle of $M^\natural\cdot$ So

$$\dim M \leq dim \overline{F_z^\natural}\cdot$$

Thus

$$\dim M = dim \overline{F_z^\natural} = \dim \overline{F_z^{\omega_T}}$$

Consequently, $\overline{F_z^\natural}$ is a sub-bundle of $M^\natural$ and $\dim M = \dim \overline{F_z^\natural}\cdot$ Therefore $\overline{F_z^\natural}$ is a coating of $M\cdot$ Thus $F_z^{\omega_T}$ and $\overline{F_z^\natural}$ form a covering of M and $\overline{F_z^\natural} \subset F_z^{\omega_T}$ so

$$\overline{F_z^\natural} = F_z^{\omega_T}\cdot$$

As a result

$$\mathcal{F}^{\omega_T} = \overline{\mathcal{F}^\natural}\cdot$$

□

We note that:
$i)$ the connection of Levi civita transverse ω_T of $M^\natural$ is suitable at $\overline{F_z^\natural}$,
$ii)$ every leaft of $\overline{\mathcal{F}^\natural}$ is a covering of M having a finite number of slips,
$iii)$ any vertical vector field of $M^\natural$ tangent to the foliation $\overline{\mathcal{F}^\natural}$ is null; that is to say,

$$\mathcal{H}_m = \left\{ \lambda \in \widetilde{\mathcal{O}}\left(q, \mathbb{R}\right) / \ \widetilde{\lambda}_z \in T_z \overline{F\natural} \right\} = \{0\}$$

$iv)$ any basic vector field relative to $M^\natural$ at ω_T is tangent to foliation $\overline{\mathcal{F}^\natural}\cdot$

Corollary 3.3. *Let $\mathcal{F}$ be a Riemannian foliation having dense leaves and admitting a flag of extension on a compact manifold M admitting a finite fundamental group $\pi_1(M)$. Let ω_T be the $\mathcal{F}-$transverse Levi civita connection on the orthonormal transverse frame bundle $M^\natural\cdot$ Finally, let $\widetilde{u}$ and $\widetilde{v}$ basic vector fields on $M^\natural$ relative to ω_T and associated respectively with the vectors u and v of $\mathbb{R}^q\cdot$*
Then $[\widetilde{u}, \widetilde{v}] = 0\cdot$

Proof. According to Molino [12], for every $z \in M^\natural$,

$$[\widetilde{u}, \widetilde{v}]_z = z \cdot R_T\left(\widetilde{u}_z, \widetilde{v}_z\right)$$

where $z \cdot R_{T}(\widetilde{u}_z, \widetilde{v}_z)$ is the value at z of the fundamental field on $M^\natural$ defined by the vector $R_{T}(\widetilde{u}_z, \widetilde{v}_z)$ of the Lie algebra of $O(q, \mathbb{R}) \cdot$
The fact that the curvature R_{T} of ω_{T} is null implies that $[\widetilde{u}, \widetilde{v}]_z = 0$ for every $z \in M^\natural \cdot$ Thus

$$[\widetilde{u}, \widetilde{v}] = 0 \cdot$$

□

Proposition 3.4. *Let $\mathcal{F}$ be a Riemannian foliation having dense leaves and admitting a flag of extension on a compact manifold M admitting a finite fundamental group $\pi_1(M) \cdot$ Let $\overline{F^\natural}$ be the closure of the leaft $F^\natural$ of $\mathcal{F}^\natural \cdot$ Let $\mathcal{F}^\natural_{/\overline{F^\natural}}$ be the Lie foliation induced by $\mathcal{F}^\natural$ on $\overline{F^\natural}$, let θ_{T} be the fundamental form of fiber $\pi : M^\natural \to M$, let $j : \overline{F^\natural} \hookrightarrow M^\natural$ be the canonical injection of $\overline{F^\natural}$ in $M^\natural$ and let $\overline{u}$ be a vector field $\mathcal{F}^\natural_{/\overline{F^\natural}}-$transverse on $\overline{F^\natural} \cdot$*
Then $\overline{u}$ is $\mathcal{F}^\natural_{/\overline{F^\natural}}-$ foliated transverse if only if there exist a vector $u \in \mathbb{R}^q$ such that

$$j^*\theta_{T}(\overline{u}) = u \cdot$$

Proof. Consider $\overline{u}$ a vector field $\mathcal{F}^\natural_{/\overline{F^\natural}}-$transverse on $\overline{F^\natural} \cdot$ Suppose there is a vector u of $\mathbb{R}^q$ such that

$$j^*\theta_{T}(\overline{u}) = u \cdot$$

Let's show that $\overline{u}$ is $\mathcal{F}^\natural_{/\overline{F^\natural}}-$ foliated$\cdot$ Let

$$\delta_T = d\theta_T + \omega_T \wedge \theta_T$$

be the twist of the Levi civita transverse connection ω_{T} we have

$$\delta_T = d\theta_T + \omega_T \wedge \theta_T = 0 \text{ and } j^*\omega_{T} = 0 \cdot$$

But

$$j^*\delta_T = d(j^*\theta_T) + (j^*\omega_T) \wedge (j^*\theta_T) = d(j^*\theta_T)$$

so

$$d(j^*\theta_T) = 0 \cdot \qquad (2)$$

We have also for every $z \in \overline{F^\natural}$, $T_z\mathcal{F}^\natural_{/\overline{F^\natural}} \subset T_z\overline{F^\natural}$ therefore for every $z \in \overline{F^\natural}$ and for every $Y_z \in T_z\mathcal{F}^\natural_{/\overline{F^\natural}}$

$$j_*(Y_z) = Y_z \qquad (3)$$

Using the equivalence

$$Y_z \in T_z\mathcal{F}^{\natural} \iff i_{Y_z}\theta_T = i_{Y_z}d\theta_T = 0$$

established in Molino [12] and the equalities (2) and (3) that

$$Y_z \in T_z\mathcal{F}^{\natural} \iff i_{Y_z}j^*\theta_T = 0 \qquad (4)$$

Now consider a vector field $X^{\natural}$ tangent to $\mathcal{F}^{\natural}_{/\overline{F^{\natural}}}\cdot$ Since

$$j^*\theta_T\left(X^{\natural}\right) = 0 \text{ and } j^*\theta_T\left(\overline{u}\right) = u,$$

$$\begin{aligned} d\left(j^*\theta_T\right)\left(\overline{u}, X^{\natural}\right) &= \left(\overline{u}\left(j^*\theta_T\right)\left(X^{\natural}\right)\right) - X^{\natural}\left(j^*\theta_T\right)\left(\overline{u}\right) - j^*\theta_T\left(\left[\overline{u}, X^{\natural}\right]\right) \\ &= j^*\theta_T\left(\left[\overline{u}, X^{\natural}\right]\right). \end{aligned}$$

But

$$d\left(j^*\theta_T\right) = 0.$$

So

$$j^*\theta_T\left(\left[\overline{u}, X^{\natural}\right]\right) = 0\cdot$$

Thus the equivalence (4) implies that $\left[\overline{u}, X^{\natural}\right]$ is tangent to $\mathcal{F}^{\natural}_{/\overline{F^{\natural}}}$ which means that $\overline{u}$ is $\mathcal{F}^{\natural}_{/\overline{F^{\natural}}}$−foliated transverse.
We now suppose that $\overline{u}$ is $\mathcal{F}^{\natural}_{/\overline{F^{\natural}}}$−foliated transverse.
As $j^*\theta_T$ is $\mathcal{F}^{\natural}_{/\overline{F^{\natural}}}$−basic, $j^*\theta_T\left(\overline{u}\right)$ is a function $\mathcal{F}^{\natural}_{/\overline{F^{\natural}}}$−basic on $\overline{F^{\natural}}$ with values are in $\mathbb{R}^q\cdot$ Thus the fact that $\mathcal{F}^{\natural}_{/\overline{F^{\natural}}}$ be a foliation having dense leaves implies that $j^*\theta_T\left(\overline{u}\right)$ is a constant function on $\overline{F^{\natural}}$ with values are in $\mathbb{R}^q\cdot$ This shows that there exists a vector $u \in \mathbb{R}^q$such as

$$j^*\theta_T\left(\overline{u}\right) = u\cdot$$

Note the following:
i) if $\left(e_1, e_2, \cdots, e_q\right)$ is a base of $\mathbb{R}^q$ then the condition $j^*\theta_T\left(\overline{e}_k\right) = e_k$ allows to obtain a parallelism $\left(\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_q\right)$ $\mathcal{F}^{\natural}_{/\overline{F^{\natural}}}$−transverse canonical on $\overline{F^{\natural}}\cdot$
ii) the basic transverse field on $\overline{F^{\natural}}$ associated with a vector u of $\mathbb{R}^q$is the vector field on $\overline{F^{\natural}}$ defined by the condition $j^*\theta_T\left(\overline{u}\right) = u\cdot$
We easily check that $\overline{u}$ is the restriction to $\overline{F^{\natural}}$ of the basic field $\widetilde{u}$ on $M^{\natural}$

associated with the vector u of associated with the vector u of $\mathbb{R}^q$relatively to $\omega_{_T}\cdot$
iii) Let $\ell\left(\overline{F^\natural},\mathcal{F}^\natural_{/\overline{F^\natural}}\right)$ be the Lie structural algebra of the vectors fields $\mathcal{F}^\natural_{/\overline{F^\natural}}-$transverse foliated· We have, the application $j^*\theta_T{:}\ell\left(\overline{F^\natural},\mathcal{F}^\natural_{/\overline{F^\natural}}\right)\to\mathbb{R}^q$ is a isomorphism of vector spaces. □

Proposition 3.5. *Let $\mathcal{G}_{_\mathcal{F}}$ be the Lie Structural Algebra of a Riemannian foliation $\mathcal{F}$ having dense leaves and admitting a flag of extension on a compact manifold M admitting a finite fundamental group $\pi_1(M)\cdot$*
Then $\mathcal{G}_{_\mathcal{F}}$ is an abelian Lie algebra whose dimension is the codimension of $\mathcal{F}\cdot$

Proof. Let $(\overline{u},\overline{v})\in\mathcal{G}^2_{_\mathcal{F}}\cdot$
We have already established that

$$d\left(j^*\theta_T\right)=0\cdot$$

But

$$\begin{aligned} d\left(j^*\theta_T\right)(\overline{u},\overline{v}) &= \left(\overline{u}\left(j^*\theta_T\right)(\overline{v})\right)-\overline{v}\left(j^*\theta_T\right)(\overline{u})-j^*\theta_T\left([\overline{u},\overline{v}]\right)\\ &= j^*\theta_T\left([\overline{u},\overline{v}]\right).\end{aligned}$$

So

$$j^*\theta_T\left([\overline{u},\overline{v}]\right)=0\cdot$$

which means that $[\overline{u},\overline{v}]$ is a vertical vector field of the covering $\overline{F^\natural}$ of M · It result of this that $[\overline{u},\overline{v}]=0\cdot$ Therefore the isomorphism of vector spaces $j^*\theta_T{:}\mathcal{G}_{_\mathcal{F}}\to\mathbb{R}^q$ is an isomorphism of Lie algebras·
Consequently, the structural Lie algebra $\mathcal{G}_{_\mathcal{F}}$ of a Riemannian foliation $\mathcal{F}$ having dense leaves and admitting a flag of extension on a compact manifold M admitting a finite fundamental group $\pi_1(M)$ is a $q-$ dimensional abelian Lie algebra· □

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