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Exact Analytic Solutions for Nonlinear Diffusion Equations via Generalized Residual Power Series Method

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Abstract

The aim of the present analysis is to generalize the recently devised technique, known as the residual power series method (RPSM), for analytic treatment of higher-order non-linear partial differential equations (NPDEs). This method based on constructing series solutions in a form of rapidly convergent series with easily computable components and without need of linearization, discretization, perturbation or unrealistic assumptions. Convergence and error analysis are discussed in details. To illustrate the efficiency, reliability and simplicity of proposed method, exact solutions for some diffusion processes of power, rational and exponential law diffusivities are obtained.

1 Introduction

Many problems and scientific phenomena in physics, chemistry, biology and engineering have their mathematical setting as nonlinear differential equations. The results of solving nonlinear equations, which are receiving increasing attention, can guide researchers to draw conclusions deeply. Recently, a variety of methods have been developed to overcome the difficulties

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of finding their exact approximate, analytical and numerical solutions, such as the Adomian decomposition method [1, 2], variational iteration method [3], homotopy-perturbation method [4], differential transform method [5], reduced differential transform method [6, 7] and $e^{\Phi(\xi)}$-expansion method [8]. See also [9,10] and references therein.

Recently, Abu Arqub and others [11] have proposed a new technique for solving linear/nonlinear differential equations namely residual power series method (RPSM). The RPSM has been extensively used by many researchers for the treatment of linear and nonlinear ordinary equations of integer and fractional orders, see [12, 13, 14, 15].

The basic motivation of present study is the extension of RPSM to tackle nonlinear higher-order nonlinear partial differential equations (NPDEs). The convergence analysis of the proposed scheme is discussed and an error bound is determined.

The present paper has been organized as follows. Section 2 is devoted to describe the generalized RPSM (shortly, GRPSM) and its convergence analysis. In section 3, the nonlinear diffusion equations with several nonlinearities are solved to illustrate the efficiency, applicability and simplicity of GRPSM in finding exact solutions. Conclusions are presented in Section 4.

2 Methodology and Convergence Analysis

Consider, as a generic example, a NPDE in the operator form

$$L[u] = L_t u + L_x u + Ru + Nu = H(x,t) \tag{1}$$

subject to the initial data

$$u(x,t_0) = u_0(x), u^{(0,1)}(x,t_0) = u_1(x), ..., u^{(0,n-1)}(x,t_0) = u_{n-1}(x) \tag{2}$$

where L_t denotes the nth-linear derivative with respect to time t, L_x a linear derivative in the space x,N is a nonlinear operator and R the reminder linear operator in the analytic function $u(x,t)$ on a domain $D \subseteq \mathbb{R}^2$ contains (x_0,t_0). The analytic function H represents the inhomogeneous term and $u^{(p,q)}(x_0,t_0)$ means $[\frac{\partial^{p+q}}{\partial x^p \partial t^q} u(x,t)]_{x=x_0,t=t_0}$.

The GRPSM assumes the solution $u(x,t)$, of Eq.(1), in a form of power series centered at (x_0,t_0) as

$$u(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j}(x-x_0)^j(t-t_0)^i = \sum_{i=0}^{\infty} \xi_j(x)(t-t_0)^i, \tag{3}$$

where $\xi_j(x) = \sum\limits_{j=0}^{\infty} a_{i,j}(x-x_0)^j$.

The initial approximation of $u(x,t)$, that satisfies the initial conditions Eq.(2), follows the $1st$ Taylor's polynomial, that is

$$u_0\left(x,t\right) = \sum_{i=0}^{n-1} \frac{1}{i!} u^{(0,i)}\left(x,t\right)\left(t-t_0\right)^i = \sum_{i=0}^{n-1} \frac{1}{i!} u_i\left(x\right)\left(t-t_0\right)^i = \sum_{i=0}^{n-1} \xi_i\left(x\right)\left(t-t_0\right)^i. \tag{4}$$

For $m > n$, the mth-order approximate solution is defined by

$$u_m\left(x,t\right) = u_0\left(x,t\right) + \sum_{i=n}^{m} \xi_i\left(x\right)\left(t-t_0\right)^i. \tag{5}$$

The unknown coefficients $\xi_i\left(x\right)$, $i = n, n+1, \ldots, m$, would be determined sequentially by solving the algebraic equation

$$\lim_{t\to t_0} \mathrm{Res}_m\left(x,t\right) = 0, \tag{6}$$

where $\mathrm{Res}_m\left(x,t\right)$ represents the analytic mth-residual function defined by

$$\mathrm{Res}_m\left(x,t\right) = \partial_t^{m-n}\left(L\left[u_m\left(x,t\right)\right] - H\left(x,t\right)\right). \tag{7}$$

Theorem 2.1. *The residual function* $\mathrm{Res}_m\left(x,t\right)$ *vanishes as* m *approaches the infinity.*

Proof. Since $H(x,t)$ is an analytic about $t = t_0$, it can be represented by a power series expansion of $(t-t_0)$ as well as in the case of $u_m\left(x,t\right)$ in Eq.(7). □

Lemma 2.2. *Suppose that* $u\left(x,t\right) = \sum\limits_{i=0}^{\infty} \xi_j\left(x\right)\left(t-t_0\right)^i$ *satisfies Eq.(1), then*

$$u^{(0,k)}\left(x,t_0\right) = i!\xi_i\left(x\right) \tag{8}$$

Proof.

$$u^{(0,k)}\left(x,t_0\right) = \partial_t^k u\left(x,t\right)\Big|_{t=t_0} = \partial_t^k \lim_{t\to t_0} u\left(x,t\right),$$

by the continuity of analytic function $u\left(x,t\right)$, we get

$$u^{(0,k)}\left(x,t_0\right) = \lim_{t\to t_0} \partial_t^k u\left(x,t\right) = \lim_{t\to t_0} \partial_t^k \left(\sum_{i=0}^{\infty} \xi_i\left(x\right) \partial_t^k (t-t_0)^i\right)$$

$$= \lim_{t\to t_0} \sum_{i=0}^{\infty} \frac{i!}{(i-k)!} \xi_i\left(x\right)\left(t-t_0\right)^{i-k} = i!\xi_i\left(x\right)$$

□

Theorem 2.3. *The approximate solution $u_k(x,t)$ defined in Eq.(5) and obtained by the GRPSM for the exact solution $u(x,t)$ of Eqs.(1) and (2) is the Taylor series expansion of $u(x,t)$ about $t = t_0$.*

Proof. For $k < n$, it is obvious from Eq.(4). For $k \geq n$, it suffices to prove that

$$u^{(0,k)}(x,t_0) = -\lim_{t\to t_0} \partial_t^{(k-n)} \left(L_x u_k + R u_k + N u_k - H(x,t)\right)$$

Applying Eq.(7) to the th-order approximate solution given in Eq.(6) gives

$$\begin{aligned}
0 &= \lim_{k\to\infty} \operatorname{Res}_k(x,t) = \lim_{t\to t_0} \partial_t^{k-n} \left(L\left[u_k(x,t)\right] - H(x,t)\right) \\
&= \lim_{t\to t_0} \partial_t^{k-n} \left(L\left[u_0(x,t) + \sum_{i=n}^{k} \xi_i(x)(t-t_0)^i\right] - H(x,t)\right) \\
&= \lim_{t\to t_0} \partial_t^{k-n} \left(L_t\left(u_0(x,t) + \sum_{i=n}^{k} \xi_i(x)(t-t_0)^i\right) + L_x u_k + R u_k + N u_k - H(x,t)\right) \\
&= \lim_{t\to t_0} \left[\partial_t^{k}\left(u_0(x,t) + \sum_{i=n}^{k} \xi_i(x)(t-t_0)^i\right) + \partial_t^{k-n}\left(L_x u_k + R u_k + N u_k - H(x,t)\right)\right] \\
&= \lim_{t\to t_0} \left[k!\ \xi_k(x) + \partial_t^{k-n}\left(L_x u_k + R u_k + N u_k - H(x,t)\right)\right] \\
&= k!\ \xi_k(x) + \lim_{t\to t_0} \partial_t^{k-n}\left(L_x u_k + R u_k + N u_k - H(x,t)\right).
\end{aligned}$$

As a result of Lemma 2.2,

$$u^{(0,k)}(x,t_0) = -\lim_{t\to t_0} \partial_t^{k-n} \left(L_x u_k + R u_k + N u_k - H(x,t)\right),$$

which completes the proof. □

Corollary 2.4. *Suppose that the truncated series $u_k(x,t)$ Eq.(6) is used as an approximation to the solution $u(x,t)$ of problem Eqs.(1)- (2) on a strip*

$$S = \left\{(x,t) : x \in \mathbb{R}^2, |t - t_0| < \rho\right\},$$

then numbers $\eta(t)$, satisfies $|\eta(t) - t_0| \leq \rho$, and $\mu_k > 0$ exist with

$$|u(x,t) - u_k(x,t)| \leq \frac{\mu_k}{(k+1)!}\rho^{k+1}. \tag{9}$$

Proof. Theorem 2.1 tells that

$$u(x,t) - u_k(x,t) = \sum_{i=k+1}^{\infty} \frac{1}{i!} u^{(0,i)}(x,t)(t-t_0)^i.$$

Following the proof of Taylor's Theorem [16], a number $\eta(t) \in (t_0 - \rho, t_0 + \rho)$ exists with

$$u(x,t) - u_k(x,t) = \frac{u^{(0,k+1)}(x,\eta(t))}{(k+1)!}(t-t_0)^{k+1}.$$

Since the $(k+1)st$-derivative of the analytic function $u(x,t)$ with respect to t is continuous, which implies the boundedness on R, a number μ_k also exists with $\left|u^{(0,k+1)}(x,t)\right| \leq \mu_k$ for all $t \in [t_0 - \rho, t_0 + \rho]$. Hence the result. □

Corollary 2.5. *Suppose that the solution $u(x,t)$ is a polynomial of t, then the RPSM results the exact solution.*

3 Applications to Nonlinear Diffusion Equations

The RPSM is implemented to obtain exact analytic solutions of the nonlinear diffusion model that is governed by the initial-value problem

$$\partial_t u = \partial_x (A(u)\partial_x u) + B(u), u(x,0) = f(x), x \in \mathbb{R}, t \geq 0, \tag{10}$$

with power, rational and exponential nonlinearities. This equation is used to model a wide range of phenomena in physics, engineering, chemistry and biology [17, 18]. $A(u)$ and $B(u)$ are assumed to be sufficiently smooth functions.

Following the presented procedure of GRPSM, the mth-order approximate solution Eq.(5) would be found subject to

$$\text{Res}_m(x,t) = \partial_t^{m-1}(\partial_t u_m - \partial_x (A(u_m)\partial_x u_m) - B(u_m)).$$

Sequentially, to determine the unknown coefficients $\xi_i(x)$ of truncated series solution, solve the algebraic equations $\lim_{t \to 0} \text{Res}_m(x,t) = 0, i = 1, 2, \ldots, m,$

Next, some problems are solved to illustrate the strength of the method and establish exact solutions.

Example 3.1. *Consider Eq.(10) as a model of fast diffusion in high-polymeric systems with $A(u) = u^{-2}$, $B(u) = 0$ and $f(x) = (x^2+1)^{\frac{-1}{2}}$ [19]. For $i = 1$, we have $u_1(x,t) = f(x) + \xi_1(x)t$ with*

$$\mathrm{Res}_1(x,t) = \partial_t\left(f(x) + \xi_1(x)t\right) - \partial_x\left(\left(f(x) + \xi_1(x)t\right)^{-2}\partial_x\left(f(x) + \xi_1(x)t\right)\right)$$

Applying Eq.(6), we get $\xi_1(x) + (1+x^2)^{\frac{-3}{2}} = 0$. Therefore $\xi_1(x) = -(1+x^2)^{\frac{-3}{2}}$. Repeating this procedure for $i = 2, 3, \ldots$, we obtain the following solution terms

$$\begin{aligned}
\xi_2(x) &= \frac{1}{2!}\left(1-2x^2\right)\left(x^2+1\right)^{\frac{-5}{2}},\\
\xi_3(x) &= \frac{1}{3!}\left(1-10x^2+4x^4\right)\left(x^2+1\right)^{\frac{-7}{2}},\\
\xi_4(x) &= \frac{1}{4!}\left(1-36x^2+60x^4-8x^6\right)\left(x^2+1\right)^{\frac{-9}{2}}\\
&\vdots
\end{aligned}$$

The solution takes the form

$$u_m(x,t) = \frac{1}{\sqrt{x^2+1}} - \frac{1}{\sqrt{(x^2+1)^3}}t + \frac{1-2x^2}{\sqrt{(x^2+1)^5}}\frac{t^2}{2!} - \frac{-1+10x^2-4x^4}{\sqrt{(x^2+1)^7}}\frac{t^3}{3!} + \ldots.$$

As $m \to \infty$, the series solution leads to the exact solution $(x^2+e^{2t})^{\frac{-1}{2}}$ obtained by Taylor's expansion.

Example 3.2. *The initial-value problem*

$$\partial_t u = \partial_x\left(u^2\partial_x u\right), u(x,0) = \frac{x+a}{2c}, x \in \mathbb{R}, 0 \le t < c^2, \tag{11}$$

where a and c are arbitrary constants, governs the slow diffusion processes such as evaporation and melting [20]. The th- residual function for Eq.(11) is expressed for m = 1,2, ... as

$$\mathrm{Res}_m(x,t) =$$

$$\partial_t^m\left(f(x) + \sum_{i=1}^{m}\xi_i(x)t^i\right) - \partial_t^{m-1}\partial_x\left(\left(f(x) + \sum_{i=1}^{m}\xi_i(x)t^i\right)^2\partial_x\left(f(x) + \sum_{i=1}^{m}\xi_i(x)t^i\right)\right),$$

Proceeding as before, with $f(x) = \frac{x+a}{2c}$*, we recursively obtain the following unknown coefficients*

$$\xi_1(x) = \frac{x+a}{4c^3},$$
$$\xi_2(x) = 3\frac{x+a}{16c^5},$$
$$*\xi_3(x) = 5\frac{x+a}{32c^7},$$
$$\vdots$$

Continuing this process, we get the exact solution $u(x,t) = \frac{x+a}{2\sqrt{c^2-t}}$ *obtained upon using the Taylor's expansion.*

Example 3.3. *Consider the other case of rational nonlinearity of diffusion processes given by*

$$\partial_t u = \partial_x \left(\frac{1}{u^2-1} \partial_x u \right), u(x,0) = -\coth x, x > 0, t \geq 0. \quad (12)$$

The undetermined coefficients of series solution Eq.(3), within generalized residual power series procedure, are computed to be as following

$$\xi_0(x) = -\coth x,$$
$$\xi_2(x) = \xi_3(x) = \xi_4(x) = ... = 0.$$

The obtained solution concise the exact analytic solution [21]. In the same way, the nonlinear diffusion problem

$$\partial_t u = \partial_x \left(\frac{1}{u^2+1} \partial_x u \right), u(x,0) = \tan x, |x| < \frac{\pi}{2}, t \geq 0, \quad (13)$$

is considered and the exact analytic solution is obtained.

Example 3.4. *Finally, we study the nonlinear diffusion equation with exponential nonlinearity*

$$\partial_t u = \partial_x \left(a\, u\, e^{\alpha u} \partial_x u \right) + b\, e^{-\alpha u}, u(x,0) = \frac{1}{\alpha} \ln (C_1 x + C_2), C_1, C_2 \in \mathbb{R}. \quad (14)$$

The theoretical solution given by [22]

$$u(x,t) = \frac{1}{\alpha} \ln\left(C_1 x + \left(\frac{a\, C_1^2}{\alpha} + b\,\alpha \right) t + C_2 \right) \tag{15}$$

Using the initial condition and proceeding as before, the approximate analytic solution will be

$$u(x,t) = \frac{1}{\alpha} \ln(C_1 x + C_2) + \frac{1}{\alpha^2} \frac{b\,\alpha^2 + a\,C_1^2}{(C_1 x + C_2)} t - \frac{1}{2\alpha^3} \frac{b\,\alpha^4 + 2\,a\,b\,\alpha^2\,C_1^2 + a^2\,C_1^4}{(C_1 x + C_2)^2} t^2$$
$$+ \frac{1}{3\alpha^4} \frac{b^3\,\alpha^6 + 3\,a\,b^2\,\alpha^4\,C_1^2 + 3\,a^2\,b\,\alpha^2\,C_1^4 + a^2\,C_1^4}{(C_1 x + C_2)^3} t^3 + \ldots$$

Thus the exact solution obtained upon using the Taylors expansion.

4 Discussion and Conclusion

We have presented an analytic framework for handling nonlinear partial differential equations. The generalized residual power series method has been successfully implemented to obtain exact solutions for nonlinear diffusion models. The method is accurate and provides efficient results. Our scheme is developed for analytic treatment of nonlinear partial differential equations and systems by converting the solution of differential equations to solution of algebraic equations that overcomes the complexity of calculating more and more series solution coefficients, as in the other existing methods that based on computing integrals and the cost of the standard power series method.

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