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Q-Cubic Soft Multiset Theory

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Abstract

Molodtsov introduced the concept of the soft set as a general mathematical tool for dealing with uncertainty. We intend to further this foray into fuzzy sets and soft sets by introducing the concepts of Q-cubic set and a Q-cubic soft multiset and their basic operations, namely complement, union, intersection, OR and AND operations along with illustrative examples. Basic theorems on the operations of Q-cubic soft multi set are also given.

1 Introduction

Many fields deal with uncertain data which may not be successfully modeled by classical mathematics, since the concept of uncertainty is presumed to be complicated and not clearly defined. However, the uncertainty aspect can actually be modeled by a number of different approaches including the probability theory, fuzzy set theory [20,34,37] extension of fuzzy set by using cubic set [22] , rough set [31], and neutrosophic set [13,18,25]. However, these theories have their own difficulties as pointed out by Molodtsov [28] who then proposed a completely new approach for modeling vagueness and uncertainty, which has potential applications in many different fields. Since then, there

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has been a rapid growth of interest in soft sets and their various applications such as algebraic structures [26,27], topology [17, 29], medical diagnosis [36], and decision making under uncertainty [8,16,23,30,35]. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [14,15,16,24], rough sets [23,32], multi Q- fuzzy sets [1,2,3,4,5,6,7], interval valued fuzzy set[19, 38], vague set [10,11,12,21,33] and neutrosophic set [24,27].

In this work, we establish the definitions of a Q-cubic set and a Q-cubic soft multiset. The notions of (internal,external) Q-cubic soft multisets, P-(R-) order, P-(R-)union, P-(R-)intersection, P-OR, R-OR, P-AND and R-AND are introduced. Based on these notions, several properties and theorems are investigated.

2 Preliminaries

In this section we present the basic definitions of soft set and soft multiset required as preliminaries.

Definition 1. *[28]. A pair (F, A) is called a soft set over U, where F is a mapping*

$$F : A \longrightarrow P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A).

Definition 2. *[9]. Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i\in I} U_i = \phi$ and let $\{E_{U_i} : i \in I\}$ be a collection of sets of parameters. Let $U = \prod_{i\in I} P(U_i)$ where $P(U_i)$ denotes the power set of U_i, $E = \prod_{i\in I} E_{U_i}$ and $A \subseteq E$. A pair (F, A) is called a soft multiset over U, where F is a mapping given by*

$$F : A \longrightarrow U$$

In other words, a soft multiset over U is a parameterized family of subsets of U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft multiset (F, A). Based on the above definition, any change in the order of universes will produce a different soft multiset.

3 Q-cubic set

In this section we introduce the concept of a Q-cubic set and define some operations on a Q-cubic set.

Definition 3. *Let X be a universal set and Q be a non-empty set. By a Q-cubic set we mean a structure*

$$\mathfrak{A} = \{< (x,q), A_Q(x,q), \lambda_Q(x,q) >: x \in X, q \in Q\}.$$

where $A_Q : X \times Q \longrightarrow Int([0,1])$ is an interval valued Q-fuzzy set and $\lambda_Q : X \times Q \longrightarrow [0,1]$ is a Q-fuzzy set in X and Q.

Definition 4. *Let X be a universal set and Q be a non-empty set. A Q-cubic set $\mathfrak{A}$ is said to be an internal Q-cubic set (briefly, IQCS) if $A_Q^-(x,q) \leq \lambda_Q(x,q) \leq A_Q^+(x,q)$ for all $x \in X$ and $q \in Q$.*

Definition 5. *Let X be a universal set and Q be a non-empty set. A Q-cubic set $\mathfrak{A}$ is said to be an external Q-cubic set (briefly, EQCS) if $\lambda_Q(x,q) \notin (A_Q^-(x,q), A_Q^+(x,q))$ for all $x \in X$ and $q \in Q$.*

Definition 6. *Let $\mathfrak{A} = \{< (x,q), A_Q(x,q), \lambda_Q(x,q) >: x \in X, q \in Q\}$ and $\mathfrak{B} = \{< (x,q), B_Q(x,q), \mu_Q(x,q) >: x \in X, q \in Q\}$ be Q-cubic sets in X and Q. Then we define*

1. *(Equality) $A = B \Longleftrightarrow A_Q(x,q) = B_Q(x,q)$ and $\lambda_Q(x,q) = \mu_Q(x,q)$.*
2. *(P-order) $A \subseteq_P B \Longleftrightarrow A_Q(x,q) \subseteq_P B_Q(x,q)$ and $\lambda_Q(x,q) \leq_P \mu_Q(x,q)$.*
3. *(R-order)$A \subseteq_R B \Longleftrightarrow A_Q(x,q) \subseteq_R B_Q(x,q)$ and $\lambda_Q(x,q) \geq_R \mu_Q(x,q)$.*

Definition 7. *The complement of $\mathfrak{A} = \{< (x,q), A_Q(x,q), \lambda_Q(x,q) >: x \in X, q \in Q\}$ is defined to be the Q-cubic set $(\mathfrak{A})^c = \{< (x,q), A_Q^c(x,q), \lambda_Q^c(x,q) >: x \in X, q \in Q\}$, where $A_Q^c(x,q) = [1 - A_Q^+(x,q), 1 - A_Q^-(x,q)]$ and $\lambda_Q^c(x,q) = 1 - \lambda_Q(x,q)$.*

Definition 8. *Let $\mathfrak{A} = \{< (x,q), A_Q(x,q), \lambda_Q(x,q) >: x \in X, q \in Q\}$ and $\mathfrak{B} = \{< (x,q), B_Q(x,q), \mu_Q(x,q) >: x \in X, q \in Q\}$ be Q-cubic sets in X and Q. Then we define*

1. *$\mathfrak{A} \cup_P \mathfrak{B} = \{< (x,q), sup(A_Q(x,q), B_Q(x,q)), sup(\lambda_Q(x,q), \mu_Q(x,q)) >: x \in X, q \in Q\}$ (P-union).*

2. $\mathfrak{A} \cap_P \mathfrak{B} = \{< (x,q), inf(A_Q(x,q), B_Q(x,q)), inf(\lambda_Q(x,q), \mu_Q(x,q)) >: x \in X, q \in Q\}$ *(P-intersection).*

3. $\mathfrak{A} \cup_R \mathfrak{B} = \{< (x,q), sup(A_Q(x,q), B_Q(x,q)), inf(\lambda_Q(x,q), \mu_Q(x,q)) >: x \in X, q \in Q\}$ *(R-union).*

4. $\mathfrak{A} \cap_R \mathfrak{B} = \{< (x,q), inf(A_Q(x,q), B_Q(x,q)), sup(\lambda_Q(x,q), \mu_Q(x,q)) >: x \in X, q \in Q\}$ *(R-intersection).*

5. $\mathfrak{A} \wedge_P \mathfrak{B} = \{< min(A_Q(x,q), B_Q(x,q)), min(\lambda_Q(x,q), \mu_Q(x,q)) >: x \in X, q \in Q\}$ *(P-AND).*

6. $\mathfrak{A} \vee_P \mathfrak{B} = \{< max(A_Q(x,q), B_Q(x,q)), max(\lambda_Q(x,q), \mu_Q(x,q)) >: x \in X, q \in Q\}$ *(P-OR).*

7. $\mathfrak{A} \wedge_R \mathfrak{B} = \{< min(A_Q(x,q), B_Q(x,q)), max(\lambda_Q(x,q), \mu_Q(x,q)) >: x \in X, q \in Q\}$ *(R-AND).*

8. $\mathfrak{A} \vee_R \mathfrak{B} = \{< max(A_Q(x,q), B_Q(x,q)), min(\lambda_Q(x,q), \mu_Q(x,q)) >: x \in X, q \in Q\}$ *(R-OR).*

4 Q-cubic soft multiset

In this section we apply Q-cubic set to soft multiset and define a new extension of Q-fuzzy soft by using Q-cubic set. We then introduce the notions of (internal, external) Q-cubic soft multisets and their related properties.

Definition 9. *Let $\{U_i : i = 1, ..., n\}$ be a collection of universes such that $\bigcap_{i=1,...,n} U_i = \emptyset$ and Q be a non-empty set . Let $\{E_{U_i} : i = 1, ..., n\}$ be a collection of sets of parameters. Let $U = \prod_{i \in n} QCS(U_i)$ where $QCS(U_i)$ denotes the set of all Q-cubic subsets of U_i, $E = \prod_{i \in n} E_{U_i}$ and $A \subseteq E$. A pair $(\tilde{\mathfrak{F}}, A)$ is called a Q-cubic soft multiset over U and Q, where $\tilde{\mathfrak{F}}$ is a mapping given by $\tilde{\mathfrak{F}} : A \to U$ such that*
$(\tilde{\mathfrak{F}}, A) = \{e_k, \tilde{\mathfrak{F}}(e_k) = \{< (u_{i,j}, q), A_Q(u_{i,j}, q), \lambda_Q(u_{i,j}, q) >: u_{i,j} \in U_i,\ q \in Q\,, e_k \in A\}\}$ for all $k = 1, ..., l, i = 1, ..., n$ and $j = 1, ..., m$
where A_Q is an interval valued Q-fuzzy set and λ_Q is a Q-fuzzy set over U_i and a non-empty set Q.

In other words, a Q-cubic soft multiset over U and Q is a parameterized family of Q-cubic subsets of U and Q. For $e_k \in A$, $\tilde{\mathfrak{F}}(e_k)$ may be considered as the set of e_k-approximate elements of the Q-cubic soft multiset$(\tilde{\mathfrak{F}}, A)$. Based on the above definition, any change in the order of universes will produce a different Q-cubic soft multiset.

Example 1. *Suppose that there are three universes* U_1, U_2, *and* U_3. *Assume that an individual has a budget to buy a house, a car and rent a premise to market computers. Let us consider a Q-cubic soft multiset* $(\tilde{\mathfrak{F}}, A)$ *which describes houses, cars, and premises the person is considering for accommodation purchase, transportation purchase, and a premise to market computers, respectively. Let* $U_1 = \{h_1, h_2, h_3\}$, $U_2 = \{c_1, c_2, c_3\}$, $U_3 = \{v_1, v_2, v_3\}$ *and* $Q = \{p = Kuala\,Lumpur\}$. *Let* $\{E_{U_1}, E_{U_2}, E_{U_3}\}$ *be a collection of sets of decision parameters related to the above universes, where*
$E_{U_1} = \{e_{U_1,1} = bungalow, e_{U_1,2} = condominium, e_{U_1,3} = apartment\}$,
$E_{U_2} = \{e_{U_2,1} = sedan, e_{U_2,2} = coupe, e_{U_2,3} = sport\}$,
$E_{U_3} = \{e_{U_3,1} = posh, e_{U_3,2} = modern, e_{U_3,3} = colonial\}$,
Let $U = \prod_{i=1}^{3} QCS(U_i)$, $E = \prod_{i=1}^{3} E_{U_i}$ *and* $A \subseteq E$, *such that*
$A = \{a_1 = \{e_{U_1,1}, e_{U_2,1}, e_{U_3,1}\}, a_2 = \{e_{U_1,2}, e_{U_2,2}, e_{U_3,1}\}, a_3 = \{e_{U_1,3}, e_{U_2,1}, e_{U_3,3}\}$.
Then we can view the Q-cubic soft multiset $(\tilde{\mathfrak{F}}, A)$ *as consisting of the following collection of approximations:*
$(\tilde{\mathfrak{F}}, A) = \{(a_1, \{< (h_1, q), [0.2, 0.4], 0.3 >, < (h_2, q), [0.5, 0.7], 0.6 >\}, \{< (c_1, q), [0.5, 0.6], 0.2 >, < (c_3, q), [0.3, 0.5], 0.4 >\}, \{< (v_2, q), [0.1, 0.3], 0.2, < (v_3, q), [0.3, 0.6], 0.5 > \}),$
$(a_2, \{< (h_1, q), [0.2, 0.5], 0.1 >, < (h_2, q), [0.4, 0.5], 0.3 >\}, \{< (c_1, q), [0.3, 0.4], 0.1 >, < (c_3, q), [0.2, 0.5], 0.3 >\}, \{< (v_2, q), [0.5, 0.6], 0.8 >, < (v_3, q), [0.2, 0.5], 0.4 > \}),$
$(a_3, \{< (h_1, q), [0.4, 0.6], 0.4 >, < (h_2, q), [0.6, 0.7], 0.9 >\}, \{< (c_1, q), [0.3, 0.7], 0.2 >, < (c_3, q), [0.2, 0.6], 0.2 >\}, \{< (v_2, q), [0.3, 0.4], 0.8 >, < (v_3, q), [0.5, 0.7], 0.1 > \})\}$.

4.1 Types of Q-cubic soft multiset

1. Internal Q-cubic soft multiset (IQCSMS).

Definition 10. *A Q-cubic soft multiset* $(\tilde{\mathfrak{F}}, A)$ *is said to be an internal Q-cubic soft multiset (IQCSMS), if for all* $e_k \in A$,

$$\tilde{\mathfrak{F}}(e_k) = \{< (u_{i,j}, q), A_Q(u_{i,j}, q), \lambda_Q(u_{i,j}, q) >\}$$

such that $A_Q^-(u_{i,j}, q) \leq \lambda_Q(u_{i,j}, q) \leq A_Q^+(u_{i,j}, q)$ *for all* $u_{i,j} \in U_i$ *and* $q \in Q$, *where* $i = 1, ..., n$ *and* $j = 1, ..., m$.

Example 2. *From example 1, suppose* $(\tilde{\mathfrak{F}}, A) = \{(a_1, \{< (h_1, q), [0.2, 0.4], 0.3 >, < (h_2, q), [0.5, 0.7], 0.6 >\}, \{< (c_1, q), [0.5, 0.8], 0.7 >, < (c_3, q), [0.3, 0.5], 0.4 > \}, \{< (v_2, q), [0.1, 0.3], 0.2 >, < (v_3, q), [0.3, 0.6], 0.5 >\}),$

$(a_2, \{< (h_1, q), [0.2, 0.5], 0.3 >, < (h_2, q), [0.4, 0.7], 0.5 >\}, \{< (c_1, q), [0.3, 0.9], 0.5 >, < (c_3, q), [0.2, 0.5], 0.3 >\}, \{< (v_2, q), [0.5, 0.8], 0.6 >, < (v_3, q), [0.2, 0.5], 0.4 >\})\}$. *Obviously,* $A^-_Q(u_i, q) \leq \lambda_Q(u_i, q) \leq A^+_Q(u_i, q)$ *for all* $u_i \in U_i$, $q \in Q$ *and* $a_1, a_2 \in A$. *Thus* $(\tilde{\mathfrak{F}}, A)$ *is an internal Q-cubic soft multiset.*

2. External Q-cubic soft multiset.

Definition 11. *A Q-cubic soft multiset* $(\tilde{\mathfrak{F}}, A)$ *is said to be an external Q-cubic soft multiset (EQCSMS), if for all* $e_k \in A$,

$$\tilde{\mathfrak{F}}(e_k) = \{< (u_{i,j}, q), A_Q(u_{i,j}, q), \lambda_Q(u_{i,j}, q) >\}$$

such that $\lambda_Q(u_{i,j}, q) \notin (A^-_Q(u_{i,j}, q), A^+_Q(u_{i,j}, q))$ *for all* $u_{i,j} \in U_i$ *and* $q \in Q$, *where* $i = 1, ..., n$ *and* $j = 1, ..., m$.

Example 3. *From example 1, suppose* $(\tilde{\mathfrak{F}}, A) = \{(a_1, \{< (h_1, q), [0.2, 0.4], 0.5 >, < (h_2, q), [0.5, 0.7], 0.3 >\}, \{< (c_1, q), [0.5, 0.8], 0.2 >, < (c_3, q), [0.3, 0.5], 0.1 >\}, \{< (v_2, q), [0.1, 0.3], 0.4, < (v_3, q), [0.3, 0.6], 0.8 >\}),$
$(a_2, \{< (h_1, q), [0.2, 0.5], 0.1 >, < (h_2, q), [0.4, 0.7], 0.9 >\}, \{< (c_1, q), [0.3, 0.9], 0.2 >, < (c_3, q), [0.2, 0.5], 0.7 >\}, \{< (v_2, q), [0.5, 0.8], 0.9 >, < (v_3, q), [0.2, 0.5], 0.3 >\})\}$. *Obviously,* $\lambda_Q(u_{i,j}, q) \notin (A^-_Q(u_{i,j}, q), A^+_Q(u_{i,j}, q))$ *for all* $u_{i,j} \in U_i$, $q \in Q$ *and* $a_1, a_2 \in A$. *Thus* $(\tilde{\mathfrak{F}}, A)$ *is an external Q-cubic soft multiset.*

Theorem 1. *Let* $(\tilde{\mathfrak{F}}, A)$ *be a Q-cubic soft multiset which is not an EQCSMS. Then, there exists at least one* $a_k \in A$ *for which there exists some* $(u_{i,j}, q) \in U_i \times Q$ *such that* $\lambda_Q(u_{i,j}, q) \in (A^-_Q(u_{i,j}, q), A^+_Q(u_{i,j}, q))$.

Proof. By definition of an external Q-cubic soft multiset (EQCSMS) we know that $\lambda_Q(u_{i,j}, q) \notin (A^-_Q(u_{i,j}, q), A^+_Q(u_{i,j}, q))$ for all $u_{i,j} \in U_i$ and $q \in Q$. But given that $(\tilde{\mathfrak{F}}, A)$ is not an EQCSMS, there is at least one $a_k \in A$ whereby some $(u_{i,j}, q) \in U_i \times Q$ such that $\lambda_Q(u_{i,j}, q) \in (A^-_Q(u_{i,j}, q), A^+_Q(u_{i,j}, q))$.

Theorem 2. *Let* $(\tilde{\mathfrak{F}}, A)$ *be a Q-cubic soft multiset. If* $(\tilde{\mathfrak{F}}, A)$ *is both an IQCSMS and EQCSMS, then for all* $u_{i,j} \in U_i$ *,*$q \in Q$ *and* $a_k \in A$, $\lambda_Q(u_{i,j}, q) \in (A^-_Q(u_{i,j}, q) \cup A^+_Q(u_{i,j}, q))$.

Proof. Let $(\tilde{\mathfrak{F}}, A)$ be a Q-cubic soft multiset which is an IQCSMS and EQCSMS. Then by definition of IQCSMS we have
$\lambda_Q(u_{i,j}, q) \in (A^-_Q(u_{i,j}, q), A^+_Q(u_{i,j}, q))$ for all $u_{i,j} \in U_i$ $q \in Q$ and $a_k \in A$. By definition of EQCSMS corresponding to each $a_k \in A$, $\lambda_Q(u_{i,j}, q) \notin (A^-_Q(u_{i,j}, q), A^+_Q(u_{i,j}, q))$ for all $u_{i,j} \in U_i$, $q \in Q$. Since $(\tilde{\mathfrak{F}}, A)$ is both an IQCSMS and EQCSMS, hence the only possibility is that $\lambda_Q(u_{i,j}, q) = A^-_Q(u_{i,j}, q)$

or $\lambda_Q(u_{i,j},q) = A^{+}_{Q}(u_{i,j},q)$ for all $u_{i,j} \in U_i$ and $q \in Q$ corresponding to each $a_k \in A$. Hence $\lambda_Q(u_{i,j},q) \in (A^{-}_{Q}(u_{i,j},q) \cup A^{+}_{Q}(u_{i,j},q))$ for all $u_i \in U_i$ and $q \in Q$ corresponding to each $a_k \in A$.

Definition 12. *Let* $(\tilde{\mathfrak{F}}, A) = \{a_k, <(u_{i,j},q), A_Q(u_{i,j},q), \lambda_Q(u_{i,j},q)>: u_{i,j} \in U_i\,, q \in Q,\, a_k \in A\}$ *and* $(\tilde{\mathfrak{G}}, B) = \{b_k, <(u_{i,j},q), B_Q(u_{i,j},q), \nu_Q(u_{i,j},q)>: u_{i,j} \in U_i\,, q \in Q,\, b_k \in B\}$ *be two Q-cubic soft multisets such that* A *and* $B \in E$. *Then we have the following:*

1. $(\tilde{\mathfrak{F}}, A) = (\tilde{\mathfrak{G}}, B)$ *if and only if*
   (a) $A = B$.
   (b) $\tilde{\mathfrak{F}}(a_k) = \tilde{\mathfrak{G}}(b_k)$ *for all* $a_k \in A$, *if and only if* $A_Q(u_{i,j},q) = B_Q(u_{i,j},q)$ *and* $\lambda_Q(u_{i,j},q) = \nu_Q(u_{i,j},q)$ *for all* $a_k \in A$.
2. *If* $(\tilde{\mathfrak{F}}, A)$ *and* $(\tilde{\mathfrak{G}}, B)$ *are two Q-cubic soft multisets, then we denote and define P-order as* $(\tilde{\mathfrak{F}}, A) \subseteq_P (\tilde{\mathfrak{G}}, B)$ *if and only if the following conditions are satisfied.*
   (a) $A \subseteq B$.
   (b) $\tilde{\mathfrak{F}}(a_k) \leq_P \tilde{\mathfrak{G}}(b_k)$ *for all* $a_k \in A$, *if and only if* $A_Q(u_{i,j},q) \preceq B_Q(u_{i,j},q)$ *and* $\lambda_Q(u_{i,j},q) \leq \nu_Q(u_{i,j},q)$ *for all* $a_k \in A$.
3. *If* $(\tilde{\mathfrak{F}}, A)$ *and* $(\tilde{\mathfrak{G}}, B)$ *are two Q-cubic soft multisets, then we denote and define R-order as* $(\tilde{\mathfrak{F}}, A) \subseteq_R (\tilde{\mathfrak{G}}, B)$ *if and only if the following conditions are satisfied.*
   (a) $A \subseteq B$.
   (b) $\tilde{\mathfrak{F}}(a_k) \leq_R \tilde{\mathfrak{G}}(b_k)$ *for all* $a_k \in A$ *if and only if* $A_Q(u_{i,j},q) \preceq B_Q(u_{i,j},q)$, *and* $\lambda_Q(u_{i,j},q) \geq \nu_Q(u_{i,j},q)$ *for all* $a_k \in A$.

Example 4. *Let* $(\tilde{\mathfrak{F}}, A)$ *and* $(\tilde{\mathfrak{G}}, B)$ *be two Q-cubic soft multisets defined as follows.*
$(\tilde{\mathfrak{F}}, A) = \{(e_1, \{<(u_1,q),[0.3,0.5],0.8>, <(h_1,q),[0.2,0.5],0.4>, <(h_2,q),[0.3,0.5],$
$0.2>, <(v_1,q),[0.2,0.5],0.8>, <(v_3,q),[0.2,0.6],0.5>\}),$
$(e_2, \{<(u_1,q),[0.3,0.5],0.8>, <(h_1,q),[0.3,0.5],0.8>, <(h_2,q),[0.4,0.6],0.6>$
$, <(v_1,q),[0.1,0.4],0.7>, <(v_3,q),[0.3,0.8],0.4>\})\}$
and $(\tilde{\mathfrak{G}}, B) = \{(e_1, \{<(u_1,q),[0.4,0.6],0.9>, <(h_1,q),[0.3,0.7],0.5>, <$
$(h_2,q),[0.5,0.8],$
$0.3>, <(v_1,q),[0.3,0.6],0.9>, <(v_3,q),[0.4,0.7],0.6>\}),$
$(e_2, \{<(u_1,q),[0.4,0.6],0.9>, <(h_1,q),[0.5,0.7],0.9>, <(h_2,q),[0.7,0.8],0.9>$

$, < (v_1, q), [0.4, 0.6], 0.9 >, < (v_3, q), [0.5, 0.9], 0.5 >\})\}$. *Thus clearly we have* $(\tilde{\mathfrak{F}}, A) \subseteq_P (\tilde{\mathfrak{G}}, B)$.

Example 5. *Let* $(\tilde{\mathfrak{F}}, A)$ *and* $(\tilde{\mathfrak{G}}, B)$ *be two Q-cubic soft multisets defined as follows.*
$(\tilde{\mathfrak{F}}, A) = \{(e_1, \{< (u_1, q), [0.3, 0.5], 0.8 >, < (h_1, q), [0.2, 0.5], 0.4 >, < (h_2, q), [0.3, 0.5], 0.2 >$
$, < (v_1, q), [0.2, 0.5], 0.8 >, < (v_3, q), [0.2, 0.6], 0.5 >\}),$
$(e_2, \{< (u_1, q), [0.3, 0.5], 0.8 >, < (h_1, q), [0.3, 0.5], 0.8 >, < (h_2, q), [0.4, 0.6], 0.6 >$
$, < (v_1, q), [0.1, 0.4], 0.7 >, < (v_3, q), [0.3, 0.8], 0.4 >\})\}$
and $(\tilde{\mathfrak{G}}, B) = \{(e_1, \{< (u_1, q), [0.4, 0.6], 0.5 >, < (h_1, q), [0.3, 0.7], 0.3 >, <$
$(h_2, q), [0.5, 0.8],$
$0.1 >, < (v_1, q), [0.3, 0.6], 0.4 >, < (v_3, q), [0.4, 0.7], 0.4 >\}),$
$(e_2, \{< (u_1, q), [0.4, 0.6], 0.6 >, < (h_1, q), [0.5, 0.7], 0.5 >, < (h_2, q), [0.7, 0.8], 0.3 >$
$, < (v_1, q), [0.4, 0.6], 0.5 >, < (v_3, q), [0.5, 0.9], 0.3 >\})\}$. *Thus clearly we have* $(\tilde{\mathfrak{F}}, A) \subseteq_R (\tilde{\mathfrak{G}}, B)$.

Definition 13. *The complement of a Q-cubic soft multiset* $(\tilde{\mathfrak{F}}, A) = \{e_k, < (u_{i,j}, q), A_Q(u_{i,j}, q), \lambda_Q(u_{i,j}, q) >: u_{i,j} \in U_i,\ q \in Q\}$ *is denoted by* $(\tilde{\mathfrak{F}}, A)^c$ *and defined as* $(\tilde{\mathfrak{F}}, A)^c = (\tilde{\mathfrak{F}}^c, \neg A)$ *where*

$$(\tilde{\mathfrak{F}}, A)^c = \{e_k, < (u_{i,j}, q), A^c_Q(u_{i,j}, q), \lambda^c_Q(u_{i,j}, q) >: u_{i,j} \in U_i,\ q \in Q\}$$

for all $e_k \in A$.

Example 6. *From example 5 the complement of a Q-cubic soft multiset* $(\tilde{\mathfrak{F}}, A)$ *is defined as follows.*
$(\tilde{\mathfrak{F}}, A)^c = \{(e_1, \{< (u_1, q), [0.5, 0.7], 0.2 >, < (h_1, q), [0.5, 0.8], 0.6 >, < (h_2, q), [0.5, 0.7],$
$0.8 >, < (v_1, q), [0.5, 0.8], 0.2 >, < (v_3, q), [0.4, 0.8], 0.5 >\}),$
$(e_2, \{< (u_1, q), [0.5, 0.7], 0.2 >, < (h_1, q), [0.5, 0.7], 0.2 >, < (h_2, q), [0.6, 0.4], 0.4 >$
$, < (v_1, q), [0.6, 0.9], 0.3 >, < (v_3, q), [0.2, 0.7], 0.6 >\})\}$.

Proposition 1. *Let* $(\tilde{\mathfrak{F}}, A)$ *and* $(\tilde{\mathfrak{G}}, B)$ *be two Q-cubic soft multisets.*

1. *If* $(\tilde{\mathfrak{F}}, A) \subseteq_P (\tilde{\mathfrak{G}}, B)$, *then* $(\tilde{\mathfrak{F}}, A)^c \subseteq_P (\tilde{\mathfrak{G}}, B)^c$ *if* $A = B$.
2. *If* $(\tilde{\mathfrak{F}}, A) \subseteq_R (\tilde{\mathfrak{G}}, B)$, *then* $(\tilde{\mathfrak{F}}, A)^c \subseteq_R (\tilde{\mathfrak{G}}, B)^c$ *if* $A = B$.

Proof. The proof is straight forward by using properties of a Q-cubic set.

Theorem 3. *Let* $(\tilde{\mathfrak{F}}, A)$ *be a Q-cubic soft multiset.*

1. *If $(\tilde{\mathfrak{F}}, A)$ is an internal Q-cubic soft multiset, then $(\tilde{\mathfrak{F}}, A)^c$ is also an internal Q-cubic soft multiset (IQCSMS).*

2. *If $(\tilde{\mathfrak{F}}, A)$ is an external Q-cubic soft multiset, then $(\tilde{\mathfrak{F}}, A)^c$ is also an external Q-cubic soft multiset (EQCSMS).*

Proof.

1. Since $(\tilde{\mathfrak{F}}, A)$ is an internal Q-cubic soft multiset, then we have $A_Q^-(u_{i,j}, q) \leq \lambda_Q(u_{i,j}, q) \leq A_Q^+(u_{i,j}, q)$ for all $e_k \in A$. This implies $1 - A_Q^+(u_{i,j}, q) \leq 1 - \lambda_Q(u_{i,j}, q) \leq 1 - A_Q^-(u_{i,j}, q)$ for all $e_k \in A$. Hence $(\tilde{\mathfrak{F}}, A)^c$ is an IQCSMS.

2. By definition of external Q-cubic soft multiset, we have $\lambda_Q(u_{i,j}, q) \notin (A_Q^-(u_{i,j}, q), A_Q^+(u_{i,j}, q))$ for all $e_k \in A$. Since $0 \leq A_Q^-(u_{i,j}, q) \leq A_Q^+(u_{i,j}, q) \leq 1$, we have $\lambda_Q(u_{i,j}, q) \leq A_Q^-(u_{i,j}, q)$ or $A_Q^+(u_{i,j}, q) \leq \lambda_Q(u_{i,j}, q)$. This implies $1-\lambda_Q(u_{i,j}, q) \geq 1-A_Q^-(u_{i,j}, q)$ or $1-A_Q^+(u_{i,j}, q) \geq 1-\lambda_Q(u_{i,j}, q)$. Thus $1 - \lambda_Q(u_{i,j}, q) \notin (1 - A_Q^-(u_{i,j}, q), 1 - A_Q^+(u_{i,j}, q))$ for all $e_k \in A$. Hence $(\tilde{\mathfrak{F}}, A)^c$ is an EQCSMS.

5 Union and intersection of Q-cubic soft multiset

In this section we introduce the union and intersection of Q-cubic soft multiset.

Definition 14. *Let $(\tilde{\mathfrak{F}}, A)$ and $(\tilde{\mathfrak{G}}, B)$ be two Q-cubic soft multisets. Then, we define P-union $(\tilde{\mathfrak{F}}, A) \cup_p (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, C)$ where $C = A \cup B$ and*

$$\mathfrak{H}(e_{U_{i,j}}) = \begin{cases} \tilde{\mathfrak{F}}(e_k) & , if\; e_k \in A - B \\ \tilde{\mathfrak{G}}(e_k) & , if\; e_k \in B - A \\ \tilde{\mathfrak{F}}(e_k) \vee_P \tilde{\mathfrak{G}}(e_k) & , if\; e_k \in A \cap B. \end{cases}$$

where $\tilde{\mathfrak{F}}(e_k) \vee_P \tilde{\mathfrak{G}}(e_k) = \{< (u_i, q), max(A_Q(u_{i,j}, q), B_Q(u_{i,j}, q)), (\lambda_Q(u_{i,j}, q) \vee \nu_Q(u_{i,j}, q)) >: e_k \in A \cap B\}$.

Definition 15. *Let $(\tilde{\mathfrak{F}}, A)$ and $(\tilde{\mathfrak{G}}, B)$ be two Q-cubic soft multisets. Then, we define P-intersection $(\tilde{\mathfrak{F}}, A) \cap_p (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, C)$ where $C = A \cap B$ and $\mathfrak{H}(e_k) = \tilde{\mathfrak{F}}(e_k) \wedge_P \tilde{\mathfrak{G}}(e_k)$,*
where $\tilde{\mathfrak{F}}(e_k) \wedge_P \tilde{\mathfrak{G}}(e_k) = \{< (u_{i,j}, q), min(A_Q(u_{i,j}, q), B_Q(u_{i,j}, q)), (\lambda_Q(u_{i,j}, q) \wedge \nu_Q(u_{i,j}, q)) >: e_k \in A \cap B\}$.

Definition 16. *Let $(\tilde{\mathfrak{F}}, A)$ and $(\tilde{\mathfrak{G}}, B)$ be two Q-cubic soft multisets. Then, we define R-union $(\tilde{\mathfrak{F}}, A) \cup_R (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, C)$ where $C = A \cup B$ and*

$$\mathfrak{H}(e_k) = \begin{cases} \tilde{\mathfrak{F}}(e_k) & , if\ (e_k) \in A - B \\ \tilde{\mathfrak{G}}(e_k) & , if\ (e_k) \in B - A \\ \tilde{\mathfrak{F}}(e_k) \vee_R \tilde{\mathfrak{G}}(e_k) & , if\ (e_k) \in A \cap B. \end{cases}$$

where $\tilde{\mathfrak{F}}(e_k) \vee_R \tilde{\mathfrak{G}}(e_k) = \{< (u_{i,j}, q), max(A_Q(u_{i,j}, q), B_Q(u_{i,j}, q)), (\lambda_Q(u_{i,j}, q) \wedge \nu_Q(u_{i,j}, q)) >: e_k \in A \cap B\}$.

Definition 17. *Let $(\tilde{\mathfrak{F}}, A)$ and $(\tilde{\mathfrak{G}}, B)$ be two Q-cubic soft multisets. Then, we define R-intersection $(\tilde{\mathfrak{F}}, A) \cap_R (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, C)$ where $C = A \cap B$ and $\mathfrak{H}(e_k) = \tilde{\mathfrak{F}}(e_k) \wedge_R \tilde{\mathfrak{G}}(e_k)$,*
where $\tilde{\mathfrak{F}}(e_k) \wedge_R \tilde{\mathfrak{G}}(e_k) = \{< (u_{i,j}, q), min(A_Q(u_{i,j}, q), B_Q(u_{i,j}, q)), (\lambda_Q(u_{i,j}, q) \vee \nu_Q(u_{i,j}, q)) >: e_k \in A \cap B\}$.

Example 7. *Let $(\tilde{\mathfrak{F}}, A)$ and $(\tilde{\mathfrak{G}}, B)$ be two Q-cubic soft multisets, where*
$(\tilde{\mathfrak{F}}, A) = \{(e_1, \{< (u_1, q), [0.3, 0.6], 0.4 >, < (h_1, q), [0.1, 0.6], 0.6 >, < (h_2, q), [0.3, 0.5], 0.8 >, < (v_1, q), [0.2, 0.7], 0.7 >, < (v_3, q), [0.1, 0.6], 0.9 >\}),$
$(e_2, \{< (u_1, q), [0.3, 0.4], 0.3 >, < (h_1, q), [0.3, 0.6], 0.4 >, < (h_2, q), [0.4, 0.8], 0.3 >, < (v_1, q), [0.1, 0.5], 0.7 >, < (v_3, q), [0.4, 0.8], 0.8 >\})\}$ *and*
$(\tilde{\mathfrak{G}}, B) = \{(e_2, \{< (u_1, q), [0.5, 0.7], 0.7 >, < (h_1, q), [0.7, 0.9], 0.5 >, < (h_2, q), [0.5, 0.8], 0.7 >, < (v_1, q), [0.3, 0.6], 0.4 >, < (v_3, q), [0.4, 0.7], 0.8 >\}),$
$(e_3, \{< (u_1, q), [0.7, 0.8], 0.7 >, < (h_1, q), [0.6, 0.7], 0.6 >, < (h_2, q), [0.5, 0.7], 0.8 >, < (v_1, q), [0.5, 0.7], 0.5 >, < (v_3, q), [0.4, 0.7], 0.2 >\})\}$. *Then we have the following:*
1. $(\tilde{\mathfrak{F}}, A) \cup_P (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, C)$ *, where $C = A \cup B$ is a Q-cubic soft multiset such that:*
$(\tilde{\mathfrak{H}}, C) = \{(e_1, \{< (u_1, q), [0.5, 0.7], 0.7 >, < (h_1, q), [0.7, 0.9], 0.6 >, < (h_2, q), [0.5, 0.8], 0.8 >, < (v_1, q), [0.3, 0.7], 0.7 >, < (v_3, q), [0.4, 0.7], 0.9 >\}),$
$(e_2, \{< (u_1, q), [0.3, 0.4], 0.3 >, < (h_1, q), [0.3, 0.6], 0.4 >, < (h_2, q), [0.4, 0.8], 0.3 >, < (v_1, q), [0.1, 0.5], 0.7 >, < (v_3, q), [0.4, 0.8], 0.8 >\}).$
$(e_3, \{< (u_1, q), [0.7, 0.8], 0.7 >, < (h_1, q), [0.6, 0.7], 0.6 >, < (h_2, q), [0.5, 0.7], 0.8 >, < (v_1, q), [0.5, 0.7], 0.5 >, < (v_3, q), [0.4, 0.7], 0.2 >\})\}.$
2. $(\tilde{\mathfrak{F}}, A) \cap_P (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, C)$ *, where $C = A \cap B$ is a Q-cubic soft multiset such that:*
$(\tilde{\mathfrak{H}}, C) = \{(e_1, \{< (u_1, q), [0.3, 0.6], 0.4 >, < (h_1, q), [0.1, 0.6], 0.5 >, < (h_2, q), [0.3, 0.5], 0.7 >, < (v_1, q), [0.2, 0.6], 0.4 >, < (v_3, q), [0.1, 0.6], 0.8 >\})\}.$
3. $(\tilde{\mathfrak{F}}, A) \cup_R (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, C)$ *, where $C = A \cup B$ is a Q-cubic soft multiset such that:*

$(\tilde{\mathfrak{H}}, C) = \{(e_1, \{< (u_1, q), [0.5, 0.7], 0.4 >, < (h_1, q), [0.7, 0.9], 0.5 >, < (h_2, q), [0.5, 0.8], 0.7 >$
$, < (v_1, q), [0.3, 0.7], 0.4 >, < (v_3, q), [0.4, 0.7], 0.8 >\}),$
$(e_2, \{< (u_1, q), [0.3, 0.4], 0.3 >, < (h_1, q), [0.3, 0.6], 0.4 >, < (h_2, q), [0.4, 0.8], 0.3 >$
$, < (v_1, q), [0.1, 0.5], 0.7 >, < (v_3, q), [0.4, 0.8], 0.8 >\}).$
$(e_3, \{< (u_1, q), [0.7, 0.8], 0.7 >, < (h_1, q), [0.6, 0.7], 0.6 >, < (h_2, q), [0.5, 0.7], 0.8 >$
$, < (v_1, q), [0.5, 0.7], 0.5 >, < (v_3, q), [0.4, 0.7], 0.2 >\})\}.$
4. $(\tilde{\mathfrak{F}}, A) \cap_R (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, C)$ *, where* $C = A \cap B$ *is a Q-cubic soft multiset such that:*
$(\tilde{\mathfrak{H}}, C) = \{(e_1, \{< (u_1, q), [0.3, 0.6], 0.7 >, < (h_1, q), [0.1, 0.6], 0.6 >, < (h_2, q), [0.3, 0.5], 0.8 >$
$, < (v_1, q), [0.2, 0.6], 0.7 >, < (v_3, q), [0.1, 0.6], 0.9 >\})\}.$

Definition 18. *Let* $(\tilde{\mathfrak{F}}, A) = \{e_{U_{i,j}}, < (u_i, q), A_Q(u_i, q), \lambda_Q(u_i, q) >: u_i \in U_i, q \in Q, e_{U_{i,j}} \in A\}$ *and* $(\tilde{\mathfrak{G}}, B) = \{e_{U_{i,j}}, < (u_i, q), B_Q(u_i, q), \nu_Q(u_i, q) >: u_i \in U_i, q \in Q, e_{U_{i,j}} \in B\}$ *be two Q-cubic soft multisets such that A and* $B \in E$*. Then*

1. *P-OR is denoted by* $(\tilde{\mathfrak{F}}, A) \vee_P (\tilde{\mathfrak{G}}, B)$ *and defined as* $(\tilde{\mathfrak{F}}, A) \vee_P (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, A \times B)$ *where* $\tilde{\mathfrak{H}}(a_i, b_i) = \tilde{\mathfrak{F}}(a_i) \cup_p \tilde{\mathfrak{G}}(b_i)$ *for all* $(a_i, b_i) \in A \times B$.
2. *R-OR is denoted by* $(\tilde{\mathfrak{F}}, A) \vee_R \tilde{\mathfrak{G}}, B)$ *and defined as* $(\tilde{\mathfrak{F}}, A) \vee_R (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, A \times B)$ *where* $\tilde{\mathfrak{H}}(a_i, b_i) = \tilde{\mathfrak{F}}(a_i) \cup_R \tilde{\mathfrak{G}}(b_i)$ *for all* $(a_i, b_i) \in A \times B$.
3. *P-AND is denoted by* $(\tilde{\mathfrak{F}}, A) \wedge_P (\tilde{\mathfrak{G}}, B)$ *and defined as* $(\tilde{\mathfrak{F}}, A) \wedge_P (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, A \times B)$ *where* $\tilde{\mathfrak{H}}(a_i, b_i) = \tilde{\mathfrak{F}}(a_i) \cap_p \tilde{\mathfrak{G}}(b_i)$ *for all* $(a_i, b_i) \in A \times B$.
4. *R-AND is denoted by* $(\tilde{\mathfrak{F}}, A) \wedge_R (\tilde{\mathfrak{G}}, B)$ *and defined as* $(\tilde{\mathfrak{F}}, A) \wedge_R (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, A \times B)$ *where* $\tilde{\mathfrak{H}}(a_i, b_i) = \tilde{\mathfrak{F}}(a_i) \cap_R \tilde{\mathfrak{G}}(b_i)$ *for all* $(a_i, b_i) \in A \times B$.

Example 8. *Let* $(\tilde{\mathfrak{F}}, A)$ *and* $(\tilde{\mathfrak{G}}, B)$ *be two Q-cubic soft multisets, such that*
$(\tilde{\mathfrak{F}}, A) = \{(e_2, \{< (u_1, q), [0.3, 0.4], 0.3 >, < (h_1, q), [0.3, 0.6], 0.4 >, < (h_2, q), [0.4, 0.8], 0.3 >$
$, < (v_1, q), [0.1, 0.5], 0.7 >, < (v_3, q), [0.4, 0.8], 0.8 >\})\}$ *and*
$(\tilde{\mathfrak{G}}, B) = \{(e_1, \{< (u_1, q), [0.5, 0.7], 0.7 >, < (h_1, q), [0.7, 0.9], 0.5 >, < (h_2, q), [0.5, 0.8],$
$0.7 >, < (v_1, q), [0.3, 0.6], 0.4 >, < (v_3, q), [0.4, 0.7], 0.8 >\}),$
$(e_3, \{< (u_1, q), [0.7, 0.8], 0.7 >, < (h_1, q), [0.6, 0.7], 0.6 >, < (h_2, q), [0.5, 0.7], 0.8 >$
$, < (v_1, q), [0.5, 0.7], 0.5 >, < (v_3, q), [0.4, 0.7], 0.2 >\})\}$. *Then P-OR is denoted by* $(\tilde{\mathfrak{F}}, A) \vee_P (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, C)$ *where* $C = A \times B = \{(e_1, e_2), (e_1, e_3)\}$ *is defined as*
$(\tilde{\mathfrak{H}}, C) = \{((e_1, e_2), \{< (u_1, q), [0.5, 0.7], 0.7 >, < (h_1, q), [0.7, 0.9], 0.5 >, <$
$(h_2, q), [0.5, 0.8], 0.7 >, < (v_1, q), [0.3, 0.6], 0.7 >, < (v_3, q), [0.4, 0.8], 0.8 >\}),$
$((e_1, e_3), \{< (u_1, q), [0.7, 0.8], 0.7 >, < (h_1, q), [0.6, 0.7], 0.6 >, < (h_2, q), [0.5, 0.8], 0.8 >$
$, < (v_1, q), [0.5, 0.7], 0.7 >, < (v_3, q), [0.4, 0.8], 0.8 >\})\}.$

Theorem 4. *Let* $(\tilde{\mathfrak{F}}, A) = \{e_{U_{i,j}}, <(u_i, q), A_Q(u_i, q), \lambda_Q(u_i, q) >: u_i \in U_i\,, q \in Q,\ e_{U_{i,j}} \in A\}$ *and* $(\tilde{\mathfrak{G}}, B) = \{e_{U_{i,j}}, <(u_i, q), B_Q(u_i, q), \nu_Q(u_i, q) >: u_i \in U_i\,, q \in Q,\ e_{U_{i,j}} \in B\}$ *be internal Q-cubic soft multisets such that A and* $B \in E$. *Then*

1. $(\tilde{\mathfrak{F}}, A) \cup_P (\tilde{\mathfrak{G}}, B)$ *is an IQCSMS.*

2. $(\tilde{\mathfrak{F}}, A) \cap_P (\tilde{\mathfrak{G}}, B)$ *is an IQCSMS.*

Proof. 1. Since $(\tilde{\mathfrak{F}}, A)$ and $(\tilde{\mathfrak{G}}, B)$ are internal Q-cubic soft multisets, then we have
$A_Q^-(u_{i,j}, q) \leq \lambda_Q(u_{i,j}, q) \leq A_Q^+(u_{i,j}, q)$ for all $e_k \in A$ and
$B_Q^-(u_{i,j}, q) \leq \nu_Q(u_{i,j}, q) \leq B_Q^+(u_{i,j}, q)$ for all $e_k \in B$. Then
$max(A_Q^-(u_{i,j}, q), B_Q^-(u_{i,j}, q)) \leq (\lambda_Q(u_{i,j}, q) \vee \nu_Q(u_{i,j}, q)) \leq max(A_Q^+(u_{i,j}, q),$
$B_Q^+(u_{i,j}, q))$
for all $e_k \in A \cup B$.
By definition of P-union of $(\tilde{\mathfrak{F}}, A)$ and $(\tilde{\mathfrak{G}}, B)$, we have

$$(\tilde{\mathfrak{F}}, A) \cup_P (\tilde{\mathfrak{G}}, B) = \begin{cases} \tilde{\mathfrak{F}}(e_k) & , if\ (e_k) \in A - B \\ \tilde{\mathfrak{G}}(e_k) & , if\ (e_k) \in B - A \\ \tilde{\mathfrak{F}}(e_k) \vee_P \tilde{\mathfrak{G}}(e_k) & , if\ (e_k) \in A \cap B. \end{cases}$$

If $(e_k) \in A - B$ or $(e_k) \in B - A$, then the result is trivial.
If $(e_k) \in A \cap B$ then $\tilde{\mathfrak{F}}(e_k) \vee_P \tilde{\mathfrak{G}}(e_k) = \{e_k, <(u_{i,j}, q), max(A_Q(u_{i,j}, q), B_Q(u_{i,j}, q)),$ $(\lambda_Q(u_{i,j}, q) \vee \nu_Q(u_{i,j}, q)) : e_k \in A \cap B\}$ is an IQCSMS. Hence,$(\tilde{\mathfrak{F}}, A) \cup_P (\tilde{\mathfrak{G}}, B)$ is an IQCSMS in all cases.

2. Since $(\tilde{\mathfrak{F}}, A) \cap_p (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, C)$ where $C = A \cap B$ and $\mathfrak{H}(e_k) = \tilde{\mathfrak{F}}(e_k) \wedge_P \tilde{\mathfrak{G}}(e_k)$,
then $e_k \in A \cap B$ implies$\tilde{\mathfrak{F}}(e_k) \wedge_P \tilde{\mathfrak{G}}(e_k) = \{<(u_{i,j}, q), min(A_Q(u_{i,j}, q), B_Q(u_{i,j}, q)),$ $(\lambda_Q(u_{i,j}, q) \wedge \nu_Q(u_{i,j}, q)) >\}$.
By definition of IQCSMS we have
$A_Q^-(u_{i,j}, q) \leq \lambda_Q(u_{i,j}, q) \leq A_Q^+(u_{i,j}, q)$ for all $e_k \in A$ and
$B_Q^-(u_{i,j}, q) \leq \nu_Q(u_{i,j}, q) \leq B_Q^+(u_{i,j}, q)$ for all $e_k in B$. Then
$min(A_Q^-(u_{i,j}, q), B_Q^-(u_{i,j}, q)) \leq (\lambda_Q(u_{i,j}, q), \wedge \nu_Q(u_{i,j}, q)) \leq min(A_Q^+(u_{i,j}, q),$
$B_Q^+(u_{i,j}, q))$
for all $e_k \in A \cap B$.
Hence $(\tilde{\mathfrak{F}}, A) \cap_p (\tilde{\mathfrak{G}}, B)$ is an internal Q-cubic soft multiset (IQCSMS).

Theorem 5. *If* $(\tilde{\mathfrak{F}}, A)$ *and* $(\tilde{\mathfrak{G}}, B)$ *are internal Q-cubic soft multisets such that* $max(A_Q^-(u_{i,j}, q), B_Q^-(u_{i,j}, q)) \leq (\lambda_Q(u_{i,j}, q), \wedge \nu_Q(u_{i,j}, q))$ *for all* $e_k \in A \cap B$, *then* $(\tilde{\mathfrak{F}}, A) \cup_R (\tilde{\mathfrak{G}}, B)$ *is an IQCSMS.*

Proof. From definition 10 we have
$A_Q^-(u_{i,j},q) \leq \lambda_Q(u_{i,j},q) \leq A_Q^+(u_{i,j},q)$ for all $e_k \in A$ and
$B_Q^-(u_{i,j},q) \leq \nu_Q(u_{i,j},q) \leq B_Q^+(u_{i,j},q)$ for all $e_k \in B$. Then
$(\lambda_Q(u_{i,j},q) \wedge \nu_Q(u_{i,j},q)) \leq max(A_Q^+(u_{i,j},q), B_Q^+(u_{i,j},q))$ for all $e_k \in A \cup B$.
Since $max(A_Q^-(u_{i,j},q), B_Q^-(u_{i,j},q)) \leq (\lambda_Q(u_{i,j},q), \wedge\nu_Q(u_{i,j},q))$,
and by definition of R-union of $(\tilde{\mathfrak{F}}, A)$ and $(\tilde{\mathfrak{G}}, B)$, we have

$$(\tilde{\mathfrak{F}}, A) \cup_R (\tilde{\mathfrak{G}}, B) = \begin{cases} \tilde{\mathfrak{F}}(e_k) & , if\ (e_k) \in A - B \\ \tilde{\mathfrak{G}}(e_k) & , if\ (e_k) \in B - A \\ \tilde{\mathfrak{F}}(e_k) \wedge_R \tilde{\mathfrak{G}}(e_k) & , if\ (e_k) \in A \cap B. \end{cases}$$

If $(e_{U_{i,j}}) \in A \cap B$, then $\tilde{\mathfrak{F}}(e_k) \wedge_R \tilde{\mathfrak{G}}(e_k) = \{e_k, < (u_{i,j},q), max(A_Q(u_{i,j},q), B_Q(u_{i,j},q)),$ $(\lambda_Q(u_{i,j},q) \wedge \nu_Q(u_{i,j},q)) >: (e_k) \in A \cap B\}$. Since $(\tilde{\mathfrak{F}}, A)$ and $(\tilde{\mathfrak{G}}, B)$ are an IQCSMS, we have
$max(A_Q^-(u_{i,j},q), B_Q^-(u_{i,j},q)) \leq (\lambda_Q(u_{i,j},q), \wedge\nu_Q(u_{i,j},q)) \leq max(A_Q^+(u_{i,j},q),$
$B_Q^+(u_{i,j},q))$ for all $e_k \in A \cap B$.
If $e_k \in A - B$ or $e_k \in B - A$, then the result is trivial. Thus $(\tilde{\mathfrak{F}}, A) \cup_R (\tilde{\mathfrak{G}}, B)$ is an IQCSMS if $(\lambda_Q(u_{i,j},q), \wedge\nu_Q(u_{i,j},q)) \leq max(A_Q^+(u_{i,j},q),$
$B_Q^+(u_{i,j},q))$.

Theorem 6. *If $(\tilde{\mathfrak{F}}, A)$ and $(\tilde{\mathfrak{G}}, B)$ are internal Q-cubic soft multisets such that $min(A_Q^+(u_{i,j},q), B_Q^+(u_{i,j},q)) \geq (\lambda_Q(u_{i,j},q), \vee\nu_Q(u_{i,j},q))$ for all $e_k \in A \cap B$, then $(\tilde{\mathfrak{F}}, A) \cap_R (\tilde{\mathfrak{G}}, B)$ is an IQCSMS.*

Proof. From definition 10 we have
$A_Q^-(u_{i,j},q) \leq \lambda_Q(u_{i,j},q) \leq A_Q^+(u_{i,j},q)$ for all $e_k \in A$ and
$B_Q^-(u_{i,j},q) \leq \nu_Q(u_{i,j},q) \leq B_Q^+(u_{i,j},q)$ for all $e_k \in B$. This implies that
$min(A_Q^-(u_{i,j},q), B_Q^-(u_{i,j},q)) \leq (\lambda_Q(u_{i,j},q) \vee \nu_Q(u_{i,j},q))$ for all $e_k \in A \cup B$.
Since $min(A_Q^+(u_{i,j},q), B_Q^+(u_{i,j},q)) \geq (\lambda_Q(u_{i,j},q), \vee\nu_Q(u_{i,j},q))$,
and by definition of R-intersection of $(\tilde{\mathfrak{F}}, A)$ and $(\tilde{\mathfrak{G}}, B)$, we have
$(\tilde{\mathfrak{F}}, A) \cap_R (\tilde{\mathfrak{G}}, B) = (\tilde{\mathfrak{H}}, C)$ such that $\tilde{\mathfrak{H}}(e_k) = \tilde{\mathfrak{F}}(e_k) \vee_R \tilde{\mathfrak{G}}(e_k)$,
$\tilde{\mathfrak{F}}(e_k) \vee_R \tilde{\mathfrak{G}}(e_k) = \{e_k, < (u_{i,j},q), min(A_Q(u_{i,j},q), B_Q(u_{i,j},q)),$
$(\lambda_Q(u_{i,j},q) \vee \nu_Q(u_{i,j},q)) >: e_k \in A \cap B\}$.

Since $(\tilde{\mathfrak{F}}, A)$ and $(\tilde{\mathfrak{G}}, B)$ are IQCSMS, thus we have
$min(A_Q^-(u_{i,j},q), B_Q^-(u_{i,j},q)) \leq (\lambda_Q(u_{i,j},q), \vee\nu_Q(u_{i,j},q)) \leq min(A_Q^+(u_{i,j},q),$
$B_Q^+(u_{i,j},q))$ for all $e_k \in A \cap B$. Therefore $(\tilde{\mathfrak{F}}, A) \cap_R (\tilde{\mathfrak{G}}, B)$ is an IQCSMS if $min(A_Q^+(u_{i,j},q), B_Q^+(u_{i,j},q)) \geq (\lambda_Q(u_{i,j},q), \wedge\nu_Q(u_{i,j},q))$.

6 Conclusion

Pioneering work on the soft set has been done by Molodtsov. The soft set theory is a general mathematical tool for dealing with uncertain, fuzzy, or vague objects. In this paper, the concept of the Q-cubic set and its operations is proposed. The notions of (internal, external) Q-cubic soft sets, P-(R-)order,P- (R-)union, P-(R-)intersection and P-OR, R-OR, P-AND and R-AND are introduced, and related properties and theorems are investigated. In further research, the applications of Q-cubic soft in information sciences and knowledge system is an important and interesting issue to be addressed.

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