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Complex multi-fuzzy soft expert set and its application

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Abstract

We introduce the concept of complex multi-fuzzy soft expert set (CMFSES) for which the range of its membership functions are represented in terms of complex numbers. CMFSES has the ability to realize more range of values while handling uncertainty of data that is captured by the amplitude terms and phase terms of the complex numbers, simultaneously. It also has a mechanism to incorporate the adequate parameterization capabilities and the opinions of all experts regarding the validity of the information at hand in one mode, thus making it quite appropriate for use in decision-making problems, whereas the time factor plays a key role in determining the final decision. Some operations related to this new concept have been defined, such as complement, union and intersection, AND and OR operations. We also investigate structural properties of these operations based on this concept. An algorithm is developed in complex multi-fuzzy expert soft set setting for a decision making method. Finally an illustrative example is employed to show that it can be successfully applied to problems that contain uncertainties. In addition, a comparison between CMFSES and other existing methods is made to reveal the dominance of our proposed method.

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1 Introduction

The notion of fuzzy sets [1] was generalized to multi-fuzzy sets by Sebastian and Ramakrishnan [2] in 2011. Sebastian and Ramakrishnan [2,3] introduced multi-fuzzy sets theory as a mathematical tool to deal with life problems that have multidimensional characterization properties, such as complete colour characterization of colour images, taste recognition of food items and decision making problems with multi aspects. Since then, several researchers [3-5] have proposed numerous extended versions of fuzzy set. Fuzzy set theory boosted the rise of many related theories that attempt to model specific decision problems. In particular, the hybridization of fuzzy sets with soft sets as proposed by Molodtsov [6] yields the notion of fuzzy soft set that was introduced by Maji et al. [7].

Maji et al. [8] presented the concept of intuitionistic fuzzy soft sets based on a combination of the intuitionistic fuzzy set and soft set models and applied it to decision making mechanisms. Yang et al. [9] introduced multi-fuzzy soft set by combining the multi-fuzzy set and soft set and presented an application in decision making problems. Since then, several researchers [10-13] have discussed more properties on soft sets. Over the years, researchers have recognized the need for models that incorporate the opinions of experts, so as to validate the information provided by the observers. Alkhazaleh and Salleh [14] firstly proposed the notion of a soft expert set, where the user can know the opinion of all experts in one model without any operations. Then, they came up with the notion of fuzzy soft expert sets [15] by combining fuzzy set with soft expert set. The works on soft expert set, in theories and applications, have been progressing rapidly (e.g. [16-18]).

The fuzzy set model can be combined with other mathematical models to extend the range from the real field to the complex field. Ramot et al. [19] extended the range of the membership function in fuzzy set from the interval [0,1] to the unit circle in the complex plane and called it complex fuzzy set, which has progressed rapidly to complex fuzzy logic [20]. The complex fuzzy set was used to represent information with uncertainty and periodicity. Since then, a lot of extensions of complex fuzzy set models have been developed such as complex intuitionistic fuzzy set [21], complex neutrosophic set [22], complex vague soft set [23], complex intuitionistic fuzzy soft set [24] and complex fuzzy soft expert set [25]. These models have been used to represent the uncertainty and periodicity aspects of an object together, in a single set. Al-Qudah and Hassan [26] introduced the complex multi-fuzzy set. It consists of multi-membership functions in that each membership function is composed of amplitude term and phase term. The phase term gives it the wave-like properties that could be used to describe constructive and destructive interference depending on the phase value of an element that involves periodicity and varies with time. This model is useful for handling problems with multidimensional characterization properties. To make this model more functional for improving new decision making results, we will develop it into complex multi-fuzzy soft expert sets (CMFSES) in order to incorporate the advantages of soft expert sets to the complex multi-fuzzy sets. The novelty of CMFSES appears in its ability to provide succinct, elegant and comprehensive representation of two-dimensional multi-fuzzy information as well as the adequate parameterization and the opinions of the experts, all in a single set. Hence, decision making process is no longer limited to discrete programming [27-29].

The rest of this paper is organized in the following way. Section 2 recalls some fundamentals of multi- fuzzy set theory, soft set theory, fuzzy soft expert sets, multi-fuzzy soft sets and complex multi-fuzzy sets. The concept of the multi-fuzzy soft expert set and the complex multi-fuzzy soft sets are also introduced in this section. Section 3 discusses the advantages and drawbacks of the various hybrid structures of fuzzy sets, and complex numbers which serve as the motivation for this paper. In Section 4, the concept of CMFSS with its operation rules are introduced. Section 5 presents the main set theoretic operations in this paper, namely complement, union, intersection, AND and OR and proves some useful fundamental properties of these operations. Then, in Section 6, the concept of CMFSS is used to analyze a decision making problem and an adjustable algorithm is proposed. In Section 7, a comparison of our proposed concept to two other existing methods is conducted to verify the validity of the proposed approach. We conclude in Section 8.

2 Preliminaries

In this section, we epitomize some of the significant concepts pertaining to multi- fuzzy set , soft set, soft expert set, multi-fuzzy soft set and complex multi-fuzzy set that are useful for subsequent discussions. The multi-fuzzy soft expert set and complex multi-fuzzy soft set are also introduced. These are stated below.

Definition 2.1. [2]. Let k be a positive integer and U be a non-empty set. A multi-fuzzy set $\mathcal{A}$ in U is a set of ordered sequences $\mathcal{A} = \{\langle x, \mu_1(x), ..., \mu_k(x)\rangle : x \in U\}$, where $\mu_i : U \longrightarrow L_i = [0,1]$, $i = 1, 2, ..., k$. The function $\mu_{\mathcal{A}}(x) = (\mu_1(x), ..., \mu_k(x))$ is called the multi-membership function of multi-fuzzy sets $\mathcal{A}$, k is called a dimension of $\mathcal{A}$. The set of all multi-fuzzy sets of dimension k in U is denoted by $M^k FS(U)$.

Molodtsov [6] defined soft set in the following way.

Definition 2.2. [6]. Let U be an initial universal, E be the set of parameters, $\mathcal{A} \subseteq E$, and $P(U)$ is the power set of U. Then $(F, \mathcal{A})$ is called a soft set, where

$$F : \mathcal{A} \to P(U).$$

Let U be a universe, E a set of parameters, X a set of experts (agents), and O a set of opinions. Let $\mathcal{Z} = E \times X \times O$ and $\mathcal{A} \subseteq \mathcal{Z}$.

Alkhazaleh and Salleh [14]. introduced the concept of soft expert set, whereby the user through this definition can know the opinion of all experts in one model.

Definition 2.3. [14]. A pair $(F, \mathcal{A})$ is called a soft expert set over U, where F is a mapping given by

$$F : \mathcal{A} \to P(U)$$

where $P(U)$ denotes the power set of U.

Definition 2.4. [14]. An agree - soft expert set $(F, \mathcal{A})_1$ over U is a soft expert subset of $(F, \mathcal{A})$ defined as follows:

$$(F, \mathcal{A})_1 = \{F_1(\alpha) : \alpha \in E \times X \times \{1\}\}.$$

Definition 2.5. [14]. A disagree- soft expert set $(F, \mathcal{A})_0$ over U is a soft expert subset of $(F, \mathcal{A})$ defined as follows:

$$(F, \mathcal{A})_0 = \{F_0(\alpha) : \alpha \in E \times X \times \{0\}\}.$$

By introducing the concept of multi-fuzzy set into the theory of soft set, Yang et al. [9] proposed the concept of the multi-fuzzy soft set as follows.

Definition 2.6. [9]. Let U be an initial universal and E be a set of parameters. A pair $(F, \mathcal{A})$ is called a multi-fuzzy soft set of dimension k over U, where F is a mapping given by

$$F : \mathcal{A} \to M^kFS(U).$$

A multi-fuzzy soft set is a mapping from parameters to $M^kFS(U)$. It is a parameterized family of multi-fuzzy subsets of U. For $e \in \mathcal{A}$, $F(e)$ may be considered as the set of e-approximate elements of the multi-fuzzy soft set $(F, \mathcal{A})$.

Lately, as a novel hybrid structure of complex fuzzy sets and multi fuzzy set, Al-Qudah and Hassan [26] introduced the concept of complex multi fuzzy set, which can be seen as an extension of multi-fuzzy set. They proposed the concept as below.

Definition 2.7. [26]. Let k be a positive integer and U be a non-empty set. A complex multi-fuzzy set (CMFS) $\mathcal{A}$, defined on a universe of discourse U, is characterised by a multi-membership function $\mu_{\mathcal{A}}(x) = (\mu^j_{\mathcal{A}}(x))_{j\in k}$,that assigns to any element $x \in U$ a complex-valued grade of multi- membership functions in $\mathcal{A}$. $\mu_{\mathcal{A}}(x)$ may all lie within the unit circle in the complex plane, and are thus of the form $\mu_{\mathcal{A}}(x) = (r^j_{\mathcal{A}}(x).e^{i\omega^j_{\mathcal{A}}(x)})_{j\in k}$, $(i = \sqrt{-1})$, $(r^j_{\mathcal{A}}(x))_{j\in k}$ are real-valued functions and $(r^j_{\mathcal{A}}(x))_{j\in k} \in [0, 1]$. The CMFS $\mathcal{A}$ may be represented as the set of ordered sequence

$$\mathcal{A} = \{(x\ , \left(\mu^j{}_{\mathcal{A}}(x) = a_j\right)_{j\in k}) : x \in U\} = \{x, ((r^j_{\mathcal{A}}(x).e^{i\omega^j_{\mathcal{A}}(x)})_{j\in k}) : x \in U\}.$$

where $\mu^j{}_{\mathcal{A}} : U \to L_j = \{a_j : a_j \in C, |a_j| \leq 1\}$ for $j = 1, 2, ..., k$.
The function $(\mu_{\mathcal{A}}(x) = r^j_{\mathcal{A}}(x).e^{i\omega^j_{\mathcal{A}}(x)})_{j\in k}$ is called the complex multi-membership function of complex multi-fuzzy set $\mathcal{A}$, k is called the dimension of $\mathcal{A}$. The set of all complex multi-fuzzy sets of dimension k in U is denoted by $CM^kFS(U)$.

We now present the theoretic operations of complex multi-fuzzy sets.

Definition 2.8. [26]. Let $\mathcal{A} = \{x, ((r^j_{\mathcal{A}}(x).e^{i\omega^j_{\mathcal{A}}(x)})_{j\in k}) : x \in U\}$ and $\mathcal{B} = \{x, ((r^j_{\mathcal{B}}(x).e^{i\omega^j_{\mathcal{B}}(x)})_{j\in k}\ : x \in U\}$ be two complex multi-fuzzy sets of dimension k in X. We define the following relations and operations.

1. $\mathcal{A} \subset \mathcal{B}$ if and only if $r^j_{\mathcal{A}}(x) \leq r^j_{\mathcal{B}}(x)$ and $\omega^j_{\mathcal{A}}(x) \leq \omega^j_{\mathcal{B}}(x)$, for all $x \in U$ and $j = 1, 2, ..., k$.

2. $\mathcal{A} = \mathcal{B}$ if and only if $r^j_{\mathcal{A}}(x) = r^j_{\mathcal{B}}(x)$ and $\omega^j_{\mathcal{A}}(x) = \omega^j_{\mathcal{B}}(x)$, for all $x \in U$ and $j = 1, 2, ..., k$.

3. $\mathcal{A}\cup\mathcal{B} = \{\langle x, r^j_{\mathcal{A}\cup\mathcal{B}}(x).e^{i\omega^j_{\mathcal{A}\cup\mathcal{B}}(x)}\rangle : x \in\} = \{\langle x, \max(r^j_{\mathcal{A}}(x), r^j_{\mathcal{B}}(x)).e^{i\max[\omega^j_{\mathcal{A}}(x),\omega^j_{\mathcal{B}}(x)]}\rangle : x \in U\}$, for all $j = 1, 2, ..., k$.

4. $\mathcal{A}\cap\mathcal{B} = \{\langle x, r^j_{\mathcal{A}\cap\mathcal{B}}(x).e^{i\omega^j_{\mathcal{A}\cap\mathcal{B}}(x)}\rangle : x \in\} = \{\langle x, \min(r^j_{\mathcal{A}}(x), r^j_{\mathcal{B}}(x)).e^{i\min[\omega^j_{\mathcal{A}}(x),\omega^j_{\mathcal{B}}(x)]}\rangle : x \in U\}$, for all $j = 1, 2, ..., k$.

5. $\mathcal{A}^c = \{x, [r^j_{A^c}(x).e^{i\omega^j_{\mathcal{A}^c}(x)}]_{j\in k} : x \in X\} = \{x, ([1 - r^j_{\mathcal{A}}(x)].e^{i[2\pi-\omega^j_{\mathcal{A}}(x)]})_{j\in k} : x \in U\}$, for all $j = 1, 2, ..., k$.

We will now introduce the concept of multi-fuzzy soft expert $(MFSES)$.

Definition 2.9. Let U be a universe of elements, E a set of parameters, X be a set of experts (agents), and $O = \{1 = agree, 0 = disagree\}$ a set of opinions. Let $Z = E \times X \times O$ and $\mathcal{A} \subseteq Z$. Then a pair $(\tilde{F}, \mathcal{A})$ is called a multi-fuzzy soft expert set of dimension k (M^kFSES) over U, where $\tilde{F}$ is a mapping given by $\tilde{F} : \mathcal{A} \to M^kF(U)$, where $M^kF(U)$ denotes the collection of all multi-fuzzy subsets of U.

To illustrate the above definition of the $MFSES$, let us consider the following typical example.

Example 2.10. Suppose that a company wants to manufacture new types of products and to take some experts opinions about these products. Let $U = \{x_1, x_2\}$ be a set universe consisting of two kinds of products. Suppose the parameter set $E = \{e_1, e_2\}$ i.e., we have two criteria to evaluate the performance of these products, where e_1 stands for "quality ", which includes three levels: excellent, very good and good, while e_2 stands for the parameter "price "which has three levels too: high, medium and low. Let $X = \{p, q\}$ be a set of experts who are assigned to give their opinions concerning these products.

Suppose that the multi-fuzzy soft expert set of dimension three $(\tilde{F}, \mathcal{A})$ is given as follows:

$$\begin{aligned}(\tilde{F}, \mathcal{A}) = \Big\{&(e_1, p, 1) = \{\tfrac{(0.6,0.3,0.5)}{x_1}, \tfrac{(0.9,0.6,1)}{x_2}\}, (e_1, q, 1) = \{\tfrac{(0.6,0.3,0.5)}{x_1}, \tfrac{(0.4,0.6,0.6)}{x_2}\},\\ &(e_2, p, 1) = \{\tfrac{(0.1,0.2,0.3)}{x_1}, \tfrac{(0.8,0.8,0.6)}{x_2}\}, (e_2, q, 1) = \{\tfrac{(0.4,0.3,0.7)}{x_1}, \tfrac{(0.9,0.8,0.7)}{x_2}\},\\ &(e_1, p, 0) = \{\tfrac{(0.6,0.3,0.5)}{x_1}, \tfrac{(0.8,0.7,0.62)}{x_2}\}, (e_1, q, 0) = \{\tfrac{(0.5,0.5,0.44)}{x_1}, \tfrac{(0.7,0.7,0.6)}{x_2}\},\\ &(e_2, p, 0) = \{\tfrac{(0.55,0.4,0.6)}{x_1}, \tfrac{(1,0,0.9)}{x_2}\}, (e_2, q, 0) = \{\tfrac{(0.4,0.3,0.4)}{x_1}, \tfrac{(0.5,0.7,0.6)}{x_2}\}\Big\}.\end{aligned}$$

Then $(\tilde{F}, \mathcal{A})$ is a multi-fuzzy soft expert set over the soft universe U. Each element of the multi-fuzzy soft expert set implies the opinion of each expert based on each parameter about the products with their own accompanying membership.

Next, we will now give the definition of complex multi-fuzzy soft sets.

Definition 2.11. Let U be a universe, E be a set of parameters and $\mathcal{A} \subseteq E$. Let $CM^k(U)$ be a set of all complex multi-fuzzy subsets of U. A pair $(F, \mathcal{A})$ is called a complex multi-fuzzy soft set of dimension k over U, where F is a mapping given by

$$F : \mathcal{A} \to CM^k(U),$$

A complex multi-fuzzy soft set (CM^{k}FSS) is a mapping from parameters to $CM^k(U)$. It is a parameterized family of complex multi-fuzzy subsets of U, and it can be written as:

$$(F, \mathcal{A}) = \{\langle e, F(e)\rangle : e \in \mathcal{A}, F(e) \in CM^k(U)\},$$

where

$$F(e) = \{\langle x, \mu^j_{F(e)}(x) = r^j_{F(e)}(x).e^{i\omega^j_{F(e)(x)}}\rangle : e \in \mathcal{A}, x \in U, j = 1, 2, ..., k\}.$$

3 Motivation for complex multi-fuzzy soft expert set

Fuzzy set theory [1] is the popular generalization of classical set theory, whose membership grades are within the real valued interval [0,1]. Fuzzy set is successfully used in modeling uncertainty in many fields of everyday life. Sebastian and Ramakrishnan [2,3] proposed the concept of multi-fuzzy set theory as a mathematical tool to deal with life problems that have multidimensional characterization properties. In fact, the multi-fuzzy set has an obstacle to retrieve full information with correct meaning. To overcome this obstacle, Al-Qudah and Hassan [26] generalised the range of membership function of multi-fuzzy set from [0,1] to the unit circle in the complex plane by adding an additional term called the phase term to multi-fuzzy set, enabling its elements to be described in terms of the time aspect.

Moreover, there are two major problems resulting from complex multi-fuzzy set: it lacks a sufficient parameterization tool and it does not have a mechanism to validate the values assigned to the membership functions of the elements in a set. Our proposed CMFSES has the added advantages of complex multi-fuzzy set. This model incorporates the advantages of complex multi-fuzzy sets and the adequate parameterization tool, in addition to having the added advantage of allowing the users to know the opinion of all the experts in a single model without the need for any additional cumbersome operations, whereas complex multi-fuzzy set lacks this advantage. These key advantages of the CMFSES model play a major part in improving the accuracy of the information provided by the observers, and this move improves the quality of the decisions made by the end users of the information i.e. the decision makers.

A novel approach to decision-making problems based on CMFSES is also introduced. This approach converts the CMFSES to a multi-fuzzy soft expert set using a practical and useful algorithm which highlights the role of the time factor in determining the final decision.

4 Complex multi-fuzzy soft expert set

In this section, we introduce the definition of complex multi-fuzzy soft expert set ($\mathcal{CMFSES}$) which is a combination of soft expert set and multi-fuzzy set defined in a complex setting, give definitions of its basic operations, namely subset, equality, complement, union, intersection, AND and OR. Several laws and related results have also been investigated.

Now, we begin proposing the definition of complex multi-fuzzy soft expert set followed by an example.

Let U be a universe of elements, E a set of parameters, X be a set of experts (agents), and $O = \{1 = agree, 0 = disagree\}$ a set of opinions. Let $\mathcal{Z} = E \times X \times O$ and $\mathcal{A} \subseteq \mathcal{Z}$.

Definition 4.1. A pair $(\mathcal{F}, \mathcal{A})$ is called a complex multi-fuzzy expert soft set of dimension k ($\mathcal{CM}^{\mathrm{k}}\mathcal{FSES}$) over U if and only if $\mathcal{F} : \mathcal{A} \to \mathcal{CM}^{\mathrm{k}}(U)$ is a mapping into the set of all complex multi-fuzzy sets in U. Complex multi-fuzzy soft expert set is denoted and defined as

$$(\mathcal{F}, \mathcal{A}) = \Big\{\mathcal{F}(e) = \{\langle x, \mu^j_{\mathcal{F}(e)}(x) = r^j_{\mathcal{F}(e)}(x).e^{i\omega^j_{\mathcal{F}(e)}(x)}\rangle : e \in \mathcal{A}, x \in U, j = 1, 2, ..., k\}\Big\},$$

where $(\mu^j_{\mathcal{F}(e)}(x))_{j\in K}$ is a complex-valued grade of multi- membership function $\forall x \in U$. By definition, the values of $(\mu^j_{\mathcal{F}(e)}(x))_{j\in K}$ may all lie within the unit circle in the complex plane, and are thus of the form $[\mu^j_{\mathcal{F}(e)}(x) = r^j_{\mathcal{F}(e)}(x).e^{i\omega^j_{\mathcal{F}(e)}(x)}]_{j\in K}$, where $(i = \sqrt{-1})$, each of the amplitude terms $(r^j_{\mathcal{F}(e)}(x))_{j\in K}$ and the phase terms $(\omega^j_{\mathcal{F}(e)}(x))_{j\in K}$ are both real-valued, and $(r^j_{\mathcal{F}(e)}(x))_{j\in K} \in [0,1]$.

To illustrate this notion, let us consider the following typical example.

Example 4.2. Suppose that a company wants to manufacture two different types of vehicles and to take some experts opinions before and after testing them. Suppose that $U = \{x_1, x_2\}$ is the universe consisting of two different types of vehicles. The experts team is represented by the set $\mathcal{X} = \{p, q\}$. Let $E = \{e_1, e_2\}$ be a set of parameters that the team of experts consider, where e_1 stands for "purchase cost" which can be categorized as high, medium and low, e_2 stands for "engine capacity" of 4000cc, 3000cc and 2000cc. It is to be noted that the parameters may potentially be affected and altered after the vehicles are tested. We apply the $\mathcal{CM}^{\text{k}}\mathcal{FSES}$ and consider the opinions of the experts in the first process (before testing the vehicles) as amplitude terms of multi-membership function, and set the opinions of the experts in the second process (after testing the vehicles) as phase terms of multi-membership function. Thus, the first and second process form a $\mathcal{CM}^3\mathcal{FSES}$ as a whole which is shown below:

$$(\mathcal{F}, A) =$$

$$\Big\{(e_1,p,1) = \{\tfrac{(0.9e^{i2\pi(0.7)},0.6e^{i2\pi(0.4)},0.2e^{i2\pi(0.1)})}{x_1}, \tfrac{(0.5e^{i2\pi(0.9)},0.4e^{i2\pi(0.8)},0.5e^{i2\pi(1)})}{x_2}\},$$
$$(e_1,q,1) = \{\tfrac{(0.6e^{i2\pi(0.6)},0.8e^{i2\pi(0.7)},0.5e^{i2\pi(0.8)})}{x_1}, \tfrac{(0.3e^{i2\pi(0.4)},0.5e^{i2\pi(0.5)},0.1e^{i2\pi(0)})}{x_2}\},$$
$$(e_2,p,1) = \{\tfrac{(0.5e^{i2\pi(0.5)},0.7e^{i2\pi(0.5)},0.9e^{i2\pi(0.4)})}{x_1}, \tfrac{(0.1e^{i2\pi(0.3)},0.1e^{i2\pi(0)},0.2e^{i2\pi(0.5)})}{x_2}\},$$
$$(e_2,q,1) = \{\tfrac{(0.4e^{i2\pi(0.3)},0.3e^{i2\pi(0.4)},0.1e^{i2\pi(0)})}{x_1}, \tfrac{(0.4e^{i2\pi(0.9)},0.6e^{i2\pi(0.9)},0.5e^{i2\pi(0.7)})}{x_2}\},$$
$$(e_1,p,0) = \{\tfrac{(0.6e^{i2\pi(0.5)},0.4e^{i2\pi(0.8)},0.7e^{i2\pi(0.9)})}{x_1}, \tfrac{(0.7e^{i2\pi(0.6)},0.6e^{i2\pi(0.5)},0.9e^{i2\pi(0.7)})}{x_2}\},$$
$$(e_1,q,0) = \{\tfrac{(0.6e^{i2\pi(0.5)},0.6e^{i2\pi(0.7)},0.4e^{i2\pi(0.8)})}{x_1}, \tfrac{(0.6e^{i2\pi(0.9)},0.7e^{i2\pi(0.8)},0.6e^{i2\pi(0.6)})}{x_2}\},$$
$$(e_2,p,0) = \{\tfrac{(0.7e^{i2\pi(0.7)},0.4e^{i2\pi(0.5)},0.5e^{i2\pi(0.6)})}{x_1}, \tfrac{(0.7e^{i2\pi(0.3)},0.4e^{i2\pi(0.4)},0.5e^{i2\pi(0.3)})}{x_2}\},$$
$$(e_2,q,0) = \{\tfrac{(0.7e^{i2\pi(0.9)},0.4e^{i2\pi(0.7)},0.7e^{i2\pi(0.5)})}{x_1}, \tfrac{(0.3e^{i2\pi(0.5)},0.7e^{i2\pi(0.3)},0.1e^{i2\pi(0.6)})}{x_2}\}\Big\}.$$

In the complex multi-fuzzy soft expert set above, the amplitude terms represent the opinions of the experts in process one (before testing the vehicles), whereas the phase terms portray the opinions of the experts in the second process (after testing the vehicles).

To illustrate what we meant, consider the approximation

$$\mathcal{F}(e_1, p, 1) = \{\frac{(0.9e^{i2\pi(0.7)}, 0.6e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.1)})}{x_1}\},$$

The first membership value $[0.9e^{i2\pi(0.5)}]$ under the parameter $(e_1, p, 1)$ for the first vehicle type (x_1) indicates that before testing the vehicle, expert p is of the opinion that the purchase cost of vehicles of type one is very high with degree 0.9, but after testing the same vehicle he retreated from his earlier opinion, and now opined that the purchase cost being high is only of degree 0.7. While the second membership value $[0.6e^{i2\pi(0.4)}]$ for the same parameter and vehicle indicates that before testing the vehicle, expert p is of the opinion that the purchase cost of vehicles of type one is medium with degree 0.6, but after testing the same vehicle he retreated from his earlier opinion, and now opined that the purchase cost medium is only of degree 0.4. For three membership value $[0.2e^{i2\pi(0.1)}]$ for the same parameter and vehicle indicates that before testing the vehicle, expert p is of the opinion that the purchase cost of vehicles of type one is very low with degree 0.2, but after trying out the same vehicle he retreated from his earlier opinion, and now opined that the purchase cost low is only of degree 0.1.

We present the concept of the subset and equality operations on two $\mathcal{CMFSES}s$ in the following definitions.

Definition 4.3. Let $\mathcal{A}, \mathcal{B} \in E$. Let $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ be two $\mathcal{CM}^{\mathrm{k}}\mathcal{FSES}s$ over U. Now, $(\mathcal{F}, \mathcal{A})$ is said to be a $\mathcal{CM}^{\mathrm{k}}\mathcal{FSE}$ subset of $(\mathcal{G}, \mathcal{B})$ if,

1. $\mathcal{A} \subseteq \mathcal{B}$ and

2. $\forall e \in \mathcal{A}, \mathcal{F}(e) \sqsubseteq \mathcal{G}(e)$.

In this case , we write $(\mathcal{F}, \mathcal{A}) \sqsubseteq (\mathcal{G}, \mathcal{B})$.

Definition 4.4. Let $\mathcal{A}, \mathcal{B} \in E$. Let $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ be two $\mathcal{CM}^{\mathrm{k}}\mathcal{FSES}s$ over U. $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ are said to be a $\mathcal{CM}^{\mathrm{k}}\mathcal{FSE}$ equal if $(\mathcal{F}, \mathcal{A})$ is a $\mathcal{CM}^{\mathrm{k}}\mathcal{FSE}$ subset of $(\mathcal{G}, \mathcal{B})$ and $(\mathcal{G}, \mathcal{B})$ is a $\mathcal{CM}^{\mathrm{k}}\mathcal{FSE}$ subset of $(\mathcal{F}, \mathcal{A})$.

Now, we suggest the definitions of an agree-$\mathcal{CMFSES}s$ and the disagree-$\mathcal{CMFSES}s$.

Definition 4.5. An agree-$\mathcal{CM}^{\mathrm{k}}\mathcal{FSES}$ $(\mathcal{F},\mathcal{A})_1$ over U is a $\mathcal{CM}^{\mathrm{k}}\mathcal{FSE}$ subset of $(\mathcal{F},\mathcal{A})$ defined as follows:

$$(\mathcal{F},\mathcal{A})_1 = \{\mathcal{F}_1(\alpha) : \alpha \in E \times X \times \{1\}\}.$$

Definition 4.6. A disagree-$\mathcal{CM}^{\mathrm{k}}\mathcal{FSES}$ $(\mathcal{F},\mathcal{A})_0$ over U is a $\mathcal{CM}^{\mathrm{k}}\mathcal{FSE}$ subset of $(\mathcal{F},\mathcal{A})$ defined as follows:

$$(\mathcal{F},\mathcal{A})_0 = \{\mathcal{F}_0(\alpha) : \alpha \in E \times X \times \{0\}\}.$$

5 Basic Operations on Complex Multi-Fuzzy Soft Expert Sets

In this section, we introduce some basic theoretic operations on complex multi-fuzzy soft expert sets such as the complement, union, intersection, AND and OR. We also give some theorems of complex multi-fuzzy soft expert sets, which are commutative, distributive, De Morgan's law and other pertaining laws.

5.1 Complement of Complex Multi-Fuzzy Soft Expert Sets

We define the complement operation for complex multi-fuzzy soft expert set, give an illustrative example and a proof of a proposed proposition.

Definition 5.1. $(\mathcal{F},\mathcal{A})$ be a $\mathcal{CM}^{\mathrm{k}}\mathcal{FSES}$ over U. The complement of $(\mathcal{F},\mathcal{A})$ is defined by $(\mathcal{F},\mathcal{A})^c = (\mathcal{F}^c,\mathcal{A})$, such that $\mathcal{F}^c : \mathcal{A} \to \mathcal{CM}^{\mathrm{k}}(U)$ is a mapping given by

$$\mathcal{F}^c(e) = \{\langle x, \mu^j_{\mathcal{F}^c(e)}(x) = r^j_{\mathcal{F}^c(e)}(x).e^{i\omega^j_{\mathcal{F}^c(e)}(x)}\rangle : e \in \mathcal{A}, x \in U, j = 1,2,...,k\}.$$

where the complement of the amplitude term is $r^j_{\mathcal{F}^c(e)}(x) = 1 - r^j_{\mathcal{F}(e)}(x)$ and the complement of the phase term is $\omega^j_{\mathcal{F}^c(e)}(x) = 2\pi - i\omega^j_{\mathcal{F}(e)}(x)$.

Example 5.2. Consider the approximation given in Example 4.2, where

$$\mathcal{F}(e_1,p,1)=\{\frac{(0.9e^{i2\pi(0.5)},0.6e^{i2\pi(0.4)},0.2e^{i2\pi(0.1)})}{x_1},\frac{(0.5e^{i2\pi(0.9)},0.4e^{i2\pi(0.8)},0.5e^{i2\pi(1)})}{x_2}\},$$

By using Definition 5.1, we obtain the complement of the approximation as

$$\mathcal{F}^c(e_1,p,1)=\{\frac{(0.1e^{i2\pi(0.5)},0.4e^{i2\pi(0.6)},0.8e^{i2\pi(0.9)})}{x_1},\frac{(0.5e^{i2\pi(0.1)},0.6e^{i2\pi(0.2)},0.5e^{i2\pi(0)})}{x_2}\}.$$

Proposition 5.3. *If* $(\mathcal{F},\mathcal{A})$ *is a* $\mathcal{CM}^k\mathcal{FSES}$ *over* U*, then* $((\mathcal{F},\mathcal{A})^c)^c=(\mathcal{F},\mathcal{A})$.

Proof. From Definition 5.1, we have $(\mathcal{F},\mathcal{A})^c=(\mathcal{F}^c,\mathcal{A})$ where

$$(\mathcal{F},\mathcal{A})^c=\{\langle x,r^j_{\mathcal{F}^c(e)}(x).e^{i\omega^j_{\mathcal{F}^c(e)}(x)}\rangle:e\in\mathcal{A},x\in U,j=1,2,...,k\},$$

$$=\{\langle x,[1-r^j_{\mathcal{F}(e)}(x)].e^{i[2\pi-\omega^j_{\mathcal{F}(e)}(x)]}\rangle:e\in\mathcal{A},x\in U,j=1,2,...,k\},$$

Now let $(\mathcal{F},\mathcal{A})^c=(\mathcal{G},\mathcal{B})=(\mathcal{F}^c,A)$. Then we obtain the following:

$$(\mathcal{G},\mathcal{B})^c=\{\langle x,[1-r^j_{\mathcal{F}^c(e)}(x)].e^{i[2\pi-\omega^j_{\mathcal{F}^c(e)}(x)]}\rangle:e\in\mathcal{A},x\in U,j=1,2,...,k\},$$

$$=\{\langle x,[1-(1-r^j_{\mathcal{F}(e)}(x))].e^{i[2\pi-(2\pi-\omega^j_{\mathcal{F}(e)}(x))]}\rangle:e\in\mathcal{A},x\in U,j=1,2,...,k\},$$

$$=\{\langle x,r^j_{\mathcal{F}(e)}(x).e^{i\omega^j_{\mathcal{F}(e)}(x)}\rangle:e\in\mathcal{A},x\in U,j=1,2,...,k\},$$

$$=(\mathcal{F},\mathcal{A}).$$

5.2 Union and Intersection of Complex Multi-Fuzzy Soft Expert Sets

In this part, we introduce the definitions of union and intersection operations of two $\mathcal{CMFSES}s$, and provide an illustrative example. The union operation utilizes the maximum operator, whereas the intersection operation employs the minimum operator.

Definition 5.4. The union of two $\mathcal{CM}^k\mathcal{FSES}s$ $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$ over U, denoted by $(\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B})$, is a complex multi-fuzzy soft expert set $(\mathcal{H},\mathcal{C})$, where $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, $\forall e \in \mathcal{C}$ and $x \in U$,

$$\mathcal{H}(e) = \begin{cases} \mathcal{F}(e) = [r^j_{\mathcal{F}(e)}(x) e^{i\omega^j_{\mathcal{F}(e)}(x)}]_{j\in k} & \text{if } e \in \mathcal{A} - \mathcal{B}, \\ \mathcal{G}(e) = [r^j_{\mathcal{G}(e)}(x) e^{i\omega^j_{\mathcal{G}(e)}(x)}]_{j\in k} & \text{if } e \in \mathcal{B} - \mathcal{A}, \\ \mathcal{F}(e) \sqcup \mathcal{G}(e) = [max(r^j_{\mathcal{F}(e)}(x),\ r^j_{\mathcal{G}(e)}(x)). & \\ \qquad e^{i\ max[\omega^j_{\mathcal{F}(e)}(x),\ \omega^j_{\mathcal{G}(e)}(x)]}]_{j\in k} & \text{if } e \in \mathcal{A} \cap \mathcal{B}. \end{cases}$$

We write $(\mathcal{H}, \mathcal{C}) = (\mathcal{F}, \mathcal{A}) \tilde{\cup} (\mathcal{G}, \mathcal{B})$.

Example 5.5. Let $U = \{x_1, x_2\}$ be a universal set, $E = \{e_1, e_2, e_3\}$ be a set of parameters, and $\mathcal{X} = \{p, q\}$ be the set of experts such that $\mathcal{A} = \{(e_1, p, 1), (e_2, p, 0)\}$, $\mathcal{B} = \{(e_1, p, 1), (e_2, p, 0), (e_3, p, 1)\}$. Suppose that $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ are two complex multi-fuzzy soft sets of dimension three over U defined as follows.

$$\begin{aligned}(\mathcal{F}, A) \\ &= \{(e_1, p, 1) = \{\tfrac{(0.5e^{i2\pi(0.5)}, 0.7e^{i2\pi(0.5)}, 0.9e^{i2\pi(0.4)})}{x_1}, \tfrac{(0.5e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.8)})}{x_2}\}, \\ &\quad (e_2, p, 0) = \{\tfrac{(0.6e^{i2\pi(0.6)}, 0.8e^{i2\pi(0.7)}, 0.5e^{i2\pi(0.8)})}{x_1}, \tfrac{(0.6e^{i2\pi(0.5)}, 0.7e^{i2\pi(0.4)}, 0.9e^{i2\pi(0.6)})}{x_2}\}\}.\end{aligned}$$

and

$$\begin{aligned}(\mathcal{G}, B) \\ &= \{(e_1, p, 1) = \{\tfrac{(0.6e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.3)}, 0.8e^{i2\pi(0.5)})}{x_1}, \tfrac{(0.6e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.2)}, 0.6e^{i2\pi(0.8)})}{x_2}\}, \\ &\quad (e_2, p, 0) = \{\tfrac{(0.9e^{i2\pi(0.9)}, 1e^{i2\pi(0.5)}, 0.2e^{i2\pi(0)})}{x_1}, \tfrac{(0.8e^{i2\pi(0.3)}, 0.7e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.7)})}{x_2}\}, \\ &\quad (e_3, p, 1) = \{\tfrac{(0.6e^{i2\pi(0.6)}, 0.8e^{i2\pi(0.7)}, 0.5e^{i2\pi(0.8)})}{x_1}, \tfrac{(0.1e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.8)}, 1e^{i2\pi(0.5)})}{x_2}\}\}.\end{aligned}$$

By using basic complex multi-fuzzy union, we have $(\mathcal{F}, \mathcal{A}) \tilde{\cup} (\mathcal{G}, \mathcal{B}) = (\mathcal{H}, \mathcal{C})$. where $\mathcal{C} = \mathcal{A} \cup \mathcal{B} = \{(e_1, p, 1), (e_2, p, 0), (e_3, p, 1)\}$ and

$$\begin{aligned}(\mathcal{H}, \mathcal{C}) \\ &= \{(e_1, p, 1) = \{\tfrac{(0.6e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.5)}, 0.9e^{i2\pi(0.5)})}{x_1}, \tfrac{(0.6e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.8)})}{x_2}\}, \\ &\quad (e_2, p, 0) = \{\tfrac{(0.9e^{i2\pi(0.9)}, 1e^{i2\pi(0.7)}, 0.5e^{i2\pi(0.8)})}{x_1}, \tfrac{(0.8e^{i2\pi(0.5)}, 0.7e^{i2\pi(0.4)}, 0.9e^{i2\pi(0.7)})}{x_2}\}, \\ &\quad (e_3, p, 1) = \{\tfrac{(0.6e^{i2\pi(0.6)}, 0.8e^{i2\pi(0.7)}, 0.5e^{i2\pi(0.8)})}{x_1}, \tfrac{(0.1e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.8)}, 1e^{i2\pi(0.5)})}{x_2}\}\}.\end{aligned}$$

Definition 5.6. The intersection of two $\mathcal{CM}^{\mathrm{k}}\mathcal{FSES}s$ $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ over U, denoted by $(\mathcal{F}, \mathcal{A}) \tilde{\cap} (\mathcal{G}, \mathcal{B})$, is a complex multi-fuzzy soft expert set

$(\mathcal{H},\mathcal{C})$, where $\mathcal{C}=\mathcal{A}\cap\mathcal{B}$, $\forall e\in\mathcal{C}$ and $x\in U$,

$$\mathcal{H}(e)=\mathcal{F}(e)\sqcap\mathcal{G}(e)=[min(r^j_{\mathcal{F}(e)}(x),\ r^j_{\mathcal{G}(e)}(x)).e^{i\ min[\omega^j_{\mathcal{F}(e)}(x),\ \omega^j_{\mathcal{G}(e)}(x)]}]_{j\in k}$$

we write $(\mathcal{H},C)=(\mathcal{F},\mathcal{A})\tilde{\cap}(\mathcal{G},\mathcal{B})$.

Example 5.7. Reconsider Example 5.5. We have $(\mathcal{F},\mathcal{A})\tilde{\cap}(\mathcal{G},\mathcal{B})=(\mathcal{H},\mathcal{C})$, where $\mathcal{C}=\mathcal{A}\cap\mathcal{B}=\{(e_1,p,1),(e_2,p,0)\}$, and

$$\begin{aligned}(\mathcal{H},C)&\\ =\{(e_1,p,1)&=\{\tfrac{(0.5e^{i2\pi(0.5)},0.7e^{i2\pi(0.3)},0.8e^{i2\pi(0.4)})}{x_1},\tfrac{(0.5e^{i2\pi(0.4)},0.3e^{i2\pi(0.2)},0.6e^{i2\pi(0.8)})}{x_2}\},\\ (e_2,p,0)&=\{\tfrac{(0.6e^{i2\pi(0.6)},18e^{i2\pi(0.5)},0.2e^{i2\pi(0)})}{x_1},\tfrac{(0.6e^{i2\pi(0.3)},0.7e^{i2\pi(0.4)},0.7e^{i2\pi(0.6)})}{x_2}\}\}.\end{aligned}$$

We will now give some theorems on the union, intersection and complement of complex multi-fuzzy soft expert sets. These theorems illustrate the relationship between the set theoretic operations that have been mentioned above.

Theorem 5.8. *Let $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$ be two $\mathcal{CM}^k\mathcal{FSES}s$ on U . Then the following commutative laws hold true*

1. $(\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B})=(\mathcal{G},\mathcal{B})\tilde{\cup}(\mathcal{F},\mathcal{A})$,
2. $(\mathcal{F},\mathcal{A})\tilde{\cap}(\mathcal{G},\mathcal{B})=(\mathcal{G},\mathcal{B})\tilde{\cap}(\mathcal{F},\mathcal{A})$.

. **Proof.** Suppose that $((\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B}))=(\mathcal{H},\mathcal{C})$ where $\mathcal{C}=\mathcal{A}\cup\mathcal{B}$. By Definition 5.4, then we have $((\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B}))$ is a $\mathcal{CM}^k\mathcal{FSES}$ $(\mathcal{H},\mathcal{C})$, where $\mathcal{C}=\mathcal{A}\cup\mathcal{B}$ and $\forall e\in\mathcal{C}$,

$$\mathcal{H}(e)=\begin{cases}\mathcal{F}(e)=[r^j_{\mathcal{F}(e)}(x)e^{i\omega^j_{\mathcal{F}(e)}(x)}]_{j\in k} & \text{if } e\in\mathcal{A}-\mathcal{B},\\ \mathcal{G}(e)=[r^j_{\mathcal{G}(e)}(x)e^{i\omega^j_{\mathcal{G}(e)}(x)}]_{j\in k} & \text{if } e\in\mathcal{B}-\mathcal{A},\\ \mathcal{F}(e)\sqcup\mathcal{G}(e)=[max(r^j_{\mathcal{F}(e)}(x),\ r^j_{\mathcal{G}(e)}(x)). & \\ \qquad e^{i\ max[\omega^j_{\mathcal{F}(e)}(x),\ \omega^j_{\mathcal{G}(e)}(x)]}]_{j\in k} & \text{if } e\in\ \mathcal{A}\cap\mathcal{B}.\end{cases}$$

We consider the case when $e\in\mathcal{A}\cap\mathcal{B}$ as the other cases are trivial. Then we have

$$
\begin{aligned}
(\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B}) &= \mathcal{F}(e)\sqcup\mathcal{G}(e)\\
&= [max(r^j_{\mathcal{F}(e)}(x),\ r^j_{\mathcal{G}(e)}(x)).e^{i\ max[\omega^j_{\mathcal{F}(e)}(x),\ \omega^j_{\mathcal{G}(e)}(x)]}]_{j\in k}\\
&= [max(r^j_{\mathcal{G}(e)}(x),\ r^j_{\mathcal{F}(e)}(x)).e^{i\ max[\omega^j_{\mathcal{G}(e)}(x),\ \omega^j_{\mathcal{F}(e)}(x)]}]_{j\in k}\\
&= \mathcal{G}(e)\sqcup\mathcal{F}(e)\\
&= (\mathcal{G},\mathcal{B})\tilde{\cup}(\mathcal{F},\mathcal{A})
\end{aligned}
$$

Therefore, we have $(\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B}) = (\mathcal{G},\mathcal{B})\tilde{\cup}(\mathcal{F},\mathcal{A})$. Thus the first assertion of Theorem 5.8 is proven. Likewise, we can prove the second assertion too.

Theorem 5.9. *Let $(\mathcal{F},\mathcal{A})$, $(\mathcal{G},\mathcal{B})$ and $(\mathcal{Q},\mathcal{D})$ be three $\mathcal{CM}^k\mathcal{FSES}$s on U. Then the following distributive laws hold true.*

1. $(\mathcal{F},\mathcal{A})\tilde{\cup}((\mathcal{G},\mathcal{B})\tilde{\cap}(\mathcal{Q},\mathcal{D})) = ((\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B}))\tilde{\cap}((\tilde{\mathcal{F}},\mathcal{A})\tilde{\cup}(\mathcal{Q},\mathcal{D}))$,
2. $(\mathcal{F},\mathcal{A})\tilde{\cap}((\mathcal{G},\mathcal{B})\tilde{\cup}(\mathcal{Q},\mathcal{D})) = ((\mathcal{F},\mathcal{A})\tilde{\cap}(\tilde{\mathcal{G}},\mathcal{B}))\tilde{\cup}((\tilde{\mathcal{F}},A)\tilde{\cap}(\mathcal{Q},\mathcal{D}))$.

Proof. We will provide the proof of assertion 1 since the proof of assertion 2 is straightforward from Definition 5.4 and Definition 5.6. Assume that $((\mathcal{G},\mathcal{B})\tilde{\cap}(\mathcal{Q},\mathcal{D})) = (\mathcal{T},\mathcal{P})$ where $\mathcal{P}=\mathcal{B}\cap\mathcal{D}$. By Definition 5.6 we have $((\mathcal{G},\mathcal{B})\tilde{\cap}(\mathcal{Q},\mathcal{D}))$ to be a $\mathcal{CM}^{\mathrm{k}}\mathcal{FSES}$ $(\mathcal{T},\mathcal{P})$, where $\mathcal{P}=\mathcal{B}\cap\mathcal{D}$ and $\forall e\in\mathcal{P}$, such that

$$
\mathcal{T}(e)=\mathcal{G}(e)\sqcap\mathcal{Q}(e)=[min(r^j_{\mathcal{G}(e)}(x),\ r^j_{\mathcal{Q}(e)}(x)).e^{i\ min[\omega^j_{\mathcal{G}(e)}(x),\ \omega^j_{\mathcal{Q}(e)}(x)]}]_{j\in k},
$$

Suppose that $(\mathcal{S},\mathcal{R}) = ((\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{T},\mathcal{P}))$ where $\mathcal{R}=\mathcal{A}\cup\mathcal{P}$. By Definition 5.4, then we have $((\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{T},\mathcal{P}))$ to be a $\mathcal{CM}^{\mathrm{k}}\mathcal{FSES}$ $(\mathcal{S},\mathcal{R})$, where $\mathcal{R}=\mathcal{A}\cup\mathcal{P}$ and $\forall e\in\mathcal{R}$, such that

$$
\mathcal{S}(e)=\begin{cases}
\mathcal{F}(e)=[r^j_{\mathcal{F}(e)}(x)e^{i\omega^j_{\mathcal{F}(e)}(x)}]_{j\in k} & \text{if } e\in\mathcal{A}-\mathcal{P},\\
\mathcal{T}(e)=[r^j_{\mathcal{T}(e)}(x)e^{i\omega^j_{\mathcal{T}(e)}(x)}]_{j\in k} & \text{if } e\in\mathcal{P}-\mathcal{A},\\
\mathcal{F}(e)\sqcup\mathcal{T}(e)=[max(r^j_{\mathcal{F}(e)}(x),\ r^j_{\mathcal{T}(e)}(x)). & \\
\qquad e^{i\ max[\omega^j_{\mathcal{F}(e)}(x),\ \omega^j_{\mathcal{T}(e)}(x)]}]_{j\in k} & \text{if } e\in\ \mathcal{A}\cap\mathcal{P}.
\end{cases}
$$

Now let $(\mathcal{F},\mathcal{A})\tilde{\cup}((\mathcal{G},\mathcal{B})\tilde{\cap}(\mathcal{Q},\mathcal{D})) = ((\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{T},\mathcal{P}))$. We consider the case when $e\in\mathcal{A}\cap\mathcal{P}$ as the other cases are trivial. Hence,

$$
\begin{aligned}
(\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{T},\mathcal{P}) &= \mathcal{F}(e)\sqcup\mathcal{T}(e)\\
&= [max(r^j_{\mathcal{F}(e)}(x),\ r^j_{\mathcal{T}(e)}(x)).e^{i\ max[\omega^j_{\mathcal{F}(e)}(x),\ \omega^j_{\mathcal{T}(e)}(x)]}]_{j\in k}
\end{aligned}
$$

$$
\begin{aligned}
&= [max(r^j_{\mathcal{F}(e)}(x),\ r^j_{\mathcal{G}(e)\sqcap\mathcal{Q}(e)}(x)).e^{i\ max(\omega^j_{\mathcal{F}(e)}(x),\ \omega^j_{\mathcal{G}(e)\sqcap\mathcal{Q}(e)}(x))}]_{j\in k}\\
&= [max(r^j_{\mathcal{F}(e)}(x), min[r^j_{\mathcal{G}(e)}(x), r^j_{\mathcal{Q}(e)}(x)]).e^{i\ max(\omega^j_{\mathcal{F}(e)}(x), min[\omega^j_{\mathcal{G}(e)}(x),\ \omega^j_{\mathcal{Q}(e)}(x)])}]_{j\in k}\\
&= [min(max[r^j_{\mathcal{F}(e)}(x), r^j_{\mathcal{G}(e)}(x)], max[r^j_{\mathcal{F}(e)}(x), r^j_{\mathcal{Q}(e)}(x)])\\
&\qquad .e^{i\ min(max[\omega^j_{\mathcal{F}(e)}(x),\ \omega^j_{\mathcal{G}(e)}(x)], max[\omega^j_{\mathcal{F}(e)}(x),\ \omega^j_{\mathcal{Q}(e)}(x)])}]_{j\in k}\\
&= [min(r^j_{\mathcal{F}(e)\sqcup\mathcal{G}(e)}(x), r^j_{\mathcal{F}(e)\sqcup\mathcal{Q}(e)}(x)).e^{i\ min(\omega^j_{\mathcal{F}(e)\sqcup\mathcal{G}(e)}(x), \omega^j_{\mathcal{F}(e)\sqcup\mathcal{Q}(e)}(x))}]_{j\in k}\\
&= (\mathcal{F}(e)\sqcup\mathcal{G}(e))\sqcap(\mathcal{F}(e)\sqcup\mathcal{Q}(e))\\
&= ((\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B}))\tilde{\cap}((\tilde{\mathcal{F}},\mathcal{A})\tilde{\cup}(\mathcal{Q},\mathcal{D}))
\end{aligned}
$$

Therefore, we have $(\mathcal{F},\mathcal{A})\tilde{\cup}((\mathcal{G},\mathcal{B})\tilde{\cap}(\mathcal{Q},\mathcal{D})) = ((\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B}))\tilde{\cap}((\tilde{\mathcal{F}},\mathcal{A})\tilde{\cup}(\mathcal{Q},\mathcal{D}))$.

Theorem 5.10. *Let $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$ be two $\mathcal{CM}^k\mathcal{FSES}s$ over U. Then the following De Morgans's laws hold true.*

1. $((\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B}))^c = (\mathcal{F},\mathcal{A})^c\tilde{\cap}\mathcal{G},\mathcal{B})^c$,
2. $((\mathcal{F},\mathcal{A})\tilde{\cap}(\mathcal{G},\mathcal{B}))^c = (\mathcal{F},\mathcal{A})^c\tilde{\cup}(\mathcal{G},\mathcal{B})^c$.

Proof. The proofs are straight forward by using the Definitions 5.1, 5.4 and 5.6.

5.3 AND and OR Operations for Complex Multi-Fuzzy Soft Expert Sets

We will now give the notion of AND and OR operations on two $\mathcal{CMFSES}s$ with a proposition of these two operations.

Definition 5.11. Let $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$ be two $\mathcal{CM}^k\mathcal{FSES}s$ over U. Then " $(\mathcal{F},\mathcal{A})$ AND $(\mathcal{G},\mathcal{B})$" is defined by

$$(\mathcal{F},\mathcal{A})\wedge(\mathcal{G},\mathcal{B}) = (\mathcal{O},\mathcal{A}\times\mathcal{B})$$

where $(\mathcal{O},\mathcal{A}\times\mathcal{B}) = \mathcal{O}(\alpha,\beta)$, such that $\mathcal{O}(\alpha,\beta) = \mathcal{F}(\alpha)\cap\mathcal{G}(\beta)$, for all $(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}$, and $\cap$ represents the complex multi-fuzzy intersection.

Definition 5.12. Let $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$ be two $\mathcal{CM}^k\mathcal{FSES}s$ over U. Then " $(\mathcal{F},\mathcal{A})$ OR $(\mathcal{G},\mathcal{B})$" is defined by

$$(\mathcal{F},\mathcal{A})\vee(\mathcal{G},\mathcal{B}) = (\mathcal{O},\mathcal{A}\times\mathcal{B})$$

where $(\mathcal{O}, \mathcal{A} \times \mathcal{B}) = \mathcal{O}(\alpha, \beta)$, such that $\mathcal{O}(\alpha, \beta) = \mathcal{F}(\alpha) \cup \mathcal{G}(\beta)$, for all $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$, and $\cup$ represents the complex multi-fuzzy union.

Proposition 5.13. *If* $(\mathcal{F}, \mathcal{A})$ *and* $(\mathcal{G}, \mathcal{B})$ *are two* $\mathcal{CM}^k\mathcal{FSES}s$ *over* U*, then we have the following properties.*

1. $((\mathcal{F}, \mathcal{A}) \vee (\mathcal{G}, \mathcal{B})^c = (\mathcal{F}, \mathcal{A})^c \wedge (\mathcal{G}, \mathcal{B})^c$,

2. $((\mathcal{F}, \mathcal{A}) \wedge (\mathcal{G}, \mathcal{B})^c = (\mathcal{F}, \mathcal{A})^c \vee (\mathcal{G}, \mathcal{B})^c$.

Proof. Suppose that $(\mathcal{F}, \mathcal{A}) \vee (\mathcal{G}, \mathcal{B}) = (\mathcal{O}, \mathcal{A} \times \mathcal{B})$. Then, we have $((\mathcal{F}, \mathcal{A}) \vee (\mathcal{G}, \mathcal{B}))^c = (\mathcal{O}, \mathcal{A} \times \mathcal{B})^c = (\mathcal{O}^c, (\mathcal{A} \times \mathcal{B}))$.

Hence, $(\mathcal{F}, \mathcal{A})^c \wedge (\mathcal{G}, \mathcal{B})^c = (\mathcal{F}^c, \mathcal{A}) \wedge (\mathcal{G}^c, \mathcal{B}) = (\mathcal{J}, (\mathcal{A} \times \mathcal{B})$,
where
$\mathcal{J}(\alpha, \beta) = \{\langle x, [min(r^j_{\mathcal{F}^c(\alpha)}(x), r^j_{\mathcal{G}^c(\beta)}(x)).e^{i\ min[\omega^j_{\mathcal{F}^c(\alpha)}(x),\ \omega^j_{\mathcal{G}^c(\beta)}(x)]}]_{j \in k}\rangle : (\alpha, \beta) \in (\mathcal{A} \times \mathcal{B}), x \in U\}$,

We take $(\alpha, \beta) \in (\mathcal{A} \times \mathcal{B})$, therefore,

$$
\begin{aligned}
\mathcal{O}^c(\alpha, \beta) &= \{\langle x, [r^j_{\mathcal{O}^c(\alpha,\beta)}(x).e^{i[\omega^j_{\mathcal{O}^c(\neg\alpha,\neg\beta)}(x)]}]_{j \in k}\rangle : (\alpha, \beta) \in (\mathcal{A} \times \mathcal{B}), x \in U\} \\
&= \{\langle x, [1 - r^j_{\mathcal{O}(\alpha,\beta)}(x).e^{i[2\pi - \omega^j_{\mathcal{O}(\alpha,\beta)}(x)]}]_{j \in k}\rangle : (\alpha, \beta) \in (\mathcal{A} \times \mathcal{B}), x \in U\} \\
&= \{\langle x, [1 - max(r^j_{\mathcal{F}(\alpha)}(x), r^j_{\mathcal{G}(\beta)}(x)).e^{i\ 2\pi - max[\omega^j_{\mathcal{F}(\alpha)}(x), \omega^j_{\mathcal{G}(\beta)}(x)]}]_{j \in k}\rangle : (\alpha, \beta) \in \\
&\qquad (\mathcal{A} \times \mathcal{B}), x \in U\} \\
&= \{\langle x, [min(1 - r^j_{\mathcal{F}(\alpha)}(x), 1 - r^j_{\mathcal{G}(\beta)}(x)).e^{i\ min[2\pi - \omega^j_{\mathcal{F}(\alpha)}(x),\ 2\pi - \omega^j_{\mathcal{G}(\beta)}(x)]}]_{j \in k}\rangle : \\
&\qquad (\alpha, \beta) \in (\mathcal{A} \times \mathcal{B}), x \in U\} \\
&= \{\langle x, [min(r^j_{\mathcal{F}^c(\alpha)}(x), r^j_{\mathcal{G}^c(\beta)}(x)).e^{i\ min[\omega^j_{\mathcal{F}^c(\neg\alpha)}(x),\ \omega^j_{\mathcal{G}^c(\neg\beta)}(x)]}]_{j \in k}\rangle : (\alpha, \beta) \in \\
&\qquad (\mathcal{A} \times \mathcal{B}), x \in U\} \\
&= \mathcal{J}(\alpha, \beta)
\end{aligned}
$$

Hence, $\mathcal{O}^c$ and $\mathcal{J}$ are the same operator, thus the first assertion of Proposition 2 is proven. Likewise, we can prove the second assertion too.

6 Application

In this section, we present an application of complex multi-fuzzy soft expert set theory in a decision making problem by using the mean of each multi-fuzzy soft expert set.

Example 6.1. Suppose we are interested in detecting methods that can be used in learning foreign languages. For example, teaching the English language for Malaysian school students who are non-speakers of English. Suppose we take three learning methods which are represented in the universal set $U = \{x_1, x_2, x_3\}$ where x_1 = grammar-translation method, x_2 = communicative language method, x_3 = audio-lingual method. We need to choose the most appropriate method out of these three in teaching English for non-native English speakers. Suppose $E = \{e_1, e_2\}$ is a set of parameters that represents the students' study level where e_1 stands for "primary school "studying English from Grade 2 to Grade 6 while e_2 stands for "secondary school ", of Grade 7 to Grade 11. Suppose $\mathcal{X} = \{p, q\}$ be a set of educational experts who are assigned to analyze these three methods of determining the degree and the total time of the influence of these methods on the school students as in the following complex multi-fuzzy soft expert set:

$$
\begin{aligned}
(\mathcal{F}, A) = \Big\{(e_1, p, 1) = \{ & \frac{(0.2e^{i2\pi(\frac{11}{12})},0.1e^{i2\pi(\frac{10}{12})},0.3e^{i2\pi(\frac{9}{12})},0.4e^{i2\pi(\frac{8}{12})},0.2e^{i2\pi(\frac{9}{12})})}{x_1}, \\
& \frac{(0.9e^{i2\pi(\frac{3}{12})},1e^{i2\pi(\frac{4}{12})},0.8e^{i2\pi(\frac{4}{12})}0.9e^{i2\pi(\frac{3}{12})},0.8e^{i2\pi(\frac{5}{12})})}{x_2}, \\
& \frac{(0.7e^{i2\pi(\frac{6}{12})},0.4e^{i2\pi(\frac{7}{12})},0.4e^{i2\pi(\frac{9}{12})},0.5e^{i2\pi(\frac{7}{12})},0.3e^{i2\pi(\frac{6}{12})})}{x_3}\}, \\
(e_1, q, 1) = \{ & \frac{(0.3e^{i2\pi(\frac{4}{12})},0.4e^{i2\pi(\frac{7}{12})},0.4e^{i2\pi(\frac{8}{12})},0.3e^{i2\pi(\frac{6}{12})},0.2e^{i2\pi(\frac{10}{12})})}{x_1}, \\
& \frac{(0.8e^{i2\pi(\frac{5}{12})},0.7e^{i2\pi(\frac{4}{12})},0.9e^{i2\pi(\frac{5}{12})},1e^{i2\pi(\frac{5}{12})},0.7e^{i2\pi(\frac{3}{12})})}{x_2}, \\
& \frac{(0.4e^{i2\pi(\frac{5}{12})},0.5e^{i2\pi(\frac{6}{12})},0.1e^{i2\pi(\frac{0}{12})},0.2e^{i2\pi(\frac{5}{12})},0.2e^{i2\pi(\frac{5}{12})})}{x_3}\}, \\
(e_2, p, 1) = \{ & \frac{(0.3e^{i2\pi(\frac{8}{12})},0.5e^{i2\pi(\frac{9}{12})},0.6e^{i2\pi(\frac{7}{12})},0.4e^{i2\pi(\frac{5}{12})},0.4e^{i2\pi(\frac{6}{12})})}{x_1}, \\
& \frac{(0.8e^{i2\pi(\frac{5}{12})},1e^{i2\pi(\frac{6}{12})},0.9e^{i2\pi(\frac{4}{12})},0.9e^{i2\pi(\frac{5}{12})},0.8e^{i2\pi(\frac{5}{12})})}{x_2}, \\
& \frac{(0.7e^{i2\pi(\frac{4}{12})},0.2e^{i2\pi(\frac{3}{12})},0.1e^{i2\pi(\frac{0}{12})},0.4e^{i2\pi(\frac{5}{12})},0.3e^{i2\pi(\frac{6}{12})})}{x_3}\}, \\
(e_2, q, 1) = \{ & \frac{(0.7e^{i2\pi(\frac{5}{12})},0.6e^{i2\pi(\frac{5}{12})},0.4e^{i2\pi(\frac{9}{12})},0.5e^{i2\pi(\frac{7}{12})},0.6e^{i2\pi(\frac{6}{12}})}{x_1}, \\
& \frac{(0.7e^{i2\pi(\frac{6}{12})},0.6e^{i2\pi(\frac{4}{12})},0.9e^{i2\pi(\frac{5}{12})},0.6e^{i2\pi(\frac{8}{12})},0.7e^{i2\pi(\frac{8}{12})})}{x_2}, \\
& \frac{(0.5e^{i2\pi(\frac{3}{12})},0.7e^{i2\pi(\frac{4}{12})},0.2e^{i2\pi(\frac{5}{12})},0.3e^{i2\pi(\frac{6}{12})},0.4e^{i2\pi(\frac{5}{12})})}{x_3}\},
\end{aligned}
$$

$$(e_1,p,0)=\{\frac{(0.3e^{i2\pi(\frac{6}{12})},0.2e^{i2\pi(\frac{5}{12})},0.1e^{i2\pi(\frac{6}{12})},0.3e^{i2\pi(\frac{7}{12})},0.4e^{i2\pi(\frac{8}{12})})}{x_1},$$
$$\frac{(0.8e^{i2\pi(\frac{3}{12})},0.7e^{i2\pi(\frac{5}{12})},0.6e^{i2\pi(\frac{4}{12})},0.4e^{i2\pi(\frac{3}{12})},0.8e^{i2\pi(\frac{3}{12})})}{x_2},$$
$$\frac{(0.3e^{i2\pi(\frac{6}{12})},0.6e^{i2\pi(\frac{5}{12})},0.1e^{i2\pi(\frac{4}{12})},0.3e^{i2\pi(\frac{7}{12})},0.4e^{i2\pi(\frac{7}{12})})}{x_3}\},$$
$$(e_1,q,0)=\{\frac{(0.4e^{i2\pi(\frac{7}{12})},0.4e^{i2\pi(\frac{6}{12})},0.5e^{i2\pi(\frac{8}{12})},0.3e^{i2\pi(\frac{8}{12})},0.4e^{i2\pi(\frac{9}{12})})}{x_1},$$
$$\frac{(0.7e^{i2\pi(\frac{4}{12})},0.6e^{i2\pi(\frac{4}{12})},0.9e^{i2\pi(\frac{3}{12})},0.7e^{i2\pi(\frac{5}{12})},0.6e^{i2\pi(\frac{4}{12})})}{x_2},$$
$$\frac{(0.5e^{i2\pi(\frac{3}{12})},0.3e^{i2\pi(\frac{6}{12})},0.5e^{i2\pi(\frac{7}{12})},0.6e^{i2\pi(\frac{6}{12})},0.3e^{i2\pi(\frac{4}{12})})}{x_3}\},$$
$$(e_2,p,0)=\{\frac{(0.7e^{i2\pi(\frac{4}{12})},0.5e^{i2\pi(\frac{6}{12})},0.6e^{i2\pi(\frac{7}{12})},0.4e^{i2\pi(\frac{6}{12})},0.4e^{i2\pi(\frac{7}{12})})}{x_1},$$
$$\frac{(0.6e^{i2\pi(\frac{3}{12})},0.7e^{i2\pi(\frac{2}{12})},0.6e^{i2\pi(\frac{3}{12})},0.8e^{i2\pi(\frac{4}{12})},0.8e^{i2\pi(\frac{4}{12})})}{x_2},$$
$$\frac{(0.4e^{i2\pi(\frac{5}{12})},0.2e^{i2\pi(\frac{6}{12})},0.1e^{i2\pi(\frac{7}{12})},0.3e^{i2\pi(\frac{5}{12})},0.3e^{i2\pi(\frac{4}{12})})}{x_3}\},$$
$$(e_2,q,0)=\{\frac{(0.3e^{i2\pi(\frac{6}{12})},0.4e^{i2\pi(\frac{7}{12})},0.5e^{i2\pi(\frac{6}{12})},0.3e^{i2\pi(\frac{8}{12})},0.4e^{i2\pi(\frac{7}{12})})}{x_1},$$
$$\frac{(0.7e^{i2\pi(\frac{3}{12})},0.8e^{i2\pi(\frac{4}{12})},0.7e^{i2\pi(\frac{4}{12})},0.9e^{i2\pi(\frac{3}{12})},0.8e^{i2\pi(\frac{5}{12})})}{x_2},$$
$$\frac{(0.3e^{i2\pi(\frac{9}{12})},0.2e^{i2\pi(\frac{8}{12})},0.1e^{i2\pi(\frac{5}{12})},0.5e^{i2\pi(\frac{9}{12})},0.3e^{i2\pi(\frac{5}{12})})}{x_3}\}\Big\}.$$

In this example, the amplitude term of the membership values represents the degree of influence of the above-mentioned methods on students, whereas the phase term represents the time it takes for the effects of the methods to affect the school students. The values for these amplitude and phase terms can be determined or calculated through the data obtained from educational experts, such as the teachers and moderators. Both of the amplitude and phase terms lie in [0, 1]. An amplitude term with value close to 0 (1) implies that the above- mentioned methods have a very little (strong) influence on a student's performance and a phase term with value close to 0 (1) implies that the above- mentioned methods take a very short (long) time to enable students to speak English.

To illustrate what we meant, consider the approximation

$$\mathcal{F}(e_1,p,1)=\{\frac{(0.2e^{i2\pi(\frac{11}{12})},0.1e^{i2\pi(\frac{10}{12})},0.3e^{i2\pi(\frac{9}{12})},0.4e^{i2\pi(\frac{8}{12})},0.2e^{i2\pi(\frac{9}{12})})}{x_1},$$
$$\frac{(0.9e^{i2\pi(\frac{3}{12})},1e^{i2\pi(\frac{4}{12})},0.8e^{i2\pi(\frac{4}{12})}0.9e^{i2\pi(\frac{3}{12})},0.8e^{i2\pi(\frac{5}{12})})}{x_2},\ldots\}.$$

The term $\frac{(0.2e^{i2\pi(\frac{11}{12})},0.1e^{i2\pi(\frac{10}{12})},0.3e^{i2\pi(\frac{9}{12})},0.4e^{i2\pi(\frac{8}{12})},0.2e^{i2\pi(\frac{9}{12})})}{x_1}$ reveals that "grammar translation" method has a weak influences on the students' performance who do not speak English. In other words, the complex multi-fuzzy soft expert values $(0.2e^{i2\pi(\frac{11}{12})}, 0.1e^{i2\pi(\frac{10}{12})}, 0.3e^{i2\pi(\frac{9}{12})}, 0.4e^{i2\pi(\frac{8}{12})}, 0.2e^{i2\pi(\frac{9}{12})})$ indicate that the expert p agrees that there is a weak influence of using "grammar translation" method on students from the primary Grade 2 to Grade 6 with degrees (0.2, 0.1,0.3,0.4,0.2) respectively. The time required for this effect to be evident in the students' performance who do not speak English is (11,10,9,8,9) months respectively, which is a very long time.

Similarly, the term $\frac{(0.9e^{i2\pi(\frac{3}{12})},1e^{i2\pi(\frac{4}{12})},0.8e^{i2\pi(\frac{4}{12})},0.9e^{i2\pi(\frac{3}{12})},0.8e^{i2\pi(\frac{5}{12})})}{x_2}$ implies that the "communicative language" method has a strongly influences on the students' performance. The membership values

$$(0.9e^{i2\pi(\frac{3}{12})}, 1e^{i2\pi(\frac{4}{12})}, 0.8e^{i2\pi(\frac{4}{12})}0.9e^{i2\pi(\frac{3}{12})}, 0.8e^{i2\pi(\frac{5}{12})})$$

indicate that the expert p agrees that there is a very strong influence from Grade 2 to Grade 6 with degrees (0.9,1,0.8,0.9,0.8) respectively. and this influence need (3,4,4,3,5) months respectively, which is considered a very short time for students to be able to speak English.

Next the complex multi-fuzzy soft expert set $(\mathcal{F}, A)$ will be used together with a generalized algorithm to solve the decision-making problem in determining the most effective method that can be used in teaching a foreign language. The algorithm given below converts the complex multi-fuzzy soft expert set to multi-fuzzy soft expert set using weighted aggregate values and proceeds to determine the best decision using the mean and score of each element of the multi-fuzzy soft expert set. The algorithm steps are given as follows.

Algorithm

1. Input the $\mathcal{CM}^{\mathrm{k}}\mathcal{FSES}$ $(\mathcal{F}, \mathcal{A})$.

2. Convert the complex multi-fuzzy soft expert set $(\mathcal{F}, \mathcal{A})$ to the multi-fuzzy soft expert set $(\tilde{F}, \mathcal{A})$ by obtaining the weighted aggregation values of $\mu^j_{\mathcal{F}(e)}(x)$,$\forall e \in \mathcal{A}$,$\forall x \in U$ and $j = 1, 2, ..., k$ as in the following equation.

$$\mu^j_{\tilde{F}(e)}(x) = \nu_1 r^j_{\mathcal{F}(e)}(x) + \nu_2(1/2\pi)\omega^j_{\mathcal{F}(e)}(x)$$

where $r^j_{\mathcal{F}(e)}(x)$ and $\omega^j_{\mathcal{F}(e)}(x)$ (for $j = 1, 2, ..., k$) are the amplitude and phase terms in the complex multi-fuzzy expert soft set $(\mathcal{F}, \mathcal{A})$, respectively. $\mu^j_{\tilde{F}(e)}(x)$ is the multi-membership function in the multi-fuzzy soft expert set $(\tilde{F}, \mathcal{A})$ and ν_1, ν_2 are the weights for the amplitude terms (degrees of influence) and the phase terms (times of influence), respectively, where ν_1 and $\nu_2 \in [0,1]$ and $\nu_1 + \nu_2 = 1$.

3. Find the mean of each multi-fuzzy soft expert set, $\forall e \in \mathcal{A}$ and $\forall x \in U$ using

$$\bar{M}_{\tilde{F}_{(e)}}(x) = \frac{\sum_{j=1}^{n} \mu^{j}_{\tilde{F}(e)}(x)}{n}.$$

4. Compute the score of each element $x_\ell \in U$ using $\mathcal{C}_\ell = \sum_{j=1}^{n} x_{j\ell}$ for agree-$M^{\mathrm{k}}FSES$.

5. Compute the score of each element $x_\ell \in U$ using $\mathcal{K}_\ell = \sum_{i=1}^{n} x_{j\ell}$ for disagree-$M^{\mathrm{k}}FSES$.

6. Find the value of the score $\mathcal{R}_\ell = \mathcal{C}_\ell - \mathcal{K}_\ell$ for each element $x_\ell \in U$.

7. Determine the value of the highest score $s = max_{x_\ell \in U}\{\mathcal{R}_\ell\}$. The best decision is to choose element x_ℓ as the optimal or best solution to the problem. If there are more than one element with the highest $\mathcal{R}_\ell$ score, any one of them could be chosen by the school based on its option.

As an illustration, we will now convert the complex multi-fuzzy soft expert set $(\mathcal{F}, \mathcal{A})$ to multi-fuzzy soft expert set $(\tilde{F}, \mathcal{A})$. To implement this step, we assume that the weight for the amplitude term is $\nu_1 = 0.6$ and the weight for the phase term is $\nu_2 = 0.4$ to obtain the weighted aggregation values of $\mu^{j}_{\tilde{F}(e)}(x)$,$\forall e \in A$,$\forall x \in U$ and $j = 1, 2, ..., k$. We calculate $\mu^{j}_{\tilde{F}(e)}(x)$, when $e = (e_1, p, 1)$ and $x = x_1$ as shown below.

$$\begin{aligned}\mu^{1}_{\tilde{F}(e_1,p,1)}(x_1) &= \nu_1 r^{1}_{\mathcal{F}(e_1,p,1)}(x_1) + \nu_2(1/2\pi)\omega^{1}_{\mathcal{F}(e_1,p,1)}(x_1)\\ &= (0.6)(0.2) + (0.4)(1/2\pi)(2\pi)(11/12) = 0.487\end{aligned}$$

$$\begin{aligned}\mu^{2}_{\tilde{F}(e_1,p,1)}(x_1) &= \nu_1 r^{2}_{\mathcal{F}(e_1,p,1)}(x_1) + \nu_2(1/2\pi)\omega^{2}_{\mathcal{F}(e_1,p,1)}(x_1)\\ &= (0.6)(0.1) + (0.4)(1/2\pi)(2\pi)(10/12) = 0.393\end{aligned}$$

$$\begin{aligned}\mu^{3}_{\tilde{F}(e_1,p,1)}(x_1) &= \nu_1 r^{3}_{\mathcal{F}(e_1,p,1)}(x_1) + \nu_2(1/2\pi)\omega^{3}_{\mathcal{F}(e_1,p,1)}(x_1)\\ &= (0.6)(0.3) + (0.6)(1/2\pi)(2\pi)(9/12) = 0.48\end{aligned}$$

$$\begin{aligned}\mu^4_{\tilde{F}(e_1,p,1)}(x_1) &= \nu_1 r^4_{\mathcal{F}(e_1,p,1)}(x_1) + \nu_4(1/2\pi)\omega^4_{\mathcal{F}(e_1,p,1)}(x_1)\\ &= (0.6)(0.4) + (0.4)(1/2\pi)(2\pi)(8/12) = 0.507\end{aligned}$$

$$\begin{aligned}\mu^5_{\tilde{F}(e_1,p,1)}(x_1) &= \nu_1 r^5_{\mathcal{F}(e_1,p,1)}(x_1) + \nu_5(1/2\pi)\omega^5_{\mathcal{F}(e_1,p,1)}(x_1)\\ &= (0.6)(0.2) + (0.4)(1/2\pi)(2\pi)(9/12) = 0.42.\end{aligned}$$

Then, for $e = (e_1, p, 1)$ and $x = x_1$, the multi-fuzzy soft expert values $(\mu^1_{\tilde{F}(e_1,p,1)}(x_1), \mu^2_{\tilde{F}(e_1,p,1)}(x_1), \mu^3_{\tilde{F}(e_1,p,1)}(x_1), \mu^4_{\tilde{F}(e_1,p,1)}(x_1), \mu^5_{\tilde{F}(e_1,p,1)}(x_1))$ $= (0.487, 0.393, 0.48, 0.507, 0.42)$.

In the same way, we calculate the other multi-fuzzy soft expert values, $\forall e \in \mathcal{A}$ and $\forall x \in U$ and the results are displayed in Table 1, which gives values of the mean of each multi-fuzzy soft expert set, $\forall e \in \mathcal{A}$ and $\forall x \in U$. Note that the upper and lower terms for each element in Table 1 represent the multi-fuzzy soft expert values, $\forall e \in \mathcal{A}$ and $\forall x \in U$ and the values of the mean of each multi-fuzzy soft expert set, $\forall e \in \mathcal{A}$ and $\forall x \in U$, respectively.

Table 1: Values of $(\tilde{F}, \mathcal{A})$ and the mean of each multi-fuzzy soft expert set.

U	x_1	x_2	x_3
$(e_1, p, 1)$	(0.487, 0.393, 0.48, 0.507, 0.42) 0.457	(0.64, 0.733, 0.613, 0.64, 0.647) 0.655	(0.62, 0.473, 0.54, 0.533.0.38) 0.509
$(e_1, q, 1)$	(0.313, 0.433, 0.507, 0.38, 0.453) 0.417	(0.647, 0.553, 0.707, 0.7, 0.52) 0.625	(0.407, 0.5, 0.06, 0.287, 0.287) 0.308
$(e_2, p, 1)$	(0.447, 0.6, 0.593, 0.407, 0.44) 0.497	(0.647, 0.8, 0.673, 0.706, 0.647) 0.695	(0.553, 0.06, 0.22, 0.407, 0.38) 0.329
$(e_2, q, 1)$	(0.587, 0.527, 0.54, 0.533, 0.56) 0.549	(0.62, 0.493, 0.707, 0.627, 0.687) 0.627	(0.4, 0.553, 0.287, 0.38, 0.407) 0.405
$(e_1, p, 0)$	(0.38, 0.287, 0.26, 0.413, 0.507) 0.459	(0.58, 0.587, 0.493, 0.64, 0.58) 0.576	(0.38, 0.527, 0.193, 0.413, 0.473) 0.397
$(e_1, q, 0)$	(0.473, 0.44, 0.567, 0.447, 0.54) 0.493	(0.553, 0.493, 0.64, 0.587, 0.493) 0.553	(0.4, 0.38, 0.533, 0.56, 0.313) 0.455
$(e_2, p, 0)$	(0.553, 0.5, 0.593, 0.44, 0.473) 0.512	(0.46, 0.487, 0.46, 0.613, 0.613) 0.527	(0.407, 0.32, 0.293, 0.347, 0.313) 0.336
$(e_2, q, 0)$	(0.38, 0.473, 0.5, 0.447, 0.473) 0.455	(0.52, 0.613, 0.553, 0.64, 0.647) 0.595	(0.48, 0.387, 0.229, 0.313, 0.347) 0.351

Tables 2 and 3 present the agree-multi fuzzy soft expert set and disagree-multi-fuzzy soft expert set respectively by using the mean of each multi-fuzzy soft expert set.

Let $\mathcal{C}_\ell$ and $\mathcal{K}_\ell$, represent the score of each numerical grade for the agree-$M^{\mathrm{k}}FSES$ and disagree-$M^{\mathrm{k}}FSES$, respectively. These values are computed in Tables 2 and 3, and rewritten in Table 4.

Table 2: Tabular representation of the agree-multi fuzzy soft expert set.

U	x_1	x_2	x_3
$(e_1, p, 1)$	0.457	0.655	0.509
$(e_1, q, 1)$	0.417	0.625	0.308
$(e_2, p, 1)$	0.497	0.695	0.329
$(e_2, q, 1)$	0.549	0.627	0.405
$\mathcal{C}_\ell = \sum_{j=1}^{n} x_{j\ell}$	$\mathcal{C}_1 = 1.92$	$\mathcal{C}_2 = 2.602$	$\mathcal{C}_3 = 1.551$

Table 3: Tabular representation of the disagree-multi-fuzzy soft expert set.

U	x_1	x_2	x_3
$(e_1, p, 0)$	0.459	0.576	0.397
$(e_1, q, 0)$	0.493	0.553	0.455
$(e_2, p, 0)$	0.512	0.527	0.336
$(e_2, q, 0)$	0.455	0.595	0.351
$\mathcal{K}_\ell = \sum_{j=1}^{n} x_{j\ell}$	$\mathcal{K}_1 = 1.919$	$\mathcal{K}_2 = 2.251$	$\mathcal{K}_3 = 1.539$

Table 4: The score $\mathcal{R}_\ell = \mathcal{C}_\ell - \mathcal{K}_\ell$.

ℓ	U	$\mathcal{C}_\ell$	$\mathcal{K}_\ell$	$\mathcal{R}_\ell$
1	x_1	1.92	1.919	0.001
2	x_2	2.602	2.251	0.351
3	x_3	1.551	1.539	0.012

Clearly, the maximum choice value is 0.351 from Table 4 and so the optimal decision is to select x_2. Therefore, the expert should select "communicative language" method as the most effective method based on the specified weights for the given parameters.

7 Comparison between complex multi-fuzzy soft expert set and the existing methods

We will now compare the complex multi-fuzzy soft expert set with two existing methods of multi-fuzzy soft set [9] and complex fuzzy soft expert set [25].

From Example 6.1, it can be observed that multi-fuzzy soft set is not able to solve the decision making problem presented, which involves two-dimensional data i.e. the degree of the influence and the total time of the influence since multi-fuzzy soft set lacks the phase term which represent the time frame of this problem. An additional reason is its inability to deal with more than one expert.

However, complex fuzzy soft expert set [25] can deal with problems that involve two-dimensional information, but it is not able to solve the decision making problem which involves multi-agent, multi-attribute, multi-object, multi-index and uncertainty since it is created with a single complex membership function. On the other hand, our model has the ability to represent two-dimensional information that are presented by multi-membership functions, making it clearly superior to both multi-fuzzy soft set and complex fuzzy soft expert set in terms of computational efficiency, practicality as well as ease and accuracy of representation of two-dimensional multi-fuzzy information.

Thus, the proposed method has certain advantages. Firstly, this method uses the complex multi-fuzzy soft expert to represent the decision making problem which involves multi-agent, multi-attribute, multi-object, multi-index and uncertainty utilizing multimembership functions. Secondly, complex multi-fuzzy soft expert includes evaluation information missing in the multi-fuzzy soft set model, such as the time frame which is presented by the phase term. In addition, it has the added advantage of allowing the users to know the opinion of all the experts in a single model without the need for any additional cumbersome operations. Thirdly, the complex multi-fuzzy soft expert set that is used in our method has the ability to handle the uncertainty information that is captured by the amplitude terms and phase terms of the complex numbers, simultaneously. Fourthly, a practical formula is employed to convert the complex multi-fuzzy soft expert set from the complex state to the real state which gives a decision-making with a simple computational process without the need to carry out directed operations on complex numbers. Finally, our proposed model provides a more accurate representation of two-dimensional multi-fuzzy information, i.e information presented by amplitude term and information presented by phase, where the phase term represents the time factor that may interfere, constructively or destructively, with the associated amplitude term in the decision process, thus making our model highly suitable for use in decision making problems to select the best alternative.

8 Conclusion

A novel mathematical tool is developed to represent the information which is utilizing time factor and to know the opinions of all the experts in a single model. The complex multi-fuzzy soft expert set is established by incorporating the features of both complex multi-fuzzy set and soft expert set. We defined the concept of the multi-fuzzy soft expert set and complex multi-fuzzy soft set. The fundamental operations on complex multi-fuzzy soft expert set such as complement, union, intersection, AND and OR operation were defined. Several properties have been investigated. A new general framework of complex multi-fuzzy soft expert set for dealing with uncertainty decision making has thus been proposed and its associated algorithm constructed. A comparison of our proposed model to two other existing models indicates the tenacity of our model to provide a more accurate representation of two-dimensional multi-fuzzy information. It was shown to be viable and easily applied as illustrated by the application in education which was used to demonstrate the principal steps of the algorithm to come up with the required decision.

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