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# Estimation of Parameters of Hyper Graeco Latin Sudoku Square Design under Random and Mixed Effect Models

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### Abstract

Hussain et al. [5] introduced a new design, which they called Hyper Graeco Latin Sudoku Square Design (Hyper GLaSS Design) by merging the two experimental designs; i.e., Sudoku Square Design and Hyper Graeco Latin Square Design. The Hyper GLaSS Design tests three sets of treatments all at the same time in an experiment and allow the study of six factors. In this paper, we discuss the estimation of the parameters of Hyper GLaSS Design under random effect model and mixed effect. Analysis of the design through numerical example, using hypothetical data set, has been performed. The Simulations study reveals that the proposed design is more efficient as compared to Hyper Graeco Latin Square Design. A numerical example has been given for illustration purpose.

**Key words and phrases:** Hyper GLaSS, Design, Hyper Graeco Square Design, Sudoku Design, Blocking.
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# 1 Introduction

The term “Sudoku” refers to a logical puzzle game with no mathematics involved in it. This game usually comprises nine-by-nine grid in which there are some blank cells while other are filled by numbers, colors or letters. The blank cells are filled in a manner that every single row, every single column and each grid (3x3 blocks) comprises of all the nine digits just once (Bailey et al. [1]).

The properties of Sudoku and Latin Square are almost same. Therefore, Sudoku grid is considered to be a special case of Latin square. In a Latin Square, 1 to n sets of numbers are arranged in such a way so that no column or row contains same number twice. Euler in the year 1707 to 1783 was considered the first mathematician by introducing a special type of Latin square as a different kind of magic square. Puzzles of Sudoku can be extended to statistical design of experiment. When Sudoku grid has empty cells then they are Sudoku puzzles whereas Sudoku grids with complete cells are Sudoku designs (Subramani and Ponnuswamy, [9]).

Fisher [4] worked on Latin Square designs and Mutually Orthogonal designs. Yates [10] extended Fisher’s idea. He revealed that such designs exist for prime order but not for non-prime order of higher terms and also they exist for 4, 8 and 9 order. Kishen [6] mentioned that the theories of Fisher and Yates were not specified completely and generalized orthogonal Latin Square to Hyper-Graeco Latin Square design.

“Hyper-Graeco Latin Square Design” is the result of three different types of “Orthogonal Latin Square Design” when they are superimposed on one another. Where treatment 1, treatment 2 and treatment 3 are the three different treatments used at the same time (Mann, [7]). It is capable to test three different treatments in the same experiment and at the same time i.e., it permits five factors examination at the same time namely columns, rows, and three treatments (Colbourn and Dinitz, [3]).

It is a square design having same advantages and disadvantages as that of Latin Square Design. It permits the use of four different blocking (Bose, [2]). Introducing blocks in designs can increase their accuracy (Sorana et al., [8]). Hyper GLaSS design has been introduced as a new efficient design which reduces mean square error. By merging the blocking property of Sudoku Square design in Hyper Graeco Latin Square Design results in minimum mean square error.

# 2 Hyper GLaSS Design

Hyper GLaSS Design is the result of overlapping three orthogonal Sudoku Square designs; one with treatment of type (1), the other with treatment of type (2) and third is treatment of type (3). Hyper GLaSS Design tests three treatments in the same experiment at the same time and allows investigation of six elements. The introduction of blocking factor in the proposed design results in minimum error.

## 2.1 Layout of Hyper GlaSS Design

The model of the Hyper GLaSS design is as follows

$$Y_{ij(klrp)} = \mu + \alpha_i + \beta_j + \tau_k + \gamma_l + \theta_r + \lambda_p + \epsilon_{ij(klrp)}$$

where $i, j, k, l, r, p = 1, 2, 3, \ldots, m$ , $Y_{ij(klrp)}$ is the experimental field resulting which have $i^{th}$ row, $j^{th}$ column, $k^{th}$ treatments type 1, $l^{th}$ block, $r^{th}$ treatments type 2 and $p^{th}$ treatments type 3 are applied to this field. Blocking in Hyper GlaSS Design is only effective if the variance between blocks is larger than the variance within blocks (Hussain et al., [5]).

# 3 Hyper GLaSS Design Under Random Effects Model (MODEL 1)

In $m \times m$ Hyper GLaSS design, an observation $y_{ij(klrp)}$belonging to the $i^{th}$ row, $j^{th}$ column, $k^{th}$ treatments type (1), $l^{th}$ block, $r^{th}$ treatments type (2) and $p^{th}$ treatments type (3) can be described as:

$$y_{ij(klrp)} = \mu + \alpha_i + \beta_j + \tau_k + \gamma_l + \theta_r + \lambda_p + \epsilon_{ij(klrp)} \tag{3.1}$$

where $i = 1, 2, 3, \ldots, m$ (Rows), $j = 1, 2, ..., m,$ (Columns), $k = 1, 2, ..., m$ (treatments type (1)), $l = 1, 2, ..., m$ (blocks), $r = 1, 2, ...,$ treatments type (2), $p = 1, 2, ..., m$ (treatments type (3)) and $\epsilon_{ij(klrp)} \sim I.I.N. (0, \sigma^2)$.

$$E\left(\epsilon_{ij(klrp)}\right) = 0, E\left(\epsilon^2_{ij(klrp)}\right) = \sigma^2, E\left(\epsilon_{ij}, \epsilon_{gh}\right) = 0, \text{for } i \neq g \text{ and } j \neq h \tag{3.2}$$

The row effects, the column effects, the treatments type (1) effects, the blocks effects, the treatments type (2) effects and the treatments type (3) effects are independently and normally distributed, each with zero mean and variance $\sigma^2$. Thus,

$$\alpha_i \sim N\left(0, \sigma^2\right), \beta_j \sim N\left(0, \sigma^2\right), \tau_k \sim N\left(0, \sigma^2\right),$$
$$\gamma_l \sim N\left(0, \sigma^2\right), \theta_r \sim N\left(0, \sigma^2\right) \text{ and } \lambda_p \sim N\left(0, \sigma^2\right)$$

This means that

$$E(\alpha_i) = 0, E(\beta_j) = 0, E(\tau_k) = 0, E(\gamma_l) = 0, E(\theta_r) = 0, \text{and } E(\lambda_p) = 0$$
$$E(\alpha_i^2) = \sigma_\alpha^2, E(\beta_j^2) = \sigma_\beta^2, E(\tau_k^2) = \sigma_\tau^2, E(\gamma_l^2) = \sigma_\gamma^2, E(\theta_r^2) = \sigma_\theta^2, \text{and } E(\lambda_p^2) = \sigma_\lambda^2$$
$$\text{Also } E\left(\alpha_{i1}, \alpha_{i2}\right) = 0 \text{ for } i1 \neq i2, E\left(\beta_{j1}, \beta_{j2}\right) = 0 \text{ for } j1 \neq j2, E\left(\tau_{k1}, \tau_{k2}\right) = 0 \text{ for } k1 \neq k2,$$
$$E\left(\gamma_{l1}, \gamma_{l2}\right) = 0 \text{ for } l1 \neq l2, E\left(\theta_{r1}, \theta_{r2}\right) = 0 \text{ for } r1 \neq r2, E\left(\lambda_{p1}, \lambda_{p2}\right) = 0 \text{ for } p1 \neq p2.$$

The rows, the column, the treatments type (1), the blocks, the treatments type (2) and the treatments type (3) are also independent of $\epsilon_{ij(klrp)}$ which is as follows:

$$E\left(\alpha_i, \varepsilon_{ij(klrp)}\right) = 0, E\left(\beta_j, \varepsilon_{ij(klrp)}\right) = 0, E\left(\tau_k, \varepsilon_{ij(klrp)}\right) = 0,$$
$$E\left(\gamma_l, \varepsilon_{ij(klrp)}\right) = 0, E\left(\theta_r, \varepsilon_{ij(klrp)}\right) = 0, E\left(\lambda_p, \varepsilon_{ij(klrp)}\right) = 0$$

The ANOVA is presented in Table 1.

# 4 Hyper GLASS Design Under Mixed Effects Model

Under mixed effect model there are several cases.

**Case 1:**

Rows effects are fixed, columns effects, treatments type (1) effects, blocks effects, treatments type (2) effects and treatments type (3) effects are random. The results are summarized in Table 2.

**Case 2:**

Columns effects are fixed, rows effects, treatments type (1) effects, blocks effects, treatments type (2) effects and treatments type (3) effects are random. The results are summarized in Table 3.

Table 1: ANOVA table of Hyper GLaSS design under random effects model (Model 1)

| Source | d.f | SS | MS | E(SS) | E(MS) |
|---|---|---|---|---|---|
| Rows | $m-1$ | SSR=$\frac{1}{m}\sum_{i=1}^{m} R^2_{i.(\ldots)} - C.F.$ | $S^2_R = \frac{SSR}{m-1}$ | $m(m-1)\sigma^2_\alpha + (m-1)\sigma^2$ | $m\sigma^2_\alpha + \sigma^2$ |
| Columns | $m-1$ | SSC=$\frac{1}{m}\sum_{j=1}^{m} C^2_{j.(\ldots)} - C.F.$ | $S^2_C = \frac{SSC}{m-1}$ | $m(m-1)\sigma^2_\beta + (m-1)\sigma^2$ | $m\sigma^2_\beta + \sigma^2$ |
| Treatments type(1) | $m-1$ | SST(1)=$\frac{1}{m}\sum_{k=1}^{m} T^2_{..k.(\ldots)} - C.F.$ | $S^2_{T(1)} = \frac{SST}{m-1}$ | $m(m-1)\sigma^2_\tau + (m-1)\sigma^2$ | $m\sigma^2_\tau + \sigma^2$ |
| Blocks | $m-1$ | SSB=$\frac{1}{m}\sum_{l=1}^{m} B^2_{..l.(\ldots)} - C.F.$ | $S^2_R = \frac{SSR}{m-1}$ | $m(m-1)\sigma^2_\gamma + (m-1)\sigma^2$ | $m\sigma^2_\gamma + \sigma^2$ |
| Treatments type(2) | $m-1$ | SST(2)=$\frac{1}{m}\sum_{r=1}^{m} T^2_{..r.(\ldots)} - C.F.$ | $S^2_{T_{(2)}} = \frac{SST(2)}{m-1}$ | $m(m-1)\sigma^2_\theta + (m-1)\sigma^2$ | $m\sigma^2_\theta + \sigma^2$ |
| Treatments type(3) | $m-1$ | SST(3)=$\frac{1}{m}\sum_{p=1}^{m} T^2_{.(\ldots p)} - C.F.$ | $S^2_{T_{(3)}} = \frac{SST(3)}{m-1}$ | $m(m-1)\sigma^2_\lambda + (m-1)\sigma^2$ | $m\sigma^2_\lambda + \sigma^2$ |
| Error | $(m-1)$ (m-5) | By subtraction | SSE | $\sigma^2(m-1)(m-5)$ | $\sigma^2$ |
| Total | $m^2-1$ | TSS=$\sum_{i,j=1}^{mm}\sum_{k,l,r,p=1}^{m} y^2_{ij(klrp)} - C.F.$ | | | |

Table 2: ANOVA Table (Rows effects are fixed, columns effects, treatments type (1) effects, blocks effects, treatments type (2) effects and treatments type (3) effects are random (Model 2))

| Source | d.f | SS | MS | E(SS) | E(MS) |
|---|---|---|---|---|---|
| Rows | $m-1$ | SSR=$\frac{1}{m}\sum_{i=1}^{m} R^2_{i.(\ldots)} - C.F.$ | $S^2_R = \frac{SSR}{m-1}$ | $m\sum_{i=1}^{m} R^2_i + (m-1)\sigma^2$ | $\frac{m}{m-1}\sum_{i=1}^{m} R^2_i + (m-1)\sigma^2$ |
| Columns | $m-1$ | SSC=$\frac{1}{m}\sum_{j=1}^{m} C^2_{j.(\ldots)} - C.F.$ | $S^2_C = \frac{SSC}{m-1}$ | $m(m-1)\sigma^2_\beta + (m-1)\sigma^2$ | $m\sigma^2_\beta + \sigma^2$ |
| Treatments type(1) | $m-1$ | SST(1)=$\frac{1}{m}\sum_{k=1}^{m} T^2_{..k.(\ldots)} - C.F.$ | $S^2_{T(1)} = \frac{SST(1)}{m-1}$ | $m(m-1)\sigma^2_\tau + (m-1)\sigma^2$ | $m\sigma^2_\tau + \sigma^2$ |
| Blocks | $m-1$ | SSB=$\frac{1}{m}\sum_{l=1}^{m} B^2_{..l.(\ldots)} - C.F.$ | $S^2_B = \frac{SSB}{m-1}$ | $m(m-1)\sigma^2_\gamma + (m-1)\sigma^2$ | $m\sigma^2_\gamma + \sigma^2$ |
| Treatments type(2) | $m-1$ | SST(2)=$\frac{1}{m}\sum_{r=1}^{m} T^2(2)_{..r.(\ldots)} - C.F.$ | $S^2_{T_{(2)}} = \frac{SST(2)}{m-1}$ | $m(m-1)\sigma^2_\theta + (m-1)\sigma^2$ | $m\sigma^2_\theta + \sigma^2$ |
| Treatments type(3) | $m-1$ | SST(3)=$\frac{1}{m}\sum_{p=1}^{m} T^2(3)_{(\ldots p)} - C.F.$ | $S^2_{T_{(3)}} = \frac{SST(3)}{m-1}$ | $m(m-1)\sigma^2_\lambda + (m-1)\sigma^2$ | $m\sigma^2_\lambda + \sigma^2$ |
| Error | (m-1) (m-5) | By subtraction | $S^2_E = \frac{SSE}{(m-1)(m-5)}$ | $\sigma^2(m-1)(m-5)$ | $\sigma^2$ |
| Total | $m^2-1$ | TSS=$\sum_{i,j=1}^{m}\sum_{k,l,r,p=1}^{m} y^2_{ij(klrp)} - C.F.$ | | | |

Table 3: ANOVA table (Columns effects are fixed, rows effects, treatments type (1) effects, blocks effects, treatments type (2) effects and treatments type (3) effects are random (Model 3))

| Source of variation | d.f | SS | MS | E(SS) | E(MS) |
|---|---|---|---|---|---|
| Rows | $m-1$ | SSR=$\frac{1}{m}\sum_{i=1}^{m} R_{i.(\ldots)}^2 - C.F.$ | $S_R^2 = \frac{SSR}{m-1}$ | $m(m-1)\sigma_\alpha^2 + (m-1)\sigma^2$ | $m\sigma_\alpha^2 + \sigma^2$ |
| Columns | $m-1$ | SSC=$\frac{1}{m}\sum_{j=1}^{m} C_{j.(\ldots)}^2 - C.F.$ | $S_C^2 = \frac{SSC}{m-1}$ | $m\sum_{j=1}^{m}\beta_j^2 + (m-1)\sigma^2$ | $\frac{m}{m-1}\sum_{j=1}^{m}$ … $\sigma^2$ |
| Treatments type(1) | $m-1$ | SST(1)=$\frac{1}{m}\sum_{k=1}^{m} T_{..k.(\ldots)}^2 - C.F.$ | $S_{T(1)}^2 = \frac{SST(1)}{m-1}$ | $m(m-1)\sigma_\tau^2 + (m-1)\sigma^2$ | $m\sigma_\tau^2 + \sigma^2$ |
| Blocks | $m-1$ | SSB=$\frac{1}{m}\sum_{l=1}^{m} B_{..l.(\ldots)}^2 - C.F.$ | $S_B^2 = \frac{SSB}{m-1}$ | $m(m-1)\sigma_\gamma^2 + (m-1)\sigma^2$ | $m\sigma_\gamma^2 + \sigma^2$ |
| Treatments type(2) | $m-1$ | SST(2)=$\frac{1}{m}\sum_{r=1}^{m} T^2(2)_{..r.(\ldots)} - C.F.$ | $S_{T_{(2)}}^2 = \frac{SST(2)}{m-1}$ | $m(m-1)\sigma_\theta^2 + (m-1)\sigma^2$ | $m\sigma_\theta^2 + \sigma^2$ |
| Treatments type(3) | $m-1$ | SST(3)=$\frac{1}{m}\sum_{p=1}^{m} T^2(3)_{(\ldots p)} - C.F.$ | $S_{T_{(3)}}^2 = \frac{SST(3)}{m-1}$ | $m(m-1)\sigma_\lambda^2 + (m-1)\sigma^2$ | $m\sigma_\lambda^2 + \sigma^2$ |
| Error | (m-1) (m-5) | By subtraction | $S_E^2=\frac{SSE}{(m-1)(m-5)}$ | $\sigma^2(m-1)(m-5)$ | $\sigma^2$ |
| Total | $m^2-1$ | TSS=$\sum_{i,j=1}^{m}\sum_{k,l,r,p=1}^{m} y_{ij(klrp)}^2 - C.F.$ | | | |

**Case 3:**

Treatments type (1) effects are fixed, rows effects, columns effects, blocks effects, treatments type (2) effects and treatments type (3) effects are random. The results are summarized in Table 4.

**Case 4:**

Blocks effects are fixed, rows effects, columns effects, treatments type (1) effects, treatments type (2) effects and treatments type (3) effects are random. The results are summarized in Table 5.

**Case 5:**

Treatment type (2) effects are fixed, rows effects, columns effects, block effects, treatment type (1) effects and treatment type (3) effects are random. The results are summarized in Table 6.

Table 4: ANOVA table (Treatments type (1) effects are fixed, rows effects, column effects, blocks effects, treatments type (2) effects and treatments type (3) effects are random (Model 4))

| Source of variation | d.f | SS | MS | E(SS) | E(MS) |
|---|---|---|---|---|---|
| Rows | $m-1$ | SSR=$\frac{1}{m}\sum_{i=1}^{m} R^2_{i.(...)} - C.F.$ | $S^2_R = \frac{SSR}{m-1}$ | $m(m-1)\sigma^2_\alpha + (m-1)\sigma^2$ | $m\sigma^2_\alpha + \sigma^2$ |
| Columns | $m-1$ | SSC=$\frac{1}{m}\sum_{j=1}^{m} C^2_{j.(...)} - C.F.$ | $S^2_C = \frac{SSC}{m-1}$ | $m(m-1)\sigma^2_\beta + (m-1)\sigma^2$ | $m\sigma^2_\beta + \sigma^2$ |
| Treatments type(1) | $m-1$ | SST(1)=$\frac{1}{m}\sum_{k=1}^{m} T^2_{..k.(...)} - C.F.$ | $S^2_{T(1)} = \frac{SST(1)}{m-1}$ | $m\sum_{k=1}^{m}\tau^2_k + (m-1)\sigma^2$ | $\frac{m}{m-1}\sum_{k=1}^{m}\tau^2_k + (m-1)\sigma^2$ |
| Blocks | $m-1$ | SSB=$\frac{1}{m}\sum_{l=1}^{m} B^2_{..l.(...)} - C.F.$ | $S^2_B = \frac{SSB}{m-1}$ | $m(m-1)\sigma^2_\gamma + (m-1)\sigma^2$ | $m\sigma^2_\gamma + \sigma^2$ |
| Treatments type(2) | $m-1$ | SST(2)=$\frac{1}{m}\sum_{r=1}^{m} T^2(2)_{..r.(...)} - C.F.$ | $S^2_{T_{(2)}} = \frac{SST(2)}{m-1}$ | $m(m-1)\sigma^2_\theta + (m-1)\sigma^2$ | $m\sigma^2_\theta + \sigma^2$ |
| Treatments type(3) | $m-1$ | SST(3)=$\frac{1}{m}\sum_{p=1}^{m} T^2(3)_{(...p)} - C.F.$ | $S^2_{T_{(3)}} = \frac{SST(3)}{m-1}$ | $m(m-1)\sigma^2_\lambda + (m-1)\sigma^2$ | $m\sigma^2_\lambda + \sigma^2$ |
| Error | (m-1) (m-5) | By subtraction | $S^2_E = \frac{SSE}{(m-1)(m-5)}$ | $\sigma^2(m-1)(m-5)$ | $\sigma^2$ |
| Total | $m^2-1$ | TSS=$\sum_{i,j=1}^{m}\sum_{k,l,r,p=1}^{m} y^2_{ij(klrp)} - C.F.$ | | | |

**Case 6:**

Treatment type (3) effects are fixed, rows effects, columns effects, block effects, treatment type (1) effects and treatment type (3) effects are random. The results are summarized in Table 7.

# 5 Numerical Illustration For Comparison

The analysis of Hyper GLaSS Design and its comparison with Hyper Graeco Latin Square Design through hypothetical data are presented in Table 8.

Table 9 presents ANOVA for Hyper Graeco-Latin Square Design of order 16.

Table 10 presents ANOVA for the Hyper-GLaSS Design.

## 5.1 Results and Discussions

The mean of error sum of squares of Hyper-Graeco-Latin square design and Hyper-GLaSS design for the same hypothetical data as shown in Table 8

Table 5: Blocks effects are fixed, rows effects, columns effects, treatments type (1) effects, treatments type (2) effects and treatments type (3) effects are random (Model 5)

| Source of variation | d.f | SS | MS | E(SS) | E(MS) |
|---|---|---|---|---|---|
| Rows | $m-1$ | SSR=$\frac{1}{m}\sum_{i=1}^{m} R^2_{i.(\ldots)} - C.F.$ | $S^2_R = \frac{SSR}{m-1}$ | $m(m-1)\sigma^2_\alpha + (m-1)\sigma^2$ | $m\sigma^2_\alpha + \sigma^2$ |
| Columns | $m-1$ | SSC=$\frac{1}{m}\sum_{j=1}^{m} C^2_{j.(\ldots)} - C.F.$ | $S^2_C = \frac{SSC}{m-1}$ | $m(m-1)\sigma^2_\beta + (m-1)\sigma^2$ | $m\sigma^2_\beta + \sigma^2$ |
| Treatments type(1) | $m-1$ | SST(1)=$\frac{1}{m}\sum_{k=1}^{m} T^2_{..k.(\ldots)} - C.F.$ | $S^2_{T(1)} = \frac{SST(1)}{m-1}$ | $m(m-1)\sigma^2_\tau + (m-1)\sigma^2$ | $m\sigma^2_\tau + \sigma^2$ |
| Blocks | $m-1$ | SSB=$\frac{1}{m}\sum_{l=1}^{m} B^2_{..l.(\ldots)} - C.F.$ | $S^2_B = \frac{SSB}{m-1}$ | $m\sum_{l=1}^{m} \gamma^2_l + (m-1)\sigma^2$ | $\frac{m}{m-1}\sum_{l=1}^{m}$ [illegible] $\sigma^2$ |
| Treatments type(2) | $m-1$ | SST(2)=$\frac{1}{m}\sum_{r=1}^{m} T^2(2)_{..r.(\ldots)} - C.F.$ | $S^2_{T_{(2)}} = \frac{SST(2)}{m-1}$ | $m(m-1)\sigma^2_\theta + (m-1)\sigma^2$ | $m\sigma^2_\theta + \sigma^2$ |
| Treatments type(3) | $m-1$ | SST(3)=$\frac{1}{m}\sum_{p=1}^{m} T^2(3)_{(\ldots p)} - C.F.$ | $S^2_{T_{(3)}} = \frac{SST(3)}{m-1}$ | $m(m-1)\sigma^2_\lambda + (m-1)\sigma^2$ | $m\sigma^2_\lambda + \sigma^2$ |
| Error | (m-1) (m-5) | By subtraction | $S^2_E = \frac{SSE}{(m-1)(m-5)}$ | $\sigma^2(m-1)(m-5)$ | $\sigma^2$ |
| Total | $m^2-1$ | TSS=$\sum_{i,j=1}^{m}\sum_{k,l,r,p=1}^{m} y^2_{ij(klrp)} - C.F.$ | | | |

have been calculated in Table 9 and in Table 10. Comparing the two designs, Hyper Graeco Latin Square Design and Hyper GLaSS Design, we see that by removing the variability due to blocking in our proposed design, the experimental error in Hyper GLaSS Design has been decreased. Thus, it is effective to use blocking. Hyper GLaSS Design, possessing the properties of both Hyper Graeco Latin Square Design and Sudoku Square design, results in lesser mean square for error.

## 5.2 Simulation Study

In this section a simulation study is presented to compare the mean of error sum of squares of the Hyper Graeco-Latin square and the new proposed design i.e. Hyper GLaSS design. Gaussian random observations have been simulated according to the following model:

$$Y_{ij(klrp)} = \mu + \alpha_i + \beta_j + \tau_k + \gamma_l + \theta_r + \lambda_p + \epsilon_{ij(klrp)}$$

where $i, j, k, l, r, p = 1, 2, 3, \ldots, m$. The model parameters $\alpha_i$ and $\beta_j$ were simulated from a Gaussian distribution with mean equals 2 and variance

Table 6: Treatment type (2) effects are fixed, rows effects, columns effects, block effects, treatment type (1) effects and treatment type (3) effects are random (Model 6)

| Source of variation | d.f | SS | MS | E(SS) | E(MS) |
|---|---|---|---|---|---|
| Rows | $m-1$ | SSR=$\frac{1}{m}\sum_{i=1}^{m} R^2_{i.(\ldots)} - C.F.$ | $S^2_R = \frac{SSR}{m-1}$ | $m(m-1)\sigma^2_\alpha + (m-1)\sigma^2$ | $m\sigma^2_\alpha + \sigma^2$ |
| Columns | $m-1$ | SSC=$\frac{1}{m}\sum_{j=1}^{m} C^2_{j.(\ldots)} - C.F.$ | $S^2_C = \frac{SSC}{m-1}$ | $m(m-1)\sigma^2_\beta + (m-1)\sigma^2$ | $m\sigma^2_\beta + \sigma^2$ |
| Treatments type(1) | $m-1$ | SST(1)=$\frac{1}{m}\sum_{k=1}^{m} T^2_{..k.(\ldots)} - C.F.$ | $S^2_{T(1)} = \frac{SST(1)}{m-1}$ | $m(m-1)\sigma^2_\tau + (m-1)\sigma^2$ | $m\sigma^2_\tau + \sigma^2$ |
| Blocks | $m-1$ | SSB=$\frac{1}{m}\sum_{l=1}^{m} B^2_{..l.(\ldots)} - C.F.$ | $S^2_B = \frac{SSB}{m-1}$ | $m(m-1)\sigma^2_\gamma + (m-1)\sigma^2$ | $m\sigma^2_\gamma + \sigma^2$ |
| Treatments type(2) | $m-1$ | SST(2)=$\frac{1}{m}\sum_{r=1}^{m} T^2(2)_{..r.(\ldots)} - C.F.$ | $S^2_{T_{(2)}} = \frac{SST(2)}{m-1}$ | $m\sum_{r=1}^{m} T^2_\theta + (m-1)\sigma^2$ | $\frac{m}{m-1}\sum_{r=1}^{m}\theta^2_r + \sigma^2$ |
| Treatments type(3) | $m-1$ | SST(3)=$\frac{1}{m}\sum_{p=1}^{m} T^2(3)_{(\ldots p)} - C.F.$ | $S^2_{T_{(3)}} = \frac{SST(3)}{m-1}$ | $m(m-1)\sigma^2_\lambda + (m-1)\sigma^2$ | $m\sigma^2_\lambda + \sigma^2$ |
| Error | (m-1) (m-5) | By subtraction | $S^2_E = \frac{SSE}{(m-1)(m-5)}$ | $\sigma^2(m-1)(m-5)$ | $\sigma^2$ |
| Total | $m^2-1$ | TSS=$\sum_{i,j=1}^{m}\sum_{k,l,r,p=1}^{m} y^2_{ij(klrp)} - C.F.$ | | | |

equals 0.0625. The parameters $\tau_k$, $\theta_r$ and $\lambda_p$ measure the effect of the Treatments 1 to 3, respectively. These parameters were generated from a Gaussian distribution with mean 1, 3 and 5 respectively and variance equals 0.0625. Finally, the parameters $\gamma_l$ measure the block effect. These parameters are of special interest in our simulation study because the block effect is the main difference between the Hyper Graeco-Latin square design and the new Hyper GlaSS design. It is expected that when the block effect is large the Hyper GlaSS design will provide smaller mean of error sum of squares than the Hyper Graeco-Latin square design. Thus, these parameters were simulated from Gaussian distributions with different expected values. The procedure is started by fixing the expectation of the Gaussian distribution at zero, which in turn represents no block effect. Then increase the expectation of the Gaussian distribution, generating the parameters $\gamma_r$, from 1 to 10 to assess the impact of the block effect on the mean of error sum of squares. For all cases, the variance of the Gaussian distribution was fixed at 0.0625. Furthermore, the variance of random error was fixed at 15 which is a value compatible with the expected values generated by the specification of $\alpha_i, \beta_j, \tau_k, \theta_r$ and $\lambda_p$ . For each parameter configuration 1000 data sets were generated and

Table 7: Treatment type (3) effects are fixed, rows effects, columns effects, block effects, treatment type (1) effects and treatment type (3) effects are random (Model 7)

| Source of variation | d.f | SS | MS | E(SS) | E(MS) |
|---|---|---|---|---|---|
| Rows | $m-1$ | SSR=$\frac{1}{m}\sum_{i=1}^{m} R_{i.(\ldots)}^2 - C.F.$ | $S_R^2 = \frac{SSR}{m-1}$ | $m(m-1)\sigma_\alpha^2 + (m-1)\sigma^2$ | $m\sigma_\alpha^2 + \sigma^2$ |
| Columns | $m-1$ | SSC=$\frac{1}{m}\sum_{j=1}^{m} C_{j.(\ldots)}^2 - C.F.$ | $S_C^2 = \frac{SSC}{m-1}$ | $m(m-1)\sigma_\beta^2 + (m-1)\sigma^2$ | $m\sigma_\beta^2 + \sigma^2$ |
| Treatments type(1) | $m-1$ | SST(1)=$\frac{1}{m}\sum_{k=1}^{m} T_{..k.(\ldots)}^2 - C.F.$ | $S_{T(1)}^2 = \frac{SST(1)}{m-1}$ | $m(m-1)\sigma_\tau^2 + (m-1)\sigma^2$ | $m\sigma_\tau^2 + \sigma^2$ |
| Blocks | $m-1$ | SSB=$\frac{1}{m}\sum_{l=1}^{m} B_{..l.(\ldots)}^2 - C.F.$ | $S_B^2 = \frac{SSB}{m-1}$ | $m(m-1)\sigma_\gamma^2 + (m-1)\sigma^2$ | $m\sigma_\gamma^2 + \sigma^2$ |
| Treatments type(2) | $m-1$ | SST(2)=$\frac{1}{m}\sum_{r=1}^{m} T^2(2)_{..r.(\ldots)} - C.F.$ | $S_{T_{(2)}}^2 = \frac{SST(2)}{m-1}$ | $m(m-1)\sigma_\theta^2 + (m-1)\sigma^2$ | $m\sigma_\theta^2 + \sigma^2$ |
| Treatments type(3) | $m-1$ | SST(3)=$\frac{1}{m}\sum_{p=1}^{m} T^2(3)_{(\ldots p)} - C.F.$ | $S_{T_{(3)}}^2 = \frac{SST(3)}{m-1}$ | $m\sum_{p=1}^{m} T_\lambda^2 + (m-1)\sigma^2$ | $\frac{m}{m-1}\sum_{p=1}^{m}$ … $\sigma^2$ |
| Error | (m-1) (m-5) | By subtraction | $S_E^2 = \frac{SSE}{(m-1)(m-5)}$ | $\sigma^2(m-1)(m-5)$ | $\sigma^2$ |
| Total | $m^2-1$ | TSS=$\sum_{i,j=1}^{m}\sum_{k,l,r,p=1}^{m} y_{ij(klrp)}^2 - C.F.$ | | | |

the average mean of error sum of squares were computed. In all cases, three treatments case was considered as presented in hypothetical data analysis (see Table 8). Table 11 presents the average mean of error sum of squares by different block size effects for the Hyper Graeco-Latin square and Hyper GlaSS designs along with the difference between the considered designs and the relative efficiency loss by considering the Hyper Graeco-Latin square instead of the Hyper GlaSS design. The relative efficiency loss is computed as the ratio of the mean square error (MSQ) obtained under the Hyper Graeco-Latin square and the MSQ obtained under the Hyper GlaSS designs

The results in Table 11 confirm that the Hyper GlaSS design provides smaller mean of error sum of squares as compared to that of Hyper Graeco-Latin square. Furthermore, the difference between the two designs increases with increasing block effect as was expected. In relative terms, results in Table 11 show that the new design can be 28.47% more efficient than the traditional design for large block effect. Similarly, Figure 1(A) presents the MSQ difference between the two considered designs along with their relative efficiency in Figure 1 (B).

From Figure 1(A), it is clear that when expected block effect increases,

Table 8: Hypothetical data of Hyper GLaSS Design of order 16

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $A_1A_2P_3$ (20) | $E_1B_2L_3$ (25) | $I_1C_2H_3$ (29) | $M_1D_2D_3$ (25) | $B_1E_2O_3$ (23) | $F_1F_2K_3$ (28) | $J_1G_2G_3$ (24) | $N_1H_2C_3$ (19) | $C_1I_2N_3$ (21) | $G_1J_2J_3$ (16) | $K_1K_2F_3$ (10) | $O_1L_2B_3$ (14) | $D_1M_2M_3$ (16) | $H_1N_2I_3$ (20) | $L_1O_2E_3$ (15) | $P_1P_2A_3$ (10) |
| 2 | $B_1E_2A_3$ (15) | $F_1F_2M_3$ (21) | $J_1G_2I_3$ (17) | $N_1H_2E_3$ (20) | $C_1I_2P_3$ (21) | $G_1J_2L_3$ (23) | $K_1K_2H_3$ (18) | $O_1L_2D_3$ (15) | $D_1M_1O_3$ (16) | $H_1N_2K_3$ (21) | $L_1O_2G_3$ (15) | $P_1P_2C_3$ (19) | $E_1A_2N_3$ (21) | $I_1B_2J_3$ (26) | $M_1C_2F_3$ (20) | $A_1D_2B_3$ (25) |
| 3 | $C_1I_2B_3$ (19) | $G_1J_2N_3$ (26) | $K_1K_2J_3$ (22) | $O_1L_2F_3$ (24) | $D_1M_2A_3$ (23) | $H_1N_2M_3$ (17) | $L_1O_2I_3$ (25) | $P_1P_2E_3$ (20) | $E_1A_2P_3$ (21) | $I_1B_1L_3$ (25) | $M_1C_2H_3$ (20) | $A_1D_2D_3$ (23) | $F_1E_2O_3$ (22) | $J_1F_2K_3$ (18) | $N_1G_2G_3$ (19) | $B_1H_2C_3$ (29) |
| 4 | $D_1M_2C_3$ (26) | $H_1N_2O_3$ (20) | $L_1O_2K_3$ (15) | $P_1P_2G_3$ (10) | $E_1A_2B_3$ (13) | $I_1B_2N_3$ (22) | $M_1G_2J_3$ (18) | $A_1D_2F_3$ (15) | $F_1E_2A_3$ (16) | $J_1F_2M_3$ (19) | $N_1G_2I_3$ (24) | $B_1H_2E_3$ (18) | $G_1I_2P_3$ (20) | $K_1J_2L_3$ (24) | $O_1K_2H_3$ (29) | $C_1L_2D_3$ $(3_3)$ |
| 5 | $F_1B_2D_3$ (28) | $I_1C_2P_3$ (19) | $M_1D_2L_3$ (14) | $A_1E_2H_3$ (9) | $F_1F_2C_3$ (12) | $J_1G_2O_3$ (21) | $N_1H_2K_3$ (16) | $B_1I_2G_3$ (10) | $G_1J_2B_3$ (18) | $K_1K_2N_3$ (22) | $O_1L_2J_3$ (19) | $C_1M_2F_3$ (22) | $H_1N_2A_3$ (21) | $L_1O_2M_3$ (23) | $P_1P_2I_3$ (27) | $D_1A_2E_3$ (30) |
| 6 | $F_1F_2E_3$ (24) | $J_1G_2A_3$ (15) | $N_1H_2M_3$ (20) | $B_1I_2I_3$ (15) | $G_1J_2D_3$ (16) | $K_1K_2P_3$ (24) | $O_1L_2L_3$ (11) | $C_1M_2H_3$ (16) | $H_1N_2C_3$ (14) | $L_1O_2O_3$ (27) | $P_1P_2K_3$ (24) | $D_1A_2G_3$ (27) | $I_1B_2B_3$ (26) | $M_1C_2N_3$ (19) | $A_1D_2J_3$ (23) | $E_1E_2F_3$ (26) |
| 7 | $G_1J_2F_3$ (27) | $K_1K_2B_3$ (24) | $O_1L_2N_3$ (17) | $C_1M_2J_3$ (20) | $H_1N_2E_3$ (22) | $L_1O_2A_3$ (19) | $P_1P_2M_3$ (19) | $D_1A_2I_3$ (20) | $I_1B_2D_3$ (19) | $M_1C_2P_3$ (24) | $A_1D_2L_3$ (19) | $E_1E_2H_3$ (33) | $J_1F_2C_3$ (30) | $N_1G_2O_3$ (25) | $B_1H_2K_3$ (19) | $F_1I_2G_3$ (21) |
| 8 | $H_1N_2G_3$ $(2_3)$ | $L_1O_2C_3$ (17) | $P_1P_2O_3$ (22) | $D_1A_2K_3$ (25) | $I_1B_2F_3$ (28) | $M_1C_2B_3$ (23) | $A_1D_2N_3$ (25) | $E_1E_2J_3$ (18) | $J_1F_2E_3$ (16) | $N_1G_2A_3$ (19) | $B_1H_2M_3$ (25) | $F_1I_2I_3$ (29) | $K_1J_2D_3$ (27) | $O_1K_2P_3$ (20) | $C_1L_2L_3$ (24) | $G_1M_2H_3$ (19) |
| 9 | $I_1C_2H_3$ $(2_2)$ | $M_1D_2D_3$ (19) | $A_1E_2P_3$ (25) | $E_1F_2L_3$ (22) | $J_1G_2G_3$ (25) | $N_1H_2C_3$ (20) | $B_1I_2O_3$ (23) | $F_1J_2K_3$ (20) | $K_1K_2F_3$ (19) | $O_1L_2B_3$ (24) | $C_1M_2N_3$ (18) | $J_1N_2J_3$ (27) | $L_1O_2E_3$ (26) | $P_1P_2A_3$ (19) | $D_1A_2M_3$ (25) | $H_1B_2I_3$ (20) |
| 10 | $J_1G_2I_3$ (18) | $N_1H_2E_3$ (14) | $B_1I_2A_3$ (19) | $F_1J_2M_3$ (15) | $K_1K_2H_3$ (17) | $O_1L_2D_3$ (24) | $C_1M_2P_3$ (28) | $G_1N_2L_3$ (24) | $L_1O_2G_3$ (23) | $P_1P_2C_3$ (29) | $D_1A_2O_3$ (22) | $H_1B_2K_3$ (21) | $M_1C_2F_3$ (22) | $A_1D_2B_3$ (24) | $E_1E_2N_3$ (20) | $I_1F_2J_3$ (16) |
| 11 | $K_1K_2J_3$ $(2_3)$ | $O_1L_2F_3$ (20) | $C_1M_2B_3$ (15) | $G_1N_2N_3$ (20) | $L_1O_2I_3$ (16) | $P_1P_2E_3$ (20) | $D_1A_2A_3$ (25) | $H_1B_2M_3$ (20) | $M_1C_2H_3$ (24) | $A_1D_2D_3$ (23) | $E_1E_2P_3$ (19) | $I_1F_2L_3$ (15) | $N_1G_2G_3$ (17) | $B_1H_2C_3$ (20) | $F_1I_2O_3$ (15) | $J_1J_2K_3$ (21) |
| 12 | $L_1O_2K_3$ (19) | $P_1P_2G_3$ (25) | $D_1A_2C_3$ (26) | $H_1B_2O_3$ (17) | $M_1C_2J_3$ (19) | $A_1D_2F_3$ (25) | $E_1E_2B_3$ (19) | $I_1F_2N_3$ (25) | $N_1G_2I_3$ (20) | $B_1H_2E_3$ (27) | $F_1I_2A_3$ (23) | $J_1J_2M_3$ (10) | $O_1K_2H_3$ (11) | $C_1L_2D_3$ (15) | $G_1M_2P_3$ (10) | $K_1N_2L_3$ (15) |
| 13 | $M_1D_2L_3$ $(2_2)$ | $A_1E_2H_3$ (24) | $E_1F_2D_3$ (27) | $I_1G_2P_3$ (20) | $N_1H_2K_3$ (21) | $B_1I_2G_3$ (26) | $F_1J_2C_3$ (21) | $J_1K_2O_3$ (26) | $O_1L_2J_3$ (24) | $C_1M_2F_3$ (26) | $G_1N_2B_3$ (22) | $K_1O_2N_3$ (12) | $P_1P_2I_3$ (13) | $D_1A_2E_3$ (17) | $H_1B_2A_3$ (11) | $L_1C_2M_3$ (17) |
| 14 | $N_1H_2M_3$ (26) | $B_1I_2I_3$ (28) | $F_1J_2E_3$ $(2_2)$ | $J_1K_2A_3$ (16) | $O_1L_2L_3$ (17) | $C_1M_2H_3$ (22) | $G_1N_2D_3$ (26) | $K_1O_2P_3$ (20) | $P_1P_1K_3$ (19) | $D_1A_2G_3$ (22) | $H_1B_2C_3$ (27) | $L_1C_2O_3$ (17) | $A_1D_2J_3$ (18) | $E_1E_2F_3$ (22) | $I_1F_2B_3$ (17) | $M_1G_2N_3$ (22) |
| 15 | $O_1L_2N_3$ $(2_1)$ | $C_1M_2J_3$ (25) | $G_1N_2F_3$ (29) | $K_1O_2B_3$ (33) | $P_1P_2M_3$ (32) | $D_1A_2I_3$ (27) | $H_1B_2E_3$ (31) | $L_1C_2A_3$ (25) | $A_1D_2L_3$ (23) | $E_1E_2H_3$ (27) | $I_1F_2D_3$ (33) | $M_1G_2P_3$ (25) | $B_1H_2K_3$ (24) | $F_1I_2G_3$ (27) | $J_1J_2C_3$ (24) | $N_1K_2O_3$ (19) |
| 16 | $P_1P_2O_3$ (16) | $D_1A_2K_3$ (20) | $H_1B_2G_3$ (25) | $L_1C_2C_3$ (29) | $A_1D_2N_3$ (27) | $E_1E_2J_3$ (33) | $I_1F_2F_3$ (29) | $M_1G_2B_3$ (21) | $B_1H_2M_3$ (20) | $F_1I_2I_3$ (24) | $J_1J_2E_3$ (29) | $N_1K_2A_3$ (20) | $C_1L_2L_3$ (21) | $G_1M_2H_3$ (32) | $K_1N_2D_3$ (29) | $O_1O_2P_3$ (24) |

Table 9: ANOVA Table for 16x16 Hyper Graeco-Latin Square Design

| Source of variation | d.f | SS | MS | F |
|---|---|---|---|---|
| Rows | 15 | 997.625 | 66.5083 | 2.9888 |
| Columns | 15 | 293.7501 | 19.5833 | 0.8800 |
| Treatments type(1) | 15 | 311 | 20.7330 | 0.9317 |
| Treatments type(2) | 15 | 196.5001 | 13.1001 | 0.5887 |
| Treatments type(3) | 15 | 359.7501 | 23.9833 | 1.0777 |
| Error | 180 | 4045.8125 | 22.2521 | |
| Total | 255 | 6164 | | |

Table 10: ANOVA Table for Hyper GLaSS Design

| Source of variation | d.f | SS | MS | F |
|---|---|---|---|---|
| Rows | 15 | 997.625 | 66.5083 | 3.5035 |
| Columns | 15 | 293.7501 | 19.5833 | 1.03162 |
| Treatments type(1) | 15 | 311 | 20.7330 | 1.0921 |
| Blocks | 15 | 913.6250 | 60.9083 | 3.2085 |
| Treatments type(2) | 15 | 196.5001 | 13.1001 | 0.69 |
| Treatments type(3) | 15 | 359.7501 | 23.9833 | 1.2634 |
| Error | 165 | 3132.1875 | 18.9829 | |
| Total | 255 | 6164 | | |

Table 11: Average mean of error sum of square by different block size effects for the Hyper Graeco-Latin square and Hyper GlaSS designs along with the difference between the average mean of error sum of squares and relative efficiency loss between the two considered designs.

| Expectation of block effect | MSQ Hyper Graeco-Latin square | MSQ Hyper GLaSS | Difference | Efficiency loss |
|---|---|---|---|---|
| 0 | 197.02 | 181.56 | 15.46 | 8.51% |
| 1 | 196.37 | 181.32 | 15.05 | 8.30% |
| 2 | 197.04 | 177.62 | 19.42 | 10.93% |
| 3 | 193.84 | 173.75 | 20.09 | 11.56% |
| 4 | 197.28 | 174.48 | 22.8 | 13.06% |
| 5 | 197.98 | 174.5 | 23.48 | 13.45% |
| 6 | 198.92 | 171.45 | 27.47 | 16.02% |
| 7 | 201.27 | 170.42 | 30.85 | 18.10% |
| 8 | 199.66 | 163.66 | 36 | 21.99% |
| 9 | 204.12 | 164.47 | 39.64 | 24.10% |
| 10 | 204.74 | 159.35 | 45.39 | 28.47% |

then the difference of MSQ between the two designs also increases. In Figure 1(B) when expected block effect increases, then the relative efficiency of Hyper GLaSS design also increases.

# 6 Conclusions

Hyper Graeco Latin Sudoku Square Design (Hyper GLaSS Design) has been introduced as merger of two designs i.e., Hyper Graeco Latin Square Design and Sudoku Square Design. Introducing the Block Sum of Square in the new proposed design, the error sum of square is further reduced. The purpose of Hyper GLaSS design is to test three sets of treatments simultaneously in one experiment and to allow investigation of six factors. Parameter estimation with Random effect model and mixed effects model with ANOVA has been performed for the proposed design. The efficiency of the new proposed design was examined through numerical example by using hypothetical data set and also with the help of simulation study. From numerical illustrations

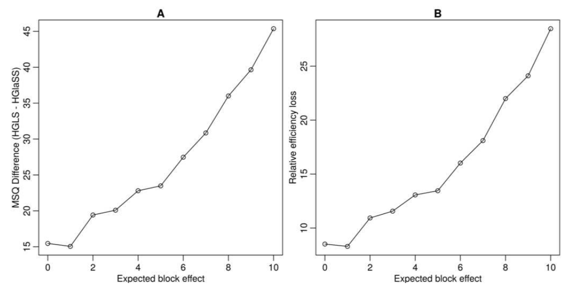


Figure 1: MSQ difference between the Hyper Graeco-Latin square and Hyper GlaSS designs 1(A) and relative efficiency loss by expected values of the block effect 1 (B).

we conclude that mean square Error of Hyper GLaSS Design is smaller than mean square error of Hyper Graeco Latin Square Design. Further, relative efficiency of the two designs have been compared and have come to the conclusion that the new proposed design is more efficient than Hyper Graeco Latin Square Design. Hence the additional blocking factor makes the result more authentic and with the use of Hyper GLaSS Design, one can control variability from six sides with lesser mean square error.

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