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Generalized Power UP-Algebras

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Abstract

The power UP-algebras of types 1 and 2 were proved by Iampan [1]. In this paper, we prove the generalized power UP-algebras of types 1 and 2, and find its cardinality.

1 Introduction and Preliminaries

In [1], Iampan introduced a new algebraic structure, called a UP-algebra, which is a generalization of a KU-algebra, and proved the following:
Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A \cdot B = B \cap A'$ and $A * B = B \cup A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras, called the *power UP-algebra of types 1 and 2*, respectively.

In this paper, we prove the generalized power UP-algebras of types 1 and 2, and find its cardinality.

Now we will recall the definition of a UP-algebra from [1].

An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra*, where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

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(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras. A mapping f from A to B is called a *UP-homomorphism* [1] if $f(x \cdot y) = f(x) * f(y)$ or all $x, y \in A$. A UP-homomorphism $f \colon A \to B$ is called a *UP-isomorphism* if it is bijective. Moreover, we say A is *UP-isomorphic* to A', symbolically, $A \cong A'$, if there is a UP-isomorphism from A to A'.

2 Generalized Power UP-algebra of Type 1

Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A' \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$.

Theorem 2.1. $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ *is a UP-algebra and we shall call it the generalized power UP-algebra of type 1 with respect to* Ω.

Proof. Let $A, B, C \in \mathcal{P}_\Omega(X)$. Then

$$
\begin{aligned}
(A \cdot B) \cdot (A \cdot C) &= (B \cap (A' \cup \Omega)) \cdot (C \cap (A' \cup \Omega)) \\
&= (C \cap (A' \cup \Omega)) \cap ((B \cap (A' \cup \Omega))' \cup \Omega) \\
&= (C \cap (A' \cup \Omega)) \cap ((B' \cup (A' \cup \Omega)') \cup \Omega) \\
&= (C \cap (A' \cup \Omega)) \cap ((B' \cup ((A')' \cap \Omega')) \cup \Omega) \\
&= (C \cap (A' \cup \Omega)) \cap ((B' \cup (A \cap \Omega')) \cup \Omega) \\
&= (C \cap (A' \cup \Omega)) \cap (B' \cup ((A \cap \Omega') \cup \Omega)) \\
&= (C \cap (A' \cup \Omega)) \cap (B' \cup ((A \cup \Omega) \cap (\Omega' \cup \Omega))) \\
&= (C \cap (A' \cup \Omega)) \cap (B' \cup (A \cap X)) \\
&= (C \cap (A' \cup \Omega)) \cap (B' \cup A) \\
&= C \cap (((A' \cup \Omega) \cap B') \cup ((A' \cup \Omega) \cap A)) \\
&= C \cap (((A' \cup \Omega) \cap B') \cup ((A' \cap A) \cup (\Omega \cap A))) \\
&= C \cap (((A' \cup \Omega) \cap B') \cup (\emptyset \cup \Omega)) && (\Omega \subseteq A) \\
&= C \cap (((A' \cup \Omega) \cap B') \cup \Omega) \\
&= C \cap (((A' \cap B') \cup (\Omega \cap B')) \cup \Omega) \\
&= C \cap ((A' \cap B') \cup ((\Omega \cap B') \cup \Omega)) \\
&= C \cap ((A' \cap B') \cup \Omega) && (\Omega \cap B' \subseteq \Omega) \\
&= (C \cap (A' \cap B')) \cup (C \cap \Omega) \\
&= (C \cap (A' \cap B')) \cup \Omega && (\Omega \subseteq C) \\
&= (C \cup \Omega) \cap ((A' \cap B') \cup \Omega) \\
&= C \cap ((A' \cap B') \cup \Omega). && (\Omega \subseteq C)
\end{aligned}
$$

Thus

$$
\begin{aligned}
(B \cdot C) \cdot ((A \cdot B) \cdot (A \cdot C)) &= (B \cdot C) \cdot (C \cap ((A' \cap B') \cup \Omega)) \\
&= (C \cap (B' \cup \Omega)) \cdot (C \cap ((A' \cap B') \cup \Omega)) \\
&= (C \cap ((A' \cap B') \cup \Omega)) \cap ((C \cap (B' \cup \Omega))' \cup \Omega).
\end{aligned}
$$

Since

$$\begin{aligned}
(C \cap (B' \cup \Omega))' \cup \Omega &= (C' \cup (B' \cup \Omega)') \cup \Omega \\
&= (C' \cup ((B')' \cap \Omega')) \cup \Omega \\
&= (C' \cup (B \cap \Omega')) \cup \Omega \\
&= (C' \cup \Omega) \cup ((B \cap \Omega') \cup \Omega) \\
&= (C' \cup \Omega) \cup ((B \cup \Omega) \cap (\Omega' \cup \Omega)) \\
&= (C' \cup \Omega) \cup (B \cap X) && (\Omega \subseteq B) \\
&= (C' \cup \Omega) \cup B,
\end{aligned}$$

we have

$$\begin{aligned}
(B \cdot C) \cdot ((A \cdot B) \cdot (A \cdot C)) &= (C \cap ((A' \cap B') \cup \Omega)) \cap ((C' \cup \Omega) \cup B) \\
&= (((A' \cap B') \cup \Omega) \cap C) \cap ((C' \cup \Omega) \cup B) \\
&= ((A' \cap B') \cup \Omega) \cap (C \cap ((C' \cup \Omega) \cup B)) \\
&= ((A' \cap B') \cup \Omega) \cap (C \cap ((C' \cup \Omega) \cup B)).
\end{aligned}$$

Because

$$\begin{aligned}
C \cap ((C' \cup \Omega) \cup B) &= (C \cap (C' \cup \Omega)) \cup (C \cap B) \\
&= ((C \cap C') \cup (C \cap \Omega)) \cup (C \cap B) \\
&= (\emptyset \cup \Omega) \cup (C \cap B) && (\Omega \subseteq C) \\
&= \Omega \cup (C \cap B) \\
&= (\Omega \cup C) \cap (\Omega \cup B) \\
&= C \cap B, && (\Omega \subseteq B, C)
\end{aligned}$$

we have

$$\begin{aligned}
(B\cdot C)\cdot((A\cdot B)\cdot(A\cdot C)) &= ((A'\cap B')\cup\Omega)\cap(C\cap B)\\
&= ((A'\cap B')\cap(C\cap B))\cup(\Omega\cap(C\cap B))\\
&= ((A'\cap B')\cap(C\cap B))\cup((\Omega\cap C)\cap B)\\
&= ((A'\cap B')\cap(C\cap B))\cup(\Omega\cap B) \qquad (\Omega\subseteq C)\\
&= ((A'\cap B')\cap(C\cap B))\cup\Omega \qquad (\Omega\subseteq B)\\
&= ((A'\cap B')\cup\Omega)\cap((C\cap B)\cup\Omega)\\
&= ((A'\cap B')\cup\Omega)\cap((C\cup\Omega)\cap(B\cup\Omega))\\
&= ((A'\cap B')\cup\Omega)\cap(C\cap B) \qquad (\Omega\subseteq B,C)\\
&= (((A'\cap B')\cup\Omega)\cap C)\cap B\\
&= (((A'\cap B')\cap C)\cup(\Omega\cap C))\cap B\\
&= (((A'\cap B')\cap C)\cup\Omega)\cap B \qquad (\Omega\subseteq C)\\
&= (((A'\cap B')\cap C)\cap B)\cup(\Omega\cap B)\\
&= (((A'\cap B')\cap C)\cap B)\cup\Omega \qquad (\Omega\subseteq B)\\
&= ((A'\cap B')\cap(C\cap B))\cup\Omega\\
&= ((A'\cap B')\cap(B\cap C))\cup\Omega\\
&= (((A'\cap B')\cap B)\cap C)\cup\Omega\\
&= (((A'\cap B)\cap(B'\cap B))\cap C)\cup\Omega\\
&= (((A'\cap B)\cap\emptyset)\cap C)\cup\Omega\\
&= (\emptyset\cap C)\cup\Omega\\
&= \emptyset\cup\Omega\\
&= \Omega.
\end{aligned}$$

Thus (UP-1) holds. Consider,

$$\Omega\cdot A = A\cap(\Omega'\cup\Omega) = A\cap X = A$$

and

$$A\cdot\Omega = \Omega\cap(A'\cup\Omega) = \Omega.$$

Thus (UP-2) and (UP-3) hold. If $A \cdot B = \Omega$ and $B \cdot A = \Omega$, then $B \cap (A' \cup \Omega) = \Omega$ and $A \cap (B' \cup \Omega) = \Omega$. Thus $(B \cap A') \cup (B \cap \Omega) = \Omega$ and $(A \cap B') \cup (A \cap \Omega) = \Omega$. Since $\Omega \subseteq A, B$, we have $(B \cap A') \cup \Omega = \Omega$ and $(A \cap B') \cup \Omega = \Omega$. So $B \cap A' \subseteq \Omega$ and $A \cap B' \subseteq \Omega$. That is, $B - A \subseteq \Omega$ and $A - B \subseteq \Omega$, so $B - A = \emptyset$ and $A - B = \emptyset$. Thus $B \subseteq A$ and $A \subseteq B$, so $A = B$. Thus (UP-4) hold as well. Hence, $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra. □

In particular, $(\mathcal{P}_\emptyset(X), \cdot, \emptyset)$ is the power UP-algebra of type 1.

Lemma 2.2. *If X is a finite set, then $|\mathcal{P}_\Omega(X)| = |\mathcal{P}(X - \Omega)| = 2^{|X|-|\Omega|}$.*

Proof. We see that for all $A \in \mathcal{P}_\Omega(X), A = \Omega \cup \widehat{A}$ for some $\widehat{A} \in X - \Omega$. Hence, $|\mathcal{P}_\Omega(X)| = |\mathcal{P}(X - \Omega)| = 2^{|X|-|\Omega|}$. □

Theorem 2.3. *Let X be a finite set with $|X| = 2^n$ and let $|\Omega| = 2^n - n$. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ induces X to a UP-algebra and $X \cong \mathcal{P}_\Omega(X)$.*

Proof. By Theorem 2.1 and Lemma 2.2, we have $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and $|\mathcal{P}_\Omega(X)| = 2^{2^n-(2^n-n)} = 2^n = |X|$. Let $f\colon \mathcal{P}_\Omega(X) \to X$ be a bijection. Define a binary operation $\circ$ on X by putting $x \circ y = f(f^{-1}(x) \cdot f^{-1}(y))$ for all $x, y \in X$. We shall show that $(X, \circ, f(\Omega))$ is a UP-algebra. Clearly, $\circ$ is a binary operation on X. Let $x, y, z \in X$.
We have,

$$\begin{aligned} &(y \circ z) \circ ((x \circ y) \circ (x \circ z)) \\ &= (f(f^{-1}(y) \cdot f^{-1}(z))) \circ ((f(f^{-1}(x) \cdot f^{-1}(y))) \circ (f(f^{-1}(x) \cdot f^{-1}(z)))) \\ &= (f(f^{-1}(y) \cdot f^{-1}(z))) \circ (f((f^{-1}(x) \cdot f^{-1}(y)) \cdot (f^{-1}(x) \cdot f^{-1}(z)))) \\ &= f((f^{-1}(y) \cdot f^{-1}(z)) \cdot ((f^{-1}(x) \cdot f^{-1}(y)) \cdot (f^{-1}(x) \cdot f^{-1}(z)))) = f(\Omega). \end{aligned}$$

Thus (UP-1) follows. Since $f(\Omega) \circ x = f(\Omega \cdot f^{-1}(x)) = f(f^{-1}(x)) = x$ and $x \circ f(\Omega) = f(f^{-1}(x) \cdot \Omega) = f(\Omega)$. Thus (UP-2) and (UP-3) follow. Let $x, y \in X$ be such that $x \circ y = f(\Omega)$ and $y \circ x = f(\Omega)$. Then $f(f^{-1}(x) \cdot f^{-1}(y)) = f(\Omega)$ and $f(f^{-1}(y) \cdot f^{-1}(x)) = f(\Omega)$. Since f is an injection, we have $f^{-1}(x) \cdot f^{-1}(y) = \Omega$ and $f^{-1}(y) \cdot f^{-1}(x) = \Omega$. Thus $f^{-1}(x) = f^{-1}(y)$, so $x = y$. Thus (UP-4) follows as well. Hence, $(X, \circ, f(\Omega))$ is a UP-algebra.

Finally, let $A, B \in \mathcal{P}_\Omega(X)$. Then $A = f^{-1}(x)$ and $B = f^{-1}(y)$ for some $x, y \in X$. Thus

$$f(A \cdot B) = f(f^{-1}(x) \cdot f^{-1}(y)) = x \circ y = f(A) \circ f(B).$$

Hence, $X \cong \mathcal{P}_\Omega(X)$. □

3 Generalized Power UP-algebra of Type 2

Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Denote $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A' \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$.

Theorem 3.1. *$(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the generalized power UP-algebra of type 2 with respect to Ω.*

Proof. Let $A, B, C \in \mathcal{P}^\Omega(X)$. Then

$$\begin{aligned}
(A * B) * (A * C) &= (B \cup (A' \cap \Omega)) * (C \cup (A' \cap \Omega)) \\
&= (C \cup (A' \cap \Omega)) \cup ((B \cup (A' \cap \Omega))' \cap \Omega) \\
&= (C \cup (A' \cap \Omega)) \cup ((B' \cap (A \cup \Omega')) \cap \Omega) \\
&= (C \cup (A' \cap \Omega)) \cup ((B' \cap \Omega) \cap (A \cup \Omega')) \\
&= (C \cup (A' \cap \Omega)) \cup (((B' \cap \Omega) \cap A) \cup ((B' \cap \Omega) \cap \Omega')) \\
&= (C \cup (A' \cap \Omega)) \cup (((B' \cap \Omega) \cap A) \cup \emptyset) \\
&= (C \cup (A' \cap \Omega)) \cup ((B' \cap \Omega) \cap A) \\
&= (C \cup (A' \cap \Omega)) \cup (B' \cap A) && (A \subseteq \Omega) \\
&= ((C \cup (A' \cap \Omega)) \cup B') \cap ((C \cup (A' \cap \Omega)) \cup A) \\
&= ((C \cup (A' \cap \Omega)) \cup B') \cap (C \cup ((A' \cap \Omega) \cup A)) \\
&= ((A' \cap \Omega) \cup B' \cup C) \cap (C \cup ((A' \cup A) \cap (\Omega \cup A))) \\
&= ((A' \cap \Omega) \cup B' \cup C) \cap (C \cup (X \cap \Omega)) && (A \subseteq \Omega) \\
&= ((A' \cap \Omega) \cup B' \cup C) \cap (C \cup \Omega) \\
&= ((A' \cap \Omega) \cup B' \cup C) \cap \Omega. && (C \subseteq \Omega)
\end{aligned}$$

Thus

$$\begin{aligned}(B*C)*((A*B)*(A*C)) &= (C\cup(B'\cap\Omega))*(((A'\cap\Omega)\cup B'\cup C)\cap\Omega)\\ &= (((A'\cap\Omega)\cup B'\cup C)\cap\Omega)\cup((C\cup(B'\cap\Omega))'\cap\Omega)\\ &= (((A'\cap\Omega)\cup B'\cup C)\cup(C\cup(B'\cap\Omega))')\cap\Omega\\ &= (((A'\cap\Omega)\cup B'\cup C)\cup(C'\cap(B\cup\Omega')))\cap\Omega\\ &= ((A'\cap\Omega)\cup B'\cup C\cup(C'\cap(B\cup\Omega')))\cap\Omega\\ &= ((A'\cap\Omega)\cup B'\cup C\cup((C'\cap B)\cup(C'\cap\Omega')))\cap\Omega\\ &= ((A'\cap\Omega)\cup B'\cup C\cup(B\cap C')\cup(C'\cap\Omega'))\cap\Omega\\ &= ((A'\cap\Omega)\cup(B\cap C')'\cup(B\cap C')\cup(C'\cap\Omega'))\cap\Omega\\ &= ((A'\cap\Omega)\cup X\cup(C'\cap\Omega'))\cap\Omega\\ &= X\cap\Omega\\ &= \Omega.\end{aligned}$$

Thus (UP-1) holds. Consider,

$$\Omega*A = A\cup(\Omega'\cap\Omega) = A\cup\emptyset = A$$

and

$$A*\Omega = \Omega\cup(A'\cap\Omega) = \Omega.$$

Thus (UP-2) and (UP-3) are holding. If $A*B=\Omega$ and $B*A=\Omega$, then $B\cup(A'\cap\Omega)=\Omega$ and $A\cup(B'\cap\Omega)=\Omega$. Thus $(B\cup A')\cap(B\cup\Omega)=\Omega$ and $(A\cup B')\cap(A\cup\Omega)=\Omega$. Since $A,B\subseteq\Omega$, we have $(B\cup A')\cap\Omega=\Omega$ and $(A\cup B')\cap\Omega=\Omega$. So $\Omega\subseteq B\cup A'$ and $\Omega\subseteq A\cup B'$. Thus $\Omega\cap B'\subseteq A'$ and $\Omega\cap A'\subseteq B'$, so $A\subseteq\Omega'\cup B$ and $B\subseteq\Omega'\cup A$. Thus $B\subseteq A$ and $A\subseteq B$, so $A=B$. Thus (UP-4) is holding. Hence, $(\mathcal{P}^{\Omega}(X),*,\Omega)$ is a UP-algebra. □

In particular, $(\mathcal{P}^{X}(X),*,X)$ is the power UP-algebra of type 2.

Lemma 3.2. *If X is a finite set, then $|\mathcal{P}^{\Omega}(X)| = |\mathcal{P}(\Omega)| = 2^{|\Omega|}$.*

Proof. We see that $\mathcal{P}^{\Omega}(X)=\mathcal{P}(\Omega)$. Hence, $|\mathcal{P}_{\Omega}(X)| = |\mathcal{P}(\Omega)| = 2^{|\Omega|}$. □

Theorem 3.3. *Let X be a finite set with $|X| = 2^n$ and let $|\Omega| = n$. Then $(\mathcal{P}^{\Omega}(X), *, \Omega)$ induces X to a UP-algebra and $X \cong \mathcal{P}^{\Omega}(X)$.*

Proof. By Theorem 3.1 and Lemma 3.2, we have $(\mathcal{P}^{\Omega}(X), *, \Omega)$ is a UP-algebra and $|\mathcal{P}_{\Omega}(X)| = 2^n = |X|$. Let $f \colon \mathcal{P}^{\Omega}(X) \to X$ be a bijection. Define a binary operation $\diamond$ on X by putting $x \diamond y = f(f^{-1}(x) * f^{-1}(y))$ for all $x, y \in X$. We shall show that $(X, \diamond, f(\Omega))$ is a UP-algebra. Clearly, $\diamond$ is a binary operation on X. Let $x, y, z \in X$. Now,

$$
\begin{aligned}
&(y \diamond z) \diamond ((x \diamond y) \diamond (x \diamond z)) \\
&= (f(f^{-1}(y) * f^{-1}(z))) \diamond ((f(f^{-1}(x) * f^{-1}(y))) \diamond (f(f^{-1}(x) * f^{-1}(z)))) \\
&= (f(f^{-1}(y) * f^{-1}(z))) \diamond (f((f^{-1}(x) * f^{-1}(y)) * (f^{-1}(x) * f^{-1}(z)))) \\
&= f((f^{-1}(y) * f^{-1}(z)) * ((f^{-1}(x) * f^{-1}(y)) * (f^{-1}(x) * f^{-1}(z)))) = f(\Omega).
\end{aligned}
$$

Thus (UP-1) holds. Since $f(\Omega) \diamond x = f(\Omega * f^{-1}(x)) = f(f^{-1}(x)) = x$ and $x \diamond f(\Omega) = f(f^{-1}(x) * \Omega) = f(\Omega)$. Thus (UP-2) and (UP-3) hold. Let $x, y \in X$ be such that $x \diamond y = f(\Omega)$ and $y \diamond x = f(\Omega)$. Then $f(f^{-1}(x) * f^{-1}(y)) = f(\Omega)$ and $f(f^{-1}(y) * f^{-1}(x)) = f(\Omega)$. Since f is an injection, we have $f^{-1}(x) * f^{-1}(y) = \Omega$ and $f^{-1}(y) * f^{-1}(x) = \Omega$. Thus $f^{-1}(x) = f^{-1}(y)$, so $x = y$. Thus (UP-4)holds as well. Hence, $(X, \diamond, f(\Omega))$ is a UP-algebra. Finally, let $A, B \in \mathcal{P}^{\Omega}(X)$. Then $A = f^{-1}(x)$ and $B = f^{-1}(y)$ for some $x, y \in X$. Thus

$$f(A * B) = f(f^{-1}(x) * f^{-1}(y)) = x \diamond y = f(A) \diamond f(B).$$

Hence, $X \cong \mathcal{P}^{\Omega}(X)$. ∎

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