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Exact Formulas for the Number of Palindromes up to a Given Positive Integer

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Abstract

Let $n \geq 0$ and $b \geq 2$ be integers. Then n is said to be a palindrome in base b (or b-adic palindrome) if $n = 0$ or $n \geq 1$ and the representation of $n = (a_k a_{k-1} \cdots a_1 a_0)_b$ in base b with $a_k \neq 0$ has the symmetric property $a_{k-i} = a_i$ for $0 \leq i \leq k$. Let $A_b(m)$ be the number of b-adic palindromes not exceeding m. In addition, let $A_b^{(even)}(m)$ and $A_b^{(odd)}(m)$ be the number of even and odd b-adic palindromes less than or equal to m, respectively. In this article, we obtain exact formulas for $A_b(m)$, $A_b^{(even)}(m)$, and $A_b^{(odd)}(m)$ for all $m \in \mathbb{N}$.

1 Introduction

Let $n \geq 0$ and $b \geq 2$ be integers. Then n is said to be a palindrome in base b (or b-adic palindrome) if $n = 0$ or $n \geq 1$ and the b-adic expansion $n = (a_k a_{k-1} \cdots a_0)_b$ satisfies $a_i = a_{k-i}$ for all $i \in \{0, 1, \ldots, k\}$. For convenience, if we write $n = (a_k a_{k-1} \cdots a_0)_b$, then it means that $n = \sum_{i=0}^{k} a_i b^i$, $a_k \neq 0$, and $0 \leq a_i < b$ for all $i = 0, 1, \ldots, k$. In addition, if we do not specify a base, it is always written in base 10. So, for example, $9 = (1001)_2 = (100)_3$ is a

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palindrome in bases 2 and 10 but is not a palindrome in base 3. See also sequence A002113 in OEIS [21] for more information.

In recent years, there has been an increasing interest in the importance of palindromes in mathematics (see e.g. [1, 2, 3, 7, 10]), theoretical computer science (see [6, 9]), and theoretical physics (see [12]). Nevertheless, many open questions concerning palindromes remain unsolved. For example, it is not known whether there are infinitely many primes which are palindromes. Let P_b be the set of all palindromes in base b and $P_b(m)$ the set of all b-adic palindromes not exceeding m. Banks, Hart and Sakata [5] obtain

$$\sum_{\substack{n\in P_b(m)\\ n \text{ is prime}}} 1 \ll |P_b(m)| \cdot \frac{\log\log\log m}{\log\log m} \quad \text{as } m \to \infty.$$

It is not difficult to show that the order of magnitude of $|P_b(m)|$ is $\sqrt{m}$ but as far as we are aware the exact formula for $|P_b(m)|$ has not appeared in the literature. In this article, we obtain exact formulas for $A_b(m)$, $A_b^{(even)}(m)$, and $A_b^{(odd)}(m)$ for all $m \geq 1$ and $b \geq 2$, where

$$A_b(m) = |P_b(m)| = \sum_{n\in P_b(m)} 1, \quad A_b^{(even)}(m) = \sum_{\substack{n\in P_b(m)\\ n \text{ is even.}}} 1, \quad A_b^{(odd)}(m) = \sum_{\substack{n\in P_b(m)\\ n \text{ is odd.}}} 1.$$

For other results concerning palindromes, we refer the reader to Korec [13] for nonpalindromic numbers having palindromic squares, Harminc and Soták [11] for b-adic palindromes in arithmetic progressions, Banks [4], Cilleruelo, Lura, and Baxter [8], and Rajasekaran, Shallit, and Smith [20] for additive properties of palindromes. For other counting formulas or some number-theoretic and combinatorial sequences, see for example in [14, 15, 16, 17, 18, 19].

2 Main Results.

We divide this section into two parts. We first count the b-adic palindromes less than or equal to m and obtain an exact formula for $A_b(m)$ in Section 2.1. Then we give the formulas for $A_b^{(even)}(m)$ and $A_b^{(odd)}(m)$ in Section 2.2. Although we can obtain $A_b(m)$ from the fact that $A_b(m) = A_b^{(even)}(m) + A_b^{(odd)}(m)$, it seems more convenient to first calculate $A_b(m)$ and extend the idea to the cases $A_b^{(even)}(m)$ and $A_b^{(odd)}(m)$.

2.1 Exact formulas for $A_b(m)$ for all $m \geq 1$ and $b \geq 2$.

Recall that for a real number x, $\lfloor x \rfloor$ is the largest integer less than or equal to x and $\lceil x \rceil$ is the smallest integer greater than or equal to x. In addition, let P be a mathematical statement. Then the *Iverson notation* $[P]$ is defined by

$$[P] = \begin{cases} 1, & \text{if } P \text{ holds;} \\ 0, & \text{otherwise.} \end{cases}$$

It is also convenient to define m^* and $C_b(m)$ for each $m \in \mathbb{N}$ as follows:

Definition 2.1. *Let $b \geq 2$. For each $m \in \mathbb{N}$, we define $C_b(m)$ to be the smallest b-adic palindrome larger than or equal to m. In addition, if $m = (a_k a_{k-1} \cdots a_1 a_0)_b$, then we define*

$$m^* = \sum_{0 \leq i \leq \lfloor \frac{k}{2} \rfloor} a_{k-i} b^{k-i} = (a_k a_{k-1} \cdots a_{k-\lfloor \frac{k}{2} \rfloor} 00 \cdots 0)_b.$$

Example 2.1. *From Definition 2.1, if $m = (a_0)_b$, $(a_1 a_0)_b$, $(a_2 a_1 a_0)_b$, then $m^* = (a_0)_b$, $(a_1 0)_b$, $(a_2 a_1 0)_b$, respectively. In general, we have $m^* \leq m$ and if $m = (a_k a_{k-1} \cdots a_1 a_0)_b$, then*

$$C_b(m^*) = (a_k a_{k-1} \cdots a_{k-\lfloor \frac{k}{2} \rfloor} \cdots a_{k-1} a_k)_b.$$

For instance, if $m = (247853)_9$, then $m^* = (247000)_9$ and $C_b(m^*) = (247742)_9$.

Theorem 2.2. *Let $b \geq 2$, $m \geq 1$, and $m = (a_k a_{k-1} \cdots a_1 a_0)_b$. Then the number of b-adic pailindromes less than or equal to m is given by*

$$A_b(m) = b^{\lceil \frac{k}{2} \rceil} + \sum_{0 \leq i \leq \lfloor \frac{k}{2} \rfloor} a_{k-i} b^{\lfloor \frac{k}{2} \rfloor - i} + \delta(m) - 1, \tag{2.1}$$

where $\delta(m) = [m \geq C_b(m^*)]$.

Proof. We first consider the case $k \leq 1$. If $m = a_0$ where $1 \leq a_0 < b$, then $A_b(m)$ counts the palindromes $0, 1, 2, \ldots, a_0$, and therefore $A_b(m) = a_0 + 1$ and the result follows. Suppose $m = (a_1 a_0)_b$. Then the possible palindromes less than or equal to m are

$$0, 1, 2, \ldots, b-1, (11)_b, (22)_b, \ldots, ((a_1 - 1)(a_1 - 1))_b, \text{ and } (a_1 a_1)_b,$$

where $(a_1a_1)_b$ is counted if and only if $a_0 \geq a_1$. So the number of such palindromes is

$$b + (a_1 - 1) + \delta(m),$$

where $\delta(m) = [a_0 \geq a_1] = [m \geq C_b(m^*)]$. This proves the theorem for $k \leq 1$. So we assume throughout that $k \geq 2$. The calculation is divided into 7 steps.

Step 1. We count the number of b-adic palindromes which have $k+1$ digits in their b-adic expansions. We show that

$$\sum_{\substack{b^k \leq n < b^{k+1} \\ n \in P_b}} 1 = (b-1)\, b^{\left\lfloor \frac{k}{2} \right\rfloor}. \tag{2.2}$$

The left hand side of (2.2) counts the number of b-adic palindromes which have $k+1$ digits. Such the numbers are of the form $n = (c_k c_{k-1} \cdots c_1 c_0)_b$ where $c_k \neq 0, c_i \in \{0, 1, 2, \ldots, b-1\}$, and $c_i = c_{k-i}$ for all $i \in \{0, 1, 2, \ldots, \left\lfloor \frac{k}{2} \right\rfloor\}$. So there are $b-1$ possible values for c_k and, after c_k is chosen, there is only one possible value for $c_0 = c_k$. There are b choices for $c_{k-1} \in \{0, 1, \ldots, b-1\}$ and there is one choice for $c_1 = c_{k-1}$. In general, there are b choices for c_{k-i} for $1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$ and exactly one choice for the corresponding c_i. Therefore

$$\sum_{\substack{b^k \leq n < b^{k+1} \\ n \in P_b}} 1 = (b-1) \cdot \underbrace{b \cdot b \cdots b}_{\left\lfloor \frac{k}{2} \right\rfloor \text{ terms}} = (b-1)\, b^{\left\lfloor \frac{k}{2} \right\rfloor}.$$

Step 2. We show that the number of b-adic palindromes which have less than $k+1$ digits is

$$\sum_{\substack{1 \leq n < b^k \\ n \in P_b}} 1 = b^{\left\lceil \frac{k}{2} \right\rceil} + b^{\left\lfloor \frac{k}{2} \right\rfloor} - 2. \tag{2.3}$$

The left hand side of (2.3) can be written and evaluated by (2.2) as

$$\sum_{\ell=0}^{k-1} \sum_{\substack{b^\ell \leq n < b^{\ell+1} \\ n \in P_b}} 1 = \sum_{\ell=0}^{k-1} (b-1) b^{\left\lfloor \frac{\ell}{2} \right\rfloor}. \tag{2.4}$$

If k is even, then the above is equal to

$$(b-1)(2 + 2b + \cdots + 2b^{\frac{k-2}{2}}) = \frac{2(b-1)(b^{\frac{k}{2}} - 1)}{b-1} = b^{\left\lceil \frac{k}{2} \right\rceil} + b^{\left\lfloor \frac{k}{2} \right\rfloor} - 2.$$

Similarly, if k is odd, then the above is

$$2(b-1)(1+b+\cdots+b^{\frac{k-3}{2}}) + (b-1)b^{\frac{k-1}{2}} = 2(b^{\frac{k-1}{2}} - 1) + (b-1)b^{\frac{k-1}{2}} = b^{\left\lceil \frac{k}{2} \right\rceil} + b^{\left\lfloor \frac{k}{2} \right\rfloor} - 2.$$

This proves (2.3).

Step 3. We show that for $1 \le a < b$,

$$\sum_{\substack{b^k \le n < ab^k \\ n \in P_b}} 1 = (a-1) \cdot b^{\lfloor \frac{k}{2} \rfloor}. \tag{2.5}$$

The left hand side of (2.5) is the number of b-adic palindromes which have $k+1$ digits in their b-adic expansions with the leading digit less than a. Such the numbers n are of the form $(c_k c_{k-1} \cdots c_1 c_0)_b$ where $1 \le c_k < a$, $0 \le c_i < b$, and $c_{k-i} = c_i$ for all $i = 0, 1, 2, \ldots, k$. So the counting is similar to that in (2.2).

There are $a - 1$ choices for c_k and so there is only one choice for c_0. There are b choices for c_{k-i} for $1 \le i \le \lfloor \frac{k}{2} \rfloor$ and exactly one choice for the corresponding c_i. Hence

$$\sum_{\substack{b^k \le n < ab^k \\ n \in P_b}} 1 = (a-1) \cdot \underbrace{b \cdot b \cdots b}_{\lfloor \frac{k}{2} \rfloor \text{ terms}} = (a-1) \cdot b^{\lfloor \frac{k}{2} \rfloor}.$$

Step 4. By (2.3) and (2.5), we immediately obtain that for $1 \le a < b$,

$$\sum_{\substack{1 \le n < ab^k \\ n \in P_b}} 1 = \sum_{\substack{1 \le n < b^k \\ n \in P_b}} 1 + \sum_{\substack{b^k \le n < ab^k \\ n \in P_b}} 1 = b^{\lceil \frac{k}{2} \rceil} + ab^{\lfloor \frac{k}{2} \rfloor} - 2. \tag{2.6}$$

Step 5. Let $a_0, a_1, \ldots, a_k \in \{0, 1, \ldots, b-1\}$ and $a_k \ne 0$. For each $j \in \{0, 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$, let $m_j = \sum_{0 \le i \le j} a_{k-i} b^{k-i}$. So $m_j = (a_k a_{k-1} \cdots a_{k-j} 00 \cdots 0)_b$ and $m_{j+1} = (a_k a_{k-1} \cdots a_{k-(j+1)} 00 \cdots 0)_b$. We show that for $0 \le j \le \lfloor \frac{k}{2} \rfloor - 1$,

$$\sum_{\substack{m_j \le n < m_{j+1} \\ n \in P_b}} 1 = a_{k-(j+1)}\, b^{\lfloor \frac{k}{2} \rfloor - (j+1)}. \tag{2.7}$$

Let $n = (c_k c_{k-1} c_{k-2} \ldots c_1 c_0)_b$ be the palindromes which are counted in the left hand side of (2.7). The counting is similar to that in (2.2) and (2.5). Clearly, there is only one choice for $c_k, c_{k-1}, \ldots, c_{k-j}$, namely, $c_k = a_k$, $c_{k-1} = a_{k-1}$, $\ldots$, $c_{k-j} = a_{k-j}$. Then there is only one choice for each $c_0, c_1, \ldots, c_j$ since $c_{k-i} = c_i$ for $i = 0, 1, \ldots, j$.

Since $c_{k-(j+1)} \in \{0, 1, 2, \ldots, a_{k-(j+1)} - 1\}$, there are $a_{k-(j+1)}$ choices for $c_{k-(j+1)}$. The remaining digits c_{k-i}, where $j + 2 \le i \le \lfloor \frac{k}{2} \rfloor$, can be chosen arbitrarily from $0, 1, \ldots, b-1$. So similar to (2.2), the left hand side of (2.7) is equal to

$$a_{k-j+1} \cdot \underbrace{b \cdot b \cdots b}_{\lfloor \frac{k}{2} \rfloor - (j+1) \text{ terms}} = a_{k-(j+1)} b^{\lfloor \frac{k}{2} \rfloor - (j+1)}.$$

Step 6. We show that

$$\sum_{\substack{1 \le n < m^* \\ n \in P_b}} 1 = b^{\lceil \frac{k}{2} \rceil} - 2 + \sum_{0 \le j \le \lfloor \frac{k}{2} \rfloor} a_{k-j} b^{\lfloor \frac{k}{2} \rfloor - j}. \tag{2.8}$$

With the notation from (2.7) and the formula from (2.6), the left hand side of (2.8) is

$$\sum_{\substack{1 \le n < a_k b^k \\ n \in P_b}} 1 + \sum_{1 \le j \le \lfloor \frac{k}{2} \rfloor} \sum_{\substack{m_{j-1} \le n < m_j \\ n \in P_b}} 1 = b^{\lceil \frac{k}{2} \rceil} + a_k b^{\lfloor \frac{k}{2} \rfloor} - 2 + \sum_{1 \le j \le \lfloor \frac{k}{2} \rfloor} a_{k-j} b^{\lfloor \frac{k}{2} \rfloor - j},$$

which gives (2.8).

Step 7. We show that

$$\sum_{\substack{m^* \le n \le m \\ n \in P_b}} 1 = \delta(m). \tag{2.9}$$

Recall that $m = (a_k a_{k-1} \cdots a_1 a_0)_b$. The only possible palindrome n such that $m^* \le n \le m$ is $n = C_b(m^*)$. So the left hand side of (2.9) is 1 if $m \ge C_b(m^*)$ and is 0 otherwise. So (2.9) follows from the definition of $\delta(m)$. Now by writing,

$$A_b(m) = 1 + \sum_{\substack{1 \le n \le m \\ n \in P_b}} 1 = 1 + \sum_{\substack{1 \le n < m^* \\ n \in P_b}} 1 + \sum_{\substack{m^* \le n \le m \\ n \in P_b}} 1,$$

we can obtain the formula for $A_b(m)$ from (2.8) and (2.9). This completes the proof. □

2.2 Counting odd and even palindromes.

In order to obtain $A_b^{(even)}(m)$ and $A_b^{(odd)}(m)$, we divide our consideration according to the parity of b.

Lemma 2.2. *Let $b \ge 2$, $m \ge 1$, and $m = (a_k a_{k-1} \cdots a_1 a_0)_b$. If b is even, then m is even if and only if a_0 is even. If b is odd, then m is even if and only if $a_k + a_{k-1} + \cdots + a_0$ is even.*

Proof. Suppose b is even. Then $b^n \equiv 0 \pmod 2$ for all $n \in \mathbb{N}$. Since $m = a_0 + a_1 b + a_2 b^2 + \cdots + a_k b^k$, it follows that $m \equiv a_0 \pmod 2$. This implies that m is even if and only if a_0 is even. Similarly, if b is odd, then $b^n \equiv 1 \pmod 2$ for all $n \in \mathbb{N}$ and

$$m = a_0 + a_1 b + a_2 b^2 + \cdots + a_k b^k \equiv a_0 + a_1 + a_2 + \cdots + a_k \pmod 2,$$

which implies the desired result. □

Theorem 2.3. *Assume that $b \geq 2$, $m \geq 1$, b is even, and $m = (a_k a_{k-1} \cdots a_1 a_0)_b$. Let*

$$m_1^* = \frac{b^{\lceil \frac{k}{2} \rceil} + b^{\lfloor \frac{k}{2} \rfloor} - 2}{b-1}, \quad m_2^* = \sum_{1 \leq j \leq \lfloor \frac{k}{2} \rfloor} a_{k-j} b^{\lfloor \frac{k}{2} \rfloor - j}, \quad \delta(m) = [m \geq C_b(m^*)],$$

$$\delta_0(m) = [a_k \equiv 0 \pmod 2], \quad \textit{and} \quad \delta_1(m) = [a_k \equiv 1 \pmod 2].$$

Then

$$A_b^{(even)}(m) = \left(\frac{b}{2} - 1\right) m_1^* + \left\lfloor \frac{a_k - 1}{2} \right\rfloor b^{\lfloor \frac{k}{2} \rfloor} + \delta_0(m)\left(m_2^* + \delta(m)\right) + 1, \quad (2.10)$$

$$A_b^{(odd)}(m) = \frac{b}{2} \cdot m_1^* + \left\lceil \frac{a_k - 1}{2} \right\rceil b^{\lfloor \frac{k}{2} \rfloor} + \delta_1(m)\left(m_2^* + \delta(m)\right). \quad (2.11)$$

Proof. Since the proof of this theorem is similar to that of Theorem 2.2, we give less details. Suppose $m = a_0$ where $1 \leq a_0 < b$. By direct counting, we obtain

$$A_b^{(even)}(m) = \left\lceil \frac{a_0 + 1}{2} \right\rceil \quad \text{and} \quad A_b^{(odd)}(m) = \left\lfloor \frac{a_0 + 1}{2} \right\rfloor.$$

In addition, $m_1^* = m_2^* = 0$, and then the right hand sides of (2.10) and (2.11) are, respectively,

$$\left\lfloor \frac{a_0 - 1}{2} \right\rfloor + [a_0 \equiv 0 \pmod 2] + 1 = \left\lceil \frac{a_0 + 1}{2} \right\rceil,$$

and

$$\left\lceil \frac{a_0 - 1}{2} \right\rceil + [a_0 \equiv 1 \pmod 2] = \left\lfloor \frac{a_0 + 1}{2} \right\rfloor.$$

Suppose $m = (a_1 a_0)_b$. Then the possible palindromes less than or equal to m are

$$0, 1, 2, \ldots, b-1, (11)_b, (22)_b, \ldots, ((a_1 - 1)(a_1 - 1))_b, \text{ and } (a_1 a_1)_b,$$

where $(a_1a_1)_b$ is counted if and only if $a_0 \geq a_1$. There are $\frac{b}{2}$ even numbers and $\frac{b}{2}$ odd numbers in $\{0, 1, 2, \ldots, b-1\}$. By Lemma 2.2, there are $\left\lfloor \frac{a_1-1}{2} \right\rfloor$ even numbers and $\left\lceil \frac{a_1-1}{2} \right\rceil$ odd numbers in $\{(11)_b, (22)_b, \ldots, ((a_1-1)(a_1-1))_b\}$. So the number of even and odd palindromes are

$$A_b^{(even)}(m) = \frac{b}{2} + \left\lfloor \frac{a_1-1}{2} \right\rfloor + [a_1 \equiv 0 \pmod 2] \cdot [a_0 \geq a_1],$$

$$A_b^{(odd)}(m) = \frac{b}{2} + \left\lceil \frac{a_1-1}{2} \right\rceil + [a_1 \equiv 1 \pmod 2] \cdot [a_0 \geq a_1],$$

which proves this theorem when $k = 1$. So we assume throughout that $k \geq 2$.

Step 1. We count the number of odd and even palindromes which have $k+1$ digits in their b-adic expansions, respectively. We obtain that

$$\sum_{\substack{b^k \leq n < b^{k+1} \\ n \in P_b \\ n \text{ is even.}}} 1 = \left(\frac{b}{2} - 1\right) b^{\left\lfloor \frac{k}{2} \right\rfloor} \tag{2.12}$$

and

$$\sum_{\substack{b^k \leq n < b^{k+1} \\ n \in P_b \\ n \text{ is odd.}}} 1 = \frac{b}{2} \cdot b^{\left\lfloor \frac{k}{2} \right\rfloor}. \tag{2.13}$$

By Lemma 2.2, the numbers counted in the left hand side of (2.12) are of the form $n = (c_kc_{k-1}\cdots c_1c_0)_b$ where $c_i = c_{k-i}$ for all $i \in \{0, 1, 2, \ldots, \left\lfloor \frac{k}{2} \right\rfloor\}$ and c_0 is even. Since $c_k = c_0$, we can choose c_k to be $2, 4, \ldots, b-2$, so there are $\frac{b}{2} - 1$ possible values for c_k. After c_k is chosen, there is only one choice for $c_0 = c_k$. There are b choices for c_{k-i} for $1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$ and exactly one choice for the corresponding c_i. Therefore

$$\sum_{\substack{b^k \leq n < b^{k+1} \\ n \in P_b \\ n \text{ is even.}}} 1 = \left(\frac{b}{2} - 1\right) \cdot \underbrace{b \cdot b \cdots b}_{\left\lfloor \frac{k}{2} \right\rfloor \text{ terms}} = \left(\frac{b}{2} - 1\right) b^{\left\lfloor \frac{k}{2} \right\rfloor}.$$

Similarly, for the left hand side of (2.13), we have $c_k = c_0$ is odd. There are $\frac{b}{2}$ possible values for c_k and only one choice for c_0. The remaining digits c_{k-i}, where $1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$, can be chosen arbitrarily from $0, 1, \ldots, b-1$. This leads to (2.13).

Step 2. We show that the number of even and odd palindromes which have less than $k+1$ digits in their b-adic expansions are, respectively,

$$\sum_{\substack{1\le n<b^k \\ n\in P_b \\ n \text{ is even.}}} 1 = \left(\frac{b}{2}-1\right)\frac{\left(b^{\lceil \frac{k}{2}\rceil}+b^{\lfloor \frac{k}{2}\rfloor}-2\right)}{b-1} = \left(\frac{b}{2}-1\right) m_1^*, \tag{2.14}$$

$$\sum_{\substack{1\le n<b^k \\ n\in P_b \\ n \text{ is odd.}}} 1 = \frac{b}{2}\cdot\frac{\left(b^{\lceil \frac{k}{2}\rceil}+b^{\lfloor \frac{k}{2}\rfloor}-2\right)}{b-1} = \frac{b}{2}\cdot m_1^*. \tag{2.15}$$

The left hand sides of (2.14) and (2.15) can be written and evaluated by (2.12) and (2.13) as

$$\sum_{\ell=0}^{k-1}\sum_{\substack{b^\ell\le n<b^{\ell+1} \\ n\in P_b \\ n \text{ is even.}}} 1 = \sum_{\ell=0}^{k-1}\left(\frac{b}{2}-1\right) b^{\lfloor \frac{\ell}{2}\rfloor}, \tag{2.16}$$

$$\sum_{\ell=0}^{k-1}\sum_{\substack{b^\ell\le n<b^{\ell+1} \\ n\in P_b \\ n \text{ is odd.}}} 1 = \sum_{\ell=0}^{k-1}\frac{b}{2}\cdot b^{\lfloor \frac{\ell}{2}\rfloor}. \tag{2.17}$$

The right hand sides of (2.16) and (2.17) can be evaluated in a similar way as (2.4), which lead to (2.14) and (2.15), respectively.

Step 3. We show that for $1\le a<b$,

$$\sum_{\substack{b^k\le n<ab^k \\ n\in P_b \\ n \text{ is even.}}} 1 = \left\lfloor\frac{a-1}{2}\right\rfloor b^{\lfloor \frac{k}{2}\rfloor}, \tag{2.18}$$

$$\sum_{\substack{b^k\le n<ab^k \\ n\in P_b \\ n \text{ is odd.}}} 1 = \left\lceil\frac{a-1}{2}\right\rceil b^{\lfloor \frac{k}{2}\rfloor}. \tag{2.19}$$

The left hand side of (2.18) is the number of even palindromes which have $k+1$ digits in their b-adic expansions with the leading digit less than a. Such the numbers n are of the form $(c_k c_{k-1}\cdots c_1 c_0)_b$ where $1\le c_k<a$,

$0 \le c_i < b$, $c_{k-i} = c_i$ for all $i = 0, 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor$, and c_0 is even. So the counting is similar to that in (2.12).

There are $\left\lfloor \frac{a-1}{2} \right\rfloor$ choices for c_k, only one choice for c_0, b choices for c_{k-i} for $1 \le i \le \left\lfloor \frac{k}{2} \right\rfloor$, and exactly one choice for the corresponding c_i. Hence

$$\sum_{\substack{b^k \le n < ab^k \\ n \in P_b \\ n \text{ is even.}}} 1 = \left\lfloor \frac{a-1}{2} \right\rfloor \cdot \underbrace{b \cdot b \cdots b}_{\left\lfloor \frac{k}{2} \right\rfloor \text{ terms}} = \left\lfloor \frac{a-1}{2} \right\rfloor b^{\left\lfloor \frac{k}{2} \right\rfloor}.$$

Similarly, n is odd if and only if c_0 is odd. So there are $\left\lceil \frac{a-1}{2} \right\rceil$ choices for c_k, which leads to (2.19).

Step 4. By (2.14), (2.15), (2.18) and (2.19), we obtain that for $1 \le a < b$,

$$\sum_{\substack{1 \le n < ab^k \\ n \in P_b \\ n \text{ is even.}}} 1 = \left(\frac{b}{2} - 1\right) m_1^* + \left\lfloor \frac{a-1}{2} \right\rfloor b^{\left\lfloor \frac{k}{2} \right\rfloor}, \tag{2.20}$$

$$\sum_{\substack{1 \le n < ab^k \\ n \in P_b \\ n \text{ is odd.}}} 1 = \frac{b}{2} \cdot m_1^* + \left\lceil \frac{a-1}{2} \right\rceil b^{\left\lfloor \frac{k}{2} \right\rfloor}. \tag{2.21}$$

Step 5. Let $a_0, a_1, \ldots, a_k \in \{0, 1, \ldots, b-1\}$ and $a_k \neq 0$. For each $j \in \left\{0, 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor\right\}$, let $m_j = \sum_{0 \le i \le j} a_{k-i} b^{k-i}$. We show that for $0 \le j \le \left\lfloor \frac{k}{2} \right\rfloor - 1$,

$$\sum_{\substack{m_j \le n < m_{j+1} \\ n \in P_b \\ n \text{ is even.}}} 1 = [a_k \equiv 0 \pmod 2]\, a_{k-(j+1)}\, b^{\left\lfloor \frac{k}{2} \right\rfloor - (j+1)}, \tag{2.22}$$

$$\sum_{\substack{m_j \le n < m_{j+1} \\ n \in P_b \\ n \text{ is odd.}}} 1 = [a_k \equiv 1 \pmod 2]\, a_{k-(j+1)}\, b^{\left\lfloor \frac{k}{2} \right\rfloor - (j+1)}. \tag{2.23}$$

Let $n = (c_k c_{k-1} c_{k-2} \cdots c_1 c_0)_b$ be the palindromes which are counted in the left hand side of (2.22). Then $c_k = a_k$. So if a_k is odd, then c_0 is odd and so n is not even, and thus the left hand side of (2.22) is equal to 0. So assume that a_k is even. The counting is similar to that in (2.12) and (2.18) and the left hand side of (2.22) is equal to

$$a_{k-j+1} \cdot \underbrace{b \cdot b \cdots b}_{\left\lfloor \frac{k}{2} \right\rfloor - (j+1) \text{ terms}} = a_{k-(j+1)} b^{\left\lfloor \frac{k}{2} \right\rfloor - (j+1)}.$$

Similarly, if a_k is even, then both sides of (2.23) are equal to 0. If a_k is odd, then the left hand side of (2.23) is $a_{k-(j+1)}b^{\lfloor \frac{k}{2} \rfloor -(j+1)}$.

Step 6. Recall the definitions of m^* and m_2^* given previously. From (2.20) and (2.22), we have

$$\sum_{\substack{1\leq n<m^* \\ n\in P_b \\ n \text{ is even}}} 1 = \sum_{\substack{1\leq n<a_k b^k \\ n\in P_b \\ n \text{ is even}}} 1 + \sum_{1\leq j\leq \lfloor \frac{k}{2} \rfloor} \sum_{\substack{m_{j-1}\leq n<m_j \\ n\in P_b \\ n \text{ is even}}} 1$$
$$= \left(\frac{b}{2}-1\right) m_1^* + \left\lfloor \frac{a_k-1}{2} \right\rfloor b^{\lfloor \frac{k}{2} \rfloor} + [a_k \equiv 0 \pmod 2]\, m_2^*. \quad (2.24)$$

Similarly, by (2.21) and (2.23), we have

$$\sum_{\substack{1\leq n<m^* \\ n\in P_b \\ n \text{ is odd.}}} 1 = \frac{b}{2}\cdot m_1^* + \left\lceil \frac{a_k-1}{2} \right\rceil b^{\lfloor \frac{k}{2} \rfloor} + [a_k \equiv 1 \pmod 2]\, m_2^*. \quad (2.25)$$

Step 7. We show that

$$\sum_{\substack{m^*\leq n\leq m \\ n\in P_b \\ n \text{ is even.}}} 1 = [a_k \equiv 0 \pmod 2]\delta(m) \quad (2.26)$$

and

$$\sum_{\substack{m^*\leq n\leq m \\ n\in P_b \\ n \text{ is odd.}}} 1 = [a_k \equiv 1 \pmod 2]\delta(m). \quad (2.27)$$

The only possible palindrome n such that $m^* \leq n \leq m$ is $n = C_b(m^*)$. So the left hand side of (2.26) is 1 if a_k is even and $m \geq C_b(m^*)$ and is 0 otherwise, which is the same as $[a_k \equiv 0 \pmod 2] \cdot \delta(m)$. Similarly, the left hand side of (2.27) is equal to $[a_k \equiv 1 \pmod 2] \cdot \delta(m)$. By writing,

$$A_b^{(even)}(m) = \sum_{\substack{0\leq n\leq m \\ n\in P_b \\ n \text{ is even.}}} 1 = 1 + \sum_{\substack{1\leq n<m^* \\ n\in P_b \\ n \text{ is even.}}} 1 + \sum_{\substack{m^*\leq n\leq m \\ n\in P_b \\ n \text{ is even.}}} 1,$$

and

$$A_b^{(odd)}(m) = \sum_{\substack{0\leq n\leq m \\ n\in P_b \\ n \text{ is odd.}}} 1 = \sum_{\substack{1\leq n<m^* \\ n\in P_b \\ n \text{ is odd.}}} 1 + \sum_{\substack{m^*\leq n\leq m \\ n\in P_b \\ n \text{ is odd.}}} 1,$$

we see that the formulas for $A_b^{(even)}(m)$ and $A_b^{(odd)}(m)$ can be obtained from (2.24), (2.25), (2.26) and (2.27). This completes the proof. □

Theorem 2.4. *Assume that $m \geq 1$, $b \geq 2$, b is odd, and $m = (a_k a_{k-1} \cdots a_1 a_0)_b$. Let*

$$m_2^* = \sum_{0 \leq i \leq \lfloor \frac{k}{2} \rfloor} a_{k-i} b^{\lfloor \frac{k}{2} \rfloor - i}, \quad m_3^* = \frac{1}{2} \sum_{0 \leq j \leq \lfloor \frac{k}{2} \rfloor - 1} a_{k-j} \left(b^{\lceil \frac{k-1}{2} \rceil - j} + b^{\lfloor \frac{k-1}{2} \rfloor - j} \right),$$

$$m_4^* = \frac{1}{2} \sum_{0 \leq j \leq \lfloor \frac{k}{2} \rfloor - 1} a_{k-j} \left(b^{\lceil \frac{k-1}{2} \rceil - j} - b^{\lfloor \frac{k-1}{2} \rfloor - j} \right), \quad \delta(m) = [m \geq C_b(m^*)],$$

$$\delta_0(m) = \left[a_{\lfloor k/2 \rfloor} \equiv 0 \pmod 2 \right], \text{ and } \delta_1(m) = \left[a_{\lfloor k/2 \rfloor} \equiv 1 \pmod 2 \right].$$

Then

$$A_b^{(even)}(m) = \begin{cases} \frac{1}{2}(b+1) b^{\frac{k-1}{2}} - 1 + m_2^* + \delta(m), & \text{if } k \text{ is odd;} \\ b^{\frac{k}{2}} - 1 + \left\lceil \frac{1}{2} a_{k/2} \right\rceil + m_3^* + \delta_0(m)\delta(m), & \text{if } k \text{ is even,} \end{cases} \tag{2.28}$$

and

$$A_b^{(odd)}(m) = \begin{cases} \frac{1}{2}(b-1) b^{\frac{k-1}{2}}, & \text{if } k \text{ is odd;} \\ \left\lfloor \frac{1}{2} a_{k/2} \right\rfloor + m_4^* + \delta_1(m)\delta(m), & \text{if } k \text{ is even.} \end{cases} \tag{2.29}$$

Proof. Similar to the proof of Theorem 2.2 and Theorem 2.3, we first consider the case $k \leq 1$. Suppose $m = a_0$ where $1 \leq a_0 < b$. By direct counting, we obtain

$$A_b^{(even)}(m) = \left\lceil \frac{a_0 + 1}{2} \right\rceil \quad \text{and} \quad A_b^{(odd)}(m) = \left\lfloor \frac{a_0 + 1}{2} \right\rfloor.$$

We also have $m_3^* = m_4^* = 0$. So the right hand sides of (2.28) and (2.29) are, respectively,

$$\left\lceil \frac{a_0}{2} \right\rceil + [a_0 \equiv 0 \pmod 2] = \left\lceil \frac{a_0 + 1}{2} \right\rceil, \quad \left\lfloor \frac{a_0}{2} \right\rfloor + [a_0 \equiv 1 \pmod 2] = \left\lfloor \frac{a_0 + 1}{2} \right\rfloor.$$

Suppose $m = (a_1 a_0)_b$. Then the possible palindromes less than or equal to m are

$$0, 1, 2, \ldots, b-1, (11)_b, (22)_b, \ldots, ((a_1 - 1)(a_1 - 1))_b, \text{ and } (a_1 a_1)_b,$$

where $(a_1 a_1)_b$ is counted if and only if $a_0 \geq a_1$. There are $\frac{b+1}{2}$ even numbers and $\frac{b-1}{2}$ odd numbers in $\{0, 1, 2, \ldots, b-1\}$. By Lemma 2.2, all numbers $(11)_b, (22)_b, \ldots, ((a_1 - 1)(a_1 - 1))_b$ are even. So the number of even and odd palindromes are

$$A_b^{(even)}(m) = \frac{b+1}{2} + (a_1 - 1) + [a_0 \geq a_1] \quad \text{and} \quad A_b^{(odd)}(m) = \frac{b-1}{2},$$

which proves the case $k = 1$. So we assume throughout that $k \geq 2$.

Step 1. We count the number of odd and even palindromes which have $k+1$ digits in their b-adic expansions, respectively. We obtain that

$$\sum_{\substack{b^k \leq n < b^{k+1} \\ n \in P_b \\ n \text{ is even.}}} 1 = \begin{cases} (b-1)b^{\lfloor \frac{k}{2} \rfloor}, & \text{if } k \text{ is odd;} \\ \frac{1}{2}(b-1)(b+1)b^{\frac{k}{2}-1}, & \text{if } k \text{ is even,} \end{cases} \tag{2.30}$$

$$\sum_{\substack{b^k \leq n < b^{k+1} \\ n \in P_b \\ n \text{ is odd.}}} 1 = \begin{cases} 0, & \text{if } k \text{ is odd;} \\ \frac{1}{2}(b-1)^2 b^{\frac{k}{2}-1}, & \text{if } k \text{ is even.} \end{cases} \tag{2.31}$$

Let $n \in P_b$, $b^k \leq n < b^{k+1}$, and $n = (c_k c_{k-1} \cdots c_1 c_0)_b$. Assume that k is odd. Then

$$\sum_{0 \leq j \leq k} c_j = \sum_{0 \leq j \leq \lfloor \frac{k}{2} \rfloor} 2c_{k-j} \equiv 0 \pmod{2}.$$

By Lemma 2.2, n is even. This shows that all palindromes which have $k+1$ digits are even. So the left hand side of (2.31) is equal to 0 while the left hand side of (2.30) counts the number of all palindromes which have $k+1$ digits, which can be obtained from (2.2) or Theorem 2.2. Assume that k is even. By Lemma 2.2, the numbers counted in the left hand side of (2.30) are the form $n = (c_k c_{k-1} \cdots c_{\frac{k}{2}+1} c_{\frac{k}{2}} c_{\frac{k}{2}-1} \cdots c_1 c_0)_b$ where $c_i = c_{k-i}$ for all $i \in \left\{0, 1, \ldots, \frac{k}{2}\right\}$ and $c_{\frac{k}{2}}$ is even. So there are $b-1$ possible values for c_k and only one possible value for $c_0 = c_k$. There are b choices for c_{k-i} for $1 \leq i \leq \frac{k}{2} - 1$ and exactly one choice for the corresponding c_i. Since $c_{\frac{k}{2}}$ is even, we can choose $c_{\frac{k}{2}}$ to be $0, 2, 4, \ldots, b-1$, so there are $\frac{b+1}{2}$ possible values for $c_{\frac{k}{2}}$ and the left hand side of (2.30) is equal to $\frac{1}{2}(b-1)(b+1)b^{\frac{k}{2}-1}$. Similarly, for (2.31), we have $c_{\frac{k}{2}}$ is odd. So there are $\frac{b-1}{2}$ choices for $c_{\frac{k}{2}}$ and the left hand side of (2.31) is equal to $\frac{1}{2}(b-1)(b-1)\, b^{\frac{k}{2}-1} = \frac{1}{2}(b-1)^2 b^{\frac{k}{2}-1}$.

Step 2. We show that the number of even and odd palindromes which have less than $k+1$ digits in their b-adic expansions are, respectively,

$$\sum_{\substack{1 \leq n < b^k \\ n \in P_b \\ n \text{ is even.}}} 1 = \frac{1}{2}(b+1)b^{\lfloor \frac{k-1}{2} \rfloor} + b^{\lfloor \frac{k}{2} \rfloor} - 2, \tag{2.32}$$

$$\sum_{\substack{1 \leq n < b^k \\ n \in P_b \\ n \text{ is odd.}}} 1 = \frac{1}{2}(b-1)b^{\lfloor \frac{k-1}{2} \rfloor}. \tag{2.33}$$

The left hand side of (2.32) can be written and evaluated by (2.30) as

$$\sum_{\substack{1\le n<b\\ n\in P_b\\ n \text{ is even.}}} 1+\sum_{\substack{1\le \ell\le k-1\\ \ell \text{ is even.}}}\sum_{\substack{b^\ell\le n<b^{\ell+1}\\ n\in P_b\\ n \text{ is even.}}} 1+\sum_{\substack{1\le \ell\le k-1\\ \ell \text{ is odd.}}}\sum_{\substack{b^\ell\le n<b^{\ell+1}\\ n\in P_b\\ n \text{ is even.}}} 1$$

$$=\frac{b-1}{2}+\frac{(b-1)(b+1)}{2}\sum_{\substack{1\le \ell\le k-1\\ \ell \text{ is even.}}} b^{\frac{\ell}{2}-1}+(b-1)\sum_{\substack{1\le \ell\le k-1\\ \ell \text{ is odd.}}} b^{\lfloor\frac{\ell}{2}\rfloor}. \tag{2.34}$$

By a straightforward calculation, we see that

$$\sum_{\substack{1\le \ell\le k-1\\ \ell \text{ is even.}}} b^{\frac{\ell}{2}-1}=\begin{cases}\frac{b^{\frac{k-1}{2}}-1}{b-1}, & \text{if } k \text{ is odd;}\\ \frac{b^{\frac{k}{2}-1}-1}{b-1}, & \text{if } k \text{ is even,}\end{cases}$$

$$\sum_{\substack{1\le \ell\le k-1\\ \ell \text{ is odd.}}} b^{\lfloor\frac{\ell}{2}\rfloor}=\begin{cases}\frac{b^{\frac{k-1}{2}}-1}{b-1}, & \text{if } k \text{ is odd;}\\ \frac{b^{\frac{k}{2}}-1}{b-1}, & \text{if } k \text{ is even,}\end{cases}$$

Then (2.34) leads to (2.32). Similarly, the left hand side of (2.33) can be written and evaluated by (2.31) as

$$\sum_{\substack{1\le n<b\\ n\in P_b\\ n \text{ is odd.}}} 1+\sum_{\substack{1\le \ell\le k-1\\ \ell \text{ is even.}}}\sum_{\substack{b^\ell\le n<b^{\ell+1}\\ n\in P_b\\ n \text{ is odd.}}} 1+\sum_{\substack{1\le \ell\le k-1\\ \ell \text{ is odd.}}}\sum_{\substack{b^\ell\le n<b^{\ell+1}\\ n\in P_b\\ n \text{ is odd.}}} 1$$

$$=\frac{b-1}{2}+\frac{(b-1)^2}{2}\sum_{\substack{1\le \ell\le k-1\\ \ell \text{ is even.}}} b^{\frac{\ell}{2}-1}=\frac{1}{2}(b-1)b^{\lfloor\frac{k-1}{2}\rfloor}. \tag{2.35}$$

Step 3. We show that for $1\le a<b$,

$$\sum_{\substack{b^k\le n<ab^k\\ n\in P_b\\ n \text{ is even.}}} 1=\begin{cases}(a-1)b^{\lfloor\frac{k}{2}\rfloor}, & \text{if } k \text{ is odd;}\\ \frac{1}{2}(a-1)(b+1)b^{\frac{k}{2}-1}, & \text{if } k \text{ is even,}\end{cases} \tag{2.36}$$

$$\sum_{\substack{b^k\le n<ab^k\\ n\in P_b\\ n \text{ is odd.}}} 1=\begin{cases}0, & \text{if } k \text{ is odd;}\\ \frac{1}{2}(a-1)(b-1)b^{\frac{k}{2}-1}, & \text{if } k \text{ is even.}\end{cases} \tag{2.37}$$

Suppose k is odd. By Lemma 2.2, all palindromes which have $k+1$ digits are even. So the left hand side of (2.37) is equal to 0 while the left hand side of (2.36) is the same as (2.5) and we are done.

Assume that k is even. The left hand side of (2.36) is the number of even palindromes which have $k+1$ digits in their b-adic expansions with the leading digit less than a. Such the numbers n are of the form $(c_k c_{k-1}\cdots c_{\frac{k}{2}}\cdots c_1 c_0)_b$ where $1 \le c_k < a$, $c_{k-i} = c_i$ for all $i = 0, 1, \ldots, \frac{k}{2}$, and $c_{\frac{k}{2}}$ is even. So the counting is similar to that in (2.30). There are $a-1$ choices for c_k and only one choice for c_0. There are b choices for c_{k-i} for $1 \le i \le \frac{k}{2}-1$ and exactly one choice for the corresponding c_i. Since $c_{\frac{k}{2}}$ is even, so there are $\frac{b+1}{2}$ possible values for $c_{\frac{k}{2}}$. Therefore the left hand side of (2.36) is equal to $\frac{1}{2}(a-1)(b+1)b^{\frac{k}{2}-1}$.

Similarly, for (2.37), we have $c_{\frac{k}{2}}$ is odd. So there are $\frac{b-1}{2}$ possible values for $c_{\frac{k}{2}}$ and the left hand side of (2.37) is equal to $\frac{1}{2}(a-1)(b-1)b^{\frac{k}{2}-1}$.

Step 4. By (2.32), (2.33), (2.36) and (2.37), we obtain that for $1 \le a < b$,

$$\sum_{\substack{1\le n<ab^k\\ n\in P_b\\ n\text{ is even.}}} 1 = \begin{cases} \frac{1}{2}(b+1)b^{\frac{k-1}{2}} + ab^{\frac{k-1}{2}} - 2, & \text{if } k \text{ is odd;}\\ b^{\frac{k}{2}} + \frac{1}{2}a(b+1)b^{\frac{k}{2}-1} - 2, & \text{if } k \text{ is even,} \end{cases} \tag{2.38}$$

$$\sum_{\substack{1\le n<ab^k\\ n\in P_b\\ n\text{ is odd.}}} 1 = \begin{cases} \frac{1}{2}(b-1)b^{\frac{k-1}{2}}, & \text{if } k \text{ is odd;}\\ \frac{1}{2}a(b-1)b^{\frac{k}{2}-1}, & \text{if } k \text{ is even.} \end{cases} \tag{2.39}$$

Step 5. Let $a_0, a_1 \ldots, a_k \in \{0, 1, \ldots, b-1\}$ and $a_k \neq 0$. For each $j \in \{0, 1, \ldots, \lfloor\frac{k}{2}\rfloor\}$, let $m_j = \sum_{0\le i\le j} a_{k-i}b^{k-i}$. We show that for $0 \le j \le \lfloor\frac{k}{2}\rfloor - 1$,

$$\sum_{\substack{m_j\le n<m_{j+1}\\ n\in P_b\\ n\text{ is even.}}} 1 = \begin{cases} a_{k-(j+1)}b^{\frac{k-1}{2}-(j+1)}, & \text{if } k \text{ is odd;}\\ \frac{1}{2}a_{k-(j+1)}\,(b+1)b^{\frac{k}{2}-j-2}, & \text{if } k \text{ is even and } j < \frac{k}{2}-1;\\ \left\lceil \frac{1}{2}a_{k/2}\right\rceil, & \text{if } k \text{ is even and } j = \frac{k}{2}-1, \end{cases} \tag{2.40}$$

$$\sum_{\substack{m_j\le n<m_{j+1}\\ n\in P_b\\ n\text{ is odd.}}} 1 = \begin{cases} 0, & \text{if } k \text{ is odd;}\\ \frac{1}{2}a_{k-(j+1)}\,(b-1)b^{\frac{k}{2}-j-2}, & \text{if } k \text{ is even and } j < \frac{k}{2}-1;\\ \left\lfloor \frac{1}{2}a_{k/2}\right\rfloor, & \text{if } k \text{ is even and } j = \frac{k}{2}-1. \end{cases} \tag{2.41}$$

If k is odd, then by a reason similar to (2.31) and (2.37), the left hand side of (2.41) is equal to 0 and (2.40) can be obtained from (2.7). So assume that k

is even. Let $n = (c_k c_{k-1} \dots c_{\frac{k}{2}+1}\, c_{\frac{k}{2}}\, c_{\frac{k}{2}-1} \dots c_1 c_0)_b$ be the palindromes which are counted in the left hand side of (2.40) By Lemma 2.2, $c_{\frac{k}{2}}$ is even.

Case 1. $j < \frac{k}{2} - 1$. Similar to (2.7), (2.22), and (2.23), there is only one choice for $c_k, c_{k-1}, \dots, c_{k-j}$ and $c_0, c_1, \dots, c_j$, there are $a_{k-(j+1)}$ choices for $c_{k-(j+1)}$, and b choices for c_{k-i}, where $j+2 \le i \le \frac{k}{2} - 1$. Since $c_{\frac{k}{2}}$ is even, there are $\frac{b+1}{2}$ possible values for $c_{\frac{k}{2}}$. Hence

$$\sum_{\substack{m_j \le n < m_{j+1} \\ n \in P_b \\ n \text{ is even.}}} 1 = \frac{1}{2} a_{k-(j+1)}\, (b+1) b^{\frac{k}{2}-j-2}.$$

Similarly, n is odd if and only if $c_{\frac{k}{2}}$ is odd. There are $\frac{b-1}{2}$ choices for $c_{\frac{k}{2}}$. Therefore

$$\sum_{\substack{m_j \le n < m_{j+1} \\ n \in P_b \\ n \text{ is odd.}}} 1 = \frac{1}{2} a_{k-(j+1)}\, (b-1) b^{\frac{k}{2}-j-2}.$$

Case 2. $j = \frac{k}{2} - 1$. Then there is only one choice for $c_k, c_{k-1}, \dots, c_{\frac{k}{2}+1}$ and $c_0, c_1, \dots, c_{\frac{k}{2}-1}$. We have $0 \le c_{\frac{k}{2}} < a_{k/2}$ and there are $\left\lceil \frac{1}{2} a_{k/2} \right\rceil$ even numbers, and $\left\lfloor \frac{1}{2} a_{k/2} \right\rfloor$ odd numbers in $\{0, 1, \dots, a_{k/2} - 1\}$. Therefore

$$\sum_{\substack{m_{k/2-1} \le n < m_{k/2} \\ n \in P_b \\ n \text{ is even.}}} 1 = \left\lceil \frac{1}{2}\, a_{k/2} \right\rceil \quad \text{and} \quad \sum_{\substack{m_{k/2-1} \le n < m_{k/2} \\ n \in P_b \\ n \text{ is odd.}}} 1 = \left\lfloor \frac{1}{2}\, a_{k/2} \right\rfloor.$$

Step 6. Recall the definitions of m^*, m_2^*, m_3^*, and m_4^* given previously. We show that

$$\sum_{\substack{1 \le n < m^* \\ n \in P_b \\ n \text{ is even.}}} 1 = \begin{cases} \frac{1}{2}(b+1) b^{\frac{k-1}{2}} - 2 + m_2^*, & \text{if } k \text{ is odd;} \\ b^{\frac{k}{2}} - 2 + \left\lceil \frac{1}{2} a_{k/2} \right\rceil + m_3^*, & \text{if } k \text{ is even,} \end{cases} \tag{2.42}$$

$$\sum_{\substack{1 \le n < m^* \\ n \in P_b \\ n \text{ is odd.}}} 1 = \begin{cases} \frac{1}{2}(b-1) b^{\frac{k-1}{2}}, & \text{if } k \text{ is odd;} \\ \left\lfloor \frac{1}{2}\, a_{k/2} \right\rfloor + m_4^*, & \text{if } k \text{ is even.} \end{cases} \tag{2.43}$$

Suppose k is odd. With the formulas from (2.38) and (2.40), the left hand side of (2.42) can be written as

$$\sum_{\substack{1\le n<a_k b^k\\ n\in P_b\\ n \text{ is even.}}} 1+\sum_{1\le j\le\lfloor\frac{k}{2}\rfloor}\sum_{\substack{m_{j-1}\le n<m_j\\ n\in P_b\\ n \text{ is even.}}} 1=\frac{1}{2}(b+1)b^{\frac{k-1}{2}}+a_k b^{\frac{k-1}{2}}-2+\sum_{1\le j\le\lfloor\frac{k}{2}\rfloor} a_{k-j}b^{\frac{k-1}{2}-j},$$

which is equal to the right hand side of (2.42) in the case k is odd. Similarly, by (2.39) and (2.41), the left hand side of (2.43) is

$$\sum_{\substack{1\le n<a_k b^k\\ n\in P_b\\ n \text{ is odd.}}} 1+\sum_{1\le j\le\lfloor\frac{k}{2}\rfloor}\sum_{\substack{m_{j-1}\le n<m_j\\ n\in P_b\\ n \text{ is odd.}}} 1=\frac{1}{2}(b-1)b^{\frac{k-1}{2}}.$$

Suppose k is even. By (2.38) and (2.40), the left hand side of (2.42) is

$$\begin{aligned}&\sum_{\substack{1\le n<a_k b^k\\ n\in P_b\\ n \text{ is even.}}} 1+\sum_{1\le j\le\frac{k}{2}-1}\sum_{\substack{m_{j-1}\le n<m_j\\ n\in P_b\\ n \text{ is even.}}} 1+\sum_{\substack{m_{k/2-1}\le n<m_{k/2}\\ n\in P_b\\ n \text{ is even.}}} 1\\ &=b^{\frac{k}{2}}+\frac{1}{2}\,a_k(b+1)b^{\frac{k}{2}-1}-2+\frac{1}{2}(b+1)\sum_{1\le j\le\frac{k}{2}-1} a_{k-j}b^{\frac{k}{2}-1-j}+\left\lceil\frac{1}{2}a_{k/2}\right\rceil\\ &=b^{\frac{k}{2}}-2+\left\lceil\frac{1}{2}a_{k/2}\right\rceil+m_3^*.\end{aligned}$$

Similarly, by (2.39) and (2.41), the left hand side of (2.43) is

$$\begin{aligned}&\sum_{\substack{1\le n<a_k b^k\\ n\in P_b\\ n \text{ is odd.}}} 1+\sum_{1\le j\le\frac{k}{2}-1}\sum_{\substack{m_{j-1}\le n<m_j\\ n\in P_b\\ n \text{ is odd.}}} 1+\sum_{\substack{m_{k/2-1}\le n<m_{k/2}\\ n\in P_b\\ n \text{ is odd.}}} 1\\ &=\frac{1}{2}\,a_k(b-1)b^{\frac{k}{2}-1}+\frac{1}{2}(b-1)\sum_{1\le j\le\frac{k}{2}-1} a_{k-j}b^{\frac{k}{2}-1-j}+\left\lfloor\frac{1}{2}\,a_{k/2}\right\rfloor\\ &=\left\lfloor\frac{1}{2}\,a_{k/2}\right\rfloor+m_4^*.\end{aligned}$$

Step 7. We show that

$$\sum_{\substack{m^*\le n\le m\\ n\in P_b\\ n \text{ is even.}}} 1=\begin{cases}\delta(m), & \text{if } k \text{ is odd;}\\ \left[a_{k/2}\equiv 0 \pmod 2\right]\,\delta(m), & \text{if } k \text{ is even,}\end{cases}\tag{2.44}$$

$$\sum_{\substack{m^* \le n \le m \\ n \in P_b \\ n \text{ is odd.}}} 1 = \begin{cases} 0, & \text{if } k \text{ is odd;} \\ \left[a_{k/2} \equiv 1 \pmod 2\right] \delta(m), & \text{if } k \text{ is even.} \end{cases} \tag{2.45}$$

Suppose k is odd. The only possible palindrome n such that $m^* \le n \le m$ is $n = C_b(m^*)$. Since k is odd, $C_b(m^*)$ is even. So the left hand side of (2.44) is 1 if $m \ge C_b(m^*)$ and is 0 otherwise. In addition, the left hand side of (2.45) is equal to 0. Suppose k is even. By Lemma 2.2, the left hand side of (2.44) is 1 if $a_{k/2}$ is even and $m \ge C_b(m^*)$ and is 0 otherwise, which is the same as $\left[a_{k/2} \equiv 0 \pmod 2\right] \cdot \delta(m)$. Similarly, the left hand side of (2.45) is equal to $\left[a_{k/2} \equiv 1 \pmod 2\right] \cdot \delta(m)$. By writing,

$$A_b^{(even)}(m) = \sum_{\substack{0 \le n \le m \\ n \in P_b \\ n \text{ is even.}}} 1 = 1 + \sum_{\substack{1 \le n < m^* \\ n \in P_b \\ n \text{ is even.}}} 1 + \sum_{\substack{m^* \le n \le m \\ n \in P_b \\ n \text{ is even.}}} 1,$$

and

$$A_b^{(odd)}(m) = \sum_{\substack{0 \le n \le m \\ n \in P_b \\ n \text{ is odd.}}} 1 = \sum_{\substack{1 \le n < m^* \\ n \in P_b \\ n \text{ is odd.}}} 1 + \sum_{\substack{m^* \le n \le m \\ n \in P_b \\ n \text{ is odd.}}} 1,$$

we see that the formula for $A_b^{(even)}(m)$ and $A_b^{(odd)}(m)$ can be obtained from (2.42), (2.43), (2.44) and (2.45). This completes the proof. □

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