International Journal of Mathematics and
Computer Science, **14**(2019), no. 1, 263–283

Fixed point theorems for $(\alpha, k, \theta)-$contractive multi-valued mapping in $b-$metric space and applications

Haitham Qawaqneh[1], **Mohd Salmi Md Noorani**[1], **Wasfi Shatanawi**[2]

[1]School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM, Selangor Darul Ehsan, Malaysia

[2]Department of Mathematics
Hashemite University
Zarqa 1315, Jordan

[2]Department of Mathematics and General Courses
Prince Sultan University
Riyadh, Saudi Arabia

email: haitham.math77@gmail.com, msn@ukm.my, swasfi@hu.edu.jo,
wshatanawi@psu.edu.sa

(Received June 29, 2018, Accepted October 20, 2018)

Abstract

In this paper, we introduce a new generalized $(\alpha, k, \theta)-$contractive multi-valued mapping in $b-$metric space, Our results generalize and extend several known results of metric. We provide an example to show the superiority of our results over corresponding fixed point results proved in $b-$metric spaces. In addition, we use our results to present an application.

Key words and phrases: Weak $\alpha-$admissible mapping, multi-valued mapping, $(\alpha, k, \theta)-$contractive multi-valued mapping, fixed point, $b-$metric space.
AMS (MOS) Subject Classifications: 47H10, 54H25.
ISSN 1814-0432, 2019, http://ijmcs.future-in-tech.net

1 Introduction and preliminaries

One of the most important tools in fixed point theory is Banach contraction principle. A lot of authors have extended or generalized this contraction and proved the existence of fixed and common fixed point theorems(see [13]-[20]). The theory of multi-valued mapping has an important tool in various fields of mathematics and several authors have also extended and presented many forms of multi-valued mapping conditions on self mappings as a generalization of Banach contraction principle by using the concept of Hausdorf metric and proved the existence of fixed and common fixed point theorems in metric space and other spaces.We recite some notations, needed definitions and elementary results, for the purpose of the sequel, and $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers respectively. Let (X, d) be a metric space, we denote $CL(X)$ the family of closed subsets of X, by $CB(X)$ the class of all nonempty closed bounded subsets of X. and $F(f)$ is the set of all fixed points of f. For $A, B \in CL(X)$, let the $H : CL(X) \times CL(X) \to \mathbb{R}^+ \cup \{\infty\}$ be defined by

$$H(A,B) = \begin{cases} \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}, & \text{if } \textit{the maximum exists} \\ \infty, & \textit{otherwise.} \end{cases}$$

such map H is called the generalized Hausdorff metric induced by the metric d.

In 1969, the study of Banach-type fixed theorems of multi-valued mappings started with the work of Nadler [27], who proved that a multi-valued contractive mapping of a complete metric space X into the family of closed bounded subsets of X has a fixed point.

Definition 1.1. [27] Let (X, d) be a metric space. A map $f : X \to CB(X)$ is said to be multi valued contraction if there exists $0 \leq \lambda < 1$ such that

$$H(fx, fy)) \leq \lambda d(x, y)),$$

for all $x, y \in X$ where $CB(X)$ denotes the family of nonempty closed subsets of X.

Definition 1.2. [27] A point of $x_0 \in X$ is said to be a fixed point of the multi-valued mapping f if $x_0 \in fx_0$.

Lemma 1.3. *[27] If $A, B \in CB(X)$ and $a \in A$, then for each $\epsilon > 0$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \epsilon$$

.

In 1989, Bakhtin [4] introduced the notion of $b-$metric spaces and extended the Banach's contraction principle to $b-$metric spaces (see also [5]).

Definition 1.4. ([4],[5]) Let X is a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a $b-$metric space on X if and only if for all $x, y, z \in X$, the following conditions hold:

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The triplet (X, d, s) is called a $b-$metric space.

It is well known that a metric space implies $b-$metric space with $t = 1$ but, in general, the converse is not true.

Example 1.5. *Consider the set $X = [0, 1]$ endowed with the function $d : X \times X \to [0, \infty)$ defined by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Clearly, $(X, d, 3)$ is a $b-$metric space but it is not a metric space.*

Let (X, d, s) be a $b-$metric space The following notions are natural deductions from their metric counterparts.

(i) A sequence $\{x_n\} \subseteq X$ converges to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$,

(ii) A sequence $\{x_n\} \subseteq X$ is said to be a Cauchy sequence if, for every given $\epsilon > 0$, there exists $n(\epsilon) \in \mathbb{N}$ such that $d(x_n, x_m) < 0$ for all $m, n \geq n(\epsilon)$,

(iii) A $b-$metric space (X, d, s) is said to be complete if and only if each Cauchy sequence converges to some $x \in X$.

Lemma 1.6. *[6] Let (X, d, s) be a $b-$metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are $b-$convergent to x, y, respectively. Then we have*

$$\frac{1}{s}d(x, y) \leq \lim_{n\to\infty} \inf d(x_n, y_n) \leq \lim_{n\to\infty} \sup d(x_n, y_n) \leq s^2 d(x, y),$$

in particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{s}d(x, y) \leq \lim_{n\to\infty} \inf d(x_n, y) \leq \lim_{n\to\infty} \sup d(x_n, y) \leq s^2 d(x, y).$$

Lemma 1.7. *[6] Let (X, d, s) be a $b-$metric space with $s \geq 1$. Let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a $b-$Cauchy sequence, then there exist $\epsilon > 0$ and two sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of positive integers such that for the following four sequences $d(x_{m(k)}, x_{n(k)})$, $d(x_{m(k)}, x_{n(k)+1})$, $d(x_{m(k)+1}, x_{n(k)})$ and $d(x_{m(k)+1}, x_{n(k)+1})$, it hold:

$$\begin{aligned}
\epsilon \leq & \lim_{n\to\infty} \inf d(x_{m(k)}, x_{n(k)}) \leq \lim_{n\to\infty} \sup d(x_{m(k)}, x_{n(k)}) && \leq s\epsilon \\
\frac{\epsilon}{s} \leq & \lim_{n\to\infty} \inf d(x_{m(k)}, x_{n(k)+1}) \leq \lim_{n\to\infty} \sup d(x_{m(k)}, x_{n(k)+1}) && \leq s^2\epsilon \\
\frac{\epsilon}{s} \leq & \lim_{n\to\infty} \inf d(x_{m(k)+1}, x_{n(k)}) \leq \lim_{n\to\infty} \sup d(x_{m(k)+1}, x_{n(k)}) && \leq s^2\epsilon \\
\frac{\epsilon}{s^2} \leq & \lim_{n\to\infty} \inf d(x_{m(k)+1}, x_{n(k)+1}) \leq \lim_{n\to\infty} \sup d(x_{m(k)+1}, x_{n(k)+1}) && \leq s^3\epsilon
\end{aligned}$$

Lemma 1.8 ([10],[11]). *Let (X, d, s) be a $b-$metric space. For any $A, B, C \in CB(X)$ and any $x, y \in X$, one has the following:*

(i) $d(x, B) \geq d(x, b)$ *for any* $b \in B$,

(ii) $d(A, B) \geq \delta(A, B)$,

(iii) $d(x, B) \geq H(A, B)$ *for any* $x \in A$,

(iv) $H(A, A) = 0$,

(v) $H(A, B) = H(B, A)$,

(vi) $H(A, C) \leq s(H(A, B) + H(B, C))$,

(vii) $H(x, A) \leq s(H(x, y) + H(y, A))$.

The concept of α-admissibility was first introduced by Samet et al. [7] and extended as admissibility of type S by Sintunavarat [14] in the framework of metric spaces and $b-$metric spaces, respectively.

Definition 1.9. [7] Let $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. f is called α-admissible when if $x, y \in X$ such that $\alpha(x, y) \geq 1$ then we have $\alpha(fx, fy) \geq 1$.

Definition 1.10. [8] Let X be a nonempty set, $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$ are two mappings, and $s \geq 1$ be a given real number. Then we say that

(*i*) A mapping f is $\alpha-$admissible of type S if
$x, y \in X$ and $\alpha(x, y) \geq s$ then we have $\alpha(fx, fy) \geq s$,
this is denoted as $f \in \mathcal{A}_s(X, \alpha)$,

(*ii*) A mapping f is weak $\alpha-$admissible if
$x \in X$ and $\alpha(x, fx) \geq 1$ then we have $\alpha(fx, f^2x) \geq 1$,
this is denoted as $f \in \mathcal{WA}(X, \alpha)$,

(*iii*) A mapping f is weak $\alpha-$admissible of type S if
$x \in X$ and $\alpha(x, fx) \geq s$ then we have $\alpha(fx, f^2x) \geq s$,
this is denoted as $f \in \mathcal{WA}_s(X, \alpha)$,

Definition 1.11. [33] Let $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then f is called a triangular α-admissible mapping if

1. f is α-admissible;
2. $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$.

After that, many authors using the concept of α-admissible contractive-type mappings to study the existence of fixed point in many spaces (see [22],[23],[24] and references cited therein).

Recently, Jleli et al.[25] presented a new type of contraction which is called the $\Theta-$contraction which it were a set of functions $\theta : [0, \infty) \to [0, \infty)$ and they established some new fixed point theorems for such a contraction in the context of generalized metric spaces.

Definition 1.12. [25]

(1) Let $\theta : (0, \infty) \to (1, \infty)$ be a function satisfying the following conditions:

(θ_1) θ is nondecreasing,

(θ_2) θ for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$,

(θ_3) θ there exist $r \in (0, 1)$ and $l \in (0, \infty)$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

(2) A mapping $f : X \to X$ is called the $\Theta-$contraction if there exists the function θ satisfying $(\theta_1) - (\theta_3)$ and a constant $k \in (0, 1)$ such that, for all for all $x, y \in X$

$$d(fx, fy) \neq 0 \Rightarrow \theta(d(fx, fy)) \leq (\theta(d(x, y)))^k.$$

Theorem 1.13. *[25] Let (X, d) be a complete metric space and $f : X \to X$ be a $\Theta-$contraction. Then F has a unique fixed point.*

Also, they showed that any Banach contraction is a particular case of $\Theta-$contraction while there exist $\Theta-$contractions which are not Banach contractions.

Very recently, Ahmad et al.[26] presented the family Θ of functions $\theta : [0, \infty) \to [0, \infty)$. Let $\theta : (0, \infty) \to (1, \infty)$ be a function satisfying the following conditions:

(θ_1) θ is nondecreasing,

(θ_2) θ for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$,

(θ_3) θ is continuous on $(0, \infty)$.

Consistent with Ahmad et al.[26], we denote by Ω the set of all functions satisfying the conditions $(\theta_1) - (\theta_3)$.

Lemma 1.14. *[26] Let (X, d) is a metric space and $\theta \in \Theta$. Then $(X, \theta \circ d)$ is a metric space.*

Lemma 1.15. *[26] Let (X, d) be a metric space, let $\theta \in \Theta$ and $C \in CL(X)$. Suppose that there exists $x \in X$ such that $\theta(d(x, C)) > 0$. Then there exists $y \in C$ such that*

$$\theta(d(x, y)) < \tau\theta(d(x, C)),$$

where $\tau > 1$.

Example 1.16. *Define some functions as follows: for all $t > 0$, $\theta_1(t) = e^t$, $\theta_2(t) = e^{\sqrt{t}}$, $\theta_3(t) = e^{\sqrt{te^t}}$, $\theta_4(t) = \cosh t$, $\theta_5(t) = 1 + \ln(1 + t)$, $\theta_6(t) = e^{te^t}$. Then θ_1, θ_2, θ_3, θ_4, θ_5, $\theta_6 \in \Omega$.*

Example 1.17. *Note that the conditions (θ_3) in [25] and [26] are independent of each other. Indeed, for $p > 1$, $\theta(t) = e^{t^p}$ satisfies the conditions (θ_1) and (θ_2), but it does not satisfy (θ_3) in [25], while it satisfies the condition (θ_3) in [26].*

In this article we motivate and inspire by ([12], [25] and [26])and define the $\Theta-$contraction for a new family of functions Ω. Also, we present the class of an $(\alpha, k, \theta)-$contractive multi-valued mapping. After that, we establish some the existence of fixed point for this class of mappings in metric space. Our work generalize and extend some theorems in the literature. An example and applications are given to support the obtained result.

2 Main result

We firstly introduce the following lemma which will be used efficiently in the proof of our main result.

Lemma 2.1. *Let $f : X \to CL(X)$ is a multi-valued mapping type function and $\alpha : X \times X \to \mathbb{R}$ is a functions such that (f) is a weak $\alpha-$admissible and satisfy transitive property. Assume that there exist $x_0 \in X$ such that $\alpha(x_0, x_1) \geq 1$and if $y \in fx_0$, $\alpha(x_0, y) \geq 1$ then $\alpha(y, z) \geq 1$ for $z \in fy$, that is denoted by $f \in \mathcal{WA}(X, \alpha)$. Define a sequence $\{x_n\}$ in X by $x_{n+1} \in fx_n$. Then $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $m > n$.*

Proof. Since f is a multi-valued weak $\alpha-$admissible mapping such that if for each $x \in X$ and $y \in fx$ with $\alpha(x, y) \geq 1$, then $\alpha(y, z) \geq 1$ for all $z \in fy$, we get

$\alpha(x_0, x_1) \geq 1$ and if $y \in fx_0$, $\alpha(x_0, y) \geq 1$ then $\alpha(y, z) \geq 1$ for $z \in fy$.
Also, $\alpha(x_1, x_2) \geq 1$ and if $y \in fx_1$, $\alpha(x_1, y) \geq 1$ then $\alpha(y, z) \geq 1$ for $z \in fy$.

By transitive property, we get $\alpha(x_0, x_2) \geq 1$. Again, since
$\alpha(x_2, x_3) \geq 1$ and if $y \in fx_2$, $\alpha(x_2, y) \geq 1$ then $\alpha(y, z) \geq 1$ for $z \in fy$.
Also, $\alpha(x_3, x_4) \geq 1$ and if $y \in fx_3$, $\alpha(x_3, y) \geq 1$ then $\alpha(y, z) \geq 1$ for $z \in fy$.

By transitive property, we get $\alpha(x_2, x_4) \geq 1$. By continuing the above process, we conclude that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, we prove that $\alpha_*(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $m > n$. For such m, n, if

$$\begin{cases} \alpha(x_n, x_{n+1}) \geq 1 \text{ and if } y \in fx_n,\ \alpha(x_n, y) \geq 1 \text{ then } \alpha(y, z) \geq 1 \text{ for } z \in fy, \\ \alpha(x_{n+1}, x_{n+2}) \geq 1 \text{ and if } y \in fx_{n+1},\ \alpha(x_{n+1}, y) \geq 1 \text{ then } \alpha(y, z) \geq 1 \text{ for } z \in fy, \end{cases}$$

then

$$\alpha(x_n, x_{n+2}) \geq s.$$

Again, since

$$\begin{cases} \alpha(x_{n+1}, x_{n+2}) \geq 1 \text{ and if } y \in fx_{n+1},\ \alpha(x_{n+1}, y) \geq 1 \text{ then } \alpha(y, z) \geq 1 \text{ for } z \in fy, \\ \alpha(x_{n+2}, x_{n+3}) \geq 1 \text{ and if } y \in fx_{n+2},\ \alpha(x_{n+2}, y) \geq 1 \text{ then } \alpha(y, z) \geq 1 \text{ for } z \in fy, \end{cases}$$

for all $x_{n+1} \in fx_n$, $x_{n+2} \in fx_{n+1}$ and $x_{n+3} \in fx_{n+2}$. We deduce that

$$\alpha(x_n, x_{n+3}) \geq s.$$

By continuing this process, we have

$$\alpha(x_n, x_m) \geq s,$$

for all $n \in \mathbb{N}$ with $m > n$. □

Now, we present the class of an $(\alpha, k, \theta)-$contractive multi-valued mapping and prove some fixed point theorems on $b-$metric space.

Definition 2.2. Let (X, d, s) be a $b-$metric space with coefficient $s \geq 1$ and $f : X \to CL(X)$ be a weak $(\alpha)-$admissible mapping satisfy transitive property. We say that f is an $(\alpha, k, \theta)-$contractive multi-valued mapping if there exists $\alpha : X \times X \to [0, +\infty)$, $\theta \in \Theta$, $k \in (0, 1)$ and constant $\lambda \in (0, \frac{1}{s}]$ such that:

$$x, y \in X,\ \alpha(x, y) \geq 1 \Rightarrow\ \theta(H(fx, fy)) \leq \lambda\theta(M(x, y))^k, \tag{2.1}$$

where

$$M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), d(y, fx), \frac{1}{2}[d(x, fy) + d(y, fx)]\}$$

for all $x, y \in X$.

Theorem 2.3. *Let (X, d, s) be a $b-$metric space with coefficient $s \geq 1$ and $f : X \to CL(X)$ be an $(\alpha, k, \theta)-$contractive multi-valued mappings. Assume that the following conditions hold:*

1. *f is a weak $(\alpha)-$admissible,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, y) \geq 1$ and $y \in fx_0$*
3. *f is a weak $(\alpha)-$continuous multi-valued mapping.*

then f has a fixed point.

Proof. By starting from x_0 and $x_1 \in fx_0$ in conditions (3), we have

$$\alpha(x_0, fx_0) \geq 1.$$

If $x_0 = x_1$, we derive that $x_1 \in F(f)$ and so the proof is done. Now, we assume that $x_0 \neq x_1$ and $x_1 \notin fx_1$. From (3.1), we have

$$\begin{aligned} 1 < \theta(d(x_1, fx_1)) &\leq \theta(H(fx_0, fx_1)) \\ &\leq \lambda\theta(M(x_0, x_1))^k \\ &\leq \theta(M(x_0, x_1))^k, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M(x_0, x_1) &= \max\{d(x_0, x_1), d(x_0, fx_0), d(x_1, fx_1), \frac{1}{2}[d(x_0, fx_1) + d(x_1, fx_0)]\} \\ &= \max\{d(x_0, x_1), d(x_0, x_1), d(x_1, fx_1), \frac{1}{2}d(x_0, fx_1)\} \\ &\leq \max\{d(x_0, x_1), d(x_1, fx_1), \frac{1}{2}[d(x_0, x_1) + d(x_1, fx_1)]\} \\ &= \max\{d(x_0, x_1), d(x_1, fx_1)\}, \end{aligned} \tag{2.3}$$

from (2.2) and (2.3), we get

$$\begin{aligned} 1 < \theta(d(x_1, fx_1)) &\leq \theta(H(fx_0, fx_1)) \\ &\leq \lambda\theta(\max\{d(x_0, x_1), d(x_1, fx_1)\})^k \\ &\leq \theta(\max\{d(x_0, x_1), d(x_1, fx_1)\})^k. \end{aligned} \tag{2.4}$$

If $\max\{d(x_0, x_1), d(x_1, fx_1)\} = d(x_1, fx_1)$, then we obtain

$$\begin{aligned} 1 < \theta(d(x_1, fx_1)) &\leq \theta(H(fx_0, fx_1)) \\ &\leq \lambda\theta(d(x_1, fx_1))^k \\ &\leq \theta(d(x_1, fx_1))^k, \end{aligned}$$

which is a contradiction. Thus $\max\{d(x_0, x_1), d(x_1, fx_1)\} = d(x_0, x_1)$. From (2.4), we obtain

$$\begin{aligned} 1 < \theta(d(x_1, fx_1)) &\leq \theta(H(fx_0, fx_1)) \\ &\leq \lambda\theta(d(x_0, x_1))^k \\ &\leq \theta(d(x_0, x_1))^k. \end{aligned} \tag{2.5}$$

Since fx_1 is compact, then there exists $x_2 \in fx_1$ such that

$$d(x_1, x_2) = d(x_1, fx_1). \tag{2.6}$$

From (2.5) and (2.6), we get

$$1 < \theta(d(x_1, x_2)) \leq \theta(d(x_0, x_1))^k. \tag{2.7}$$

If $x_1 = x_2$ or $x_2 \in fx_1$, then it follows that $x_2 \in F(f)$ and so the proof is done. Therefore, we assume that $x_2 \neq x_1$ and $x_2 \in fx_2$. Since $x_1 \in fx_0$, $x_2 \in fx_1$, $\alpha(x_0, (x_1) \geq 1$ and f is an $(\alpha, k, \theta)-$contractive multi-valued mapping, we have $\alpha(x_1, x_2) \geq 1$.

By applying an $(\alpha, k, \theta)-$contractive multi-valued condition, we have

$$\begin{aligned} \theta(H(fx_1, fx_2)) &\leq \alpha(x_1)\beta(x_2)\theta(H(fx_1, fx_2)) \\ &\leq \lambda\theta(M(x_1, x_2))^k \\ &\leq \theta(M(x_1, x_2))^k, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} M(x_1, x_2) &= \max\{d(x_1, x_2), d(x_1, fx_1), d(x_2, fx_2), \frac{1}{2}[d(x_1, fx_2) + d(x_2, fx_1)]\} \\ &= \max\{d(x_1, x_2), d(x_1, x_2), d(x_2, fx_2), \frac{1}{2}d(x_1, fx_2)\} \\ &= \max\{d(x_1, x_2), d(x_2, fx_2), \frac{1}{2}[d(x_1, x_2) + d(x_2, fx_2)]\} \\ &= \max\{d(x_1, x_2), d(x_2, fx_2)\}. \end{aligned}$$

If $M(x_1, x_2) = d(x_2, fx_2)$, we have:

$$\begin{aligned} 1 < \theta(d(x_2, fx_2)) &\leq \theta(H(fx_1, fx_2)) \\ &\leq \lambda\theta(d(x_2, fx_2))^k \\ &\leq \theta(d(x_2, fx_2))^k, \end{aligned}$$

which is a contradiction. Thus, if $M(x_1, x_2) = d(x_1, x_2)$, we get

$$\begin{aligned} 1 < \theta(d(x_2, fx_2)) &\leq \theta(H(fx_1, fx_2)) \\ &\leq \lambda(\theta(d(x_1, x_2)))^k \\ &\leq (\theta(d(x_1, x_2)))^k. \end{aligned} \tag{2.9}$$

Since fx_2 is compact, then there exists $x_3 \in fx_2$ such that

$$d(x_2, x_3) = d(x_2, fx_2). \tag{2.10}$$

From (2.9)and (2.10), we obtain

$$1 < \theta(d(x_2, x_3)) \leq \theta(d(x_1, x_2)))^k \leq \theta(d(x_0, x_1)))^{k^2}. \tag{2.11}$$

By continuing this procedure, we construct the sequence$\{x_n\}$ in X such that $x_{n+1} \neq x_n \in fx_n$, again, since f is an $(\alpha, k, \theta)-$contractive multi-valued mapping and by Lemma(1.7) , we have

$$\alpha(x_n, x_{n+1}) \geq 1,$$

and

$$\begin{aligned} 1 < \theta(d(x_{n+1}, x_{n+2})) &\leq \theta(H(fx_n, fx_{n+1})) \\ &\leq \lambda\theta(d(x_0, x_1))^{k^n} \\ &\leq \theta(d(x_0, x_1))^{k^n}, \end{aligned}$$

for all $\mathbb{N} \cup \{0\}$. This shows that $\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1$ and so $\lim_{n\to\infty}(d(x_n, x_{n+1})) = 0$. By our assumptions about θ, it follows that there exist $n_1 \in N$ and $r \in (0, 1)$ such that
$(d(x_n, x_{n+1})) \leq \frac{1}{n^{\frac{1}{r}}}$, for all $n \geq n_1$.

Now, for $m > n > n_1$ we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1}(d(x_i, x_{i+1})) \leq \sum_{i=n}^{m-1}(\frac{1}{i^{\frac{1}{r}}}).$$

Since $0 < r < 1$, $\sum_{i=n}^{m-1}(\frac{1}{i^{\frac{1}{r}}})$ converges. This implies that $\lim_{n,m\to\infty} \theta(d(x_m, x_n)) = 0$ and by the continuity of θ, we have $lim_{n,m\to\infty} d(x_m, x_n) = 0$. Thus $\{x_n\}$ is Cauchy sequence in (X, d) and there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$ for all $n \in \mathbb{N}$.

We suppose that condition(3) hold. Hence $\alpha(x_n, z) \geq 1$. From (3.1) and by the weak α-continuity of multi-valued mapping f, we have

$$\lim_{n\to\infty} H(fx_n, fz) = 0 \tag{2.12}$$

for all $n \in \mathbb{N}$, which implies that

$$\theta(d(z, fz)) = \lim_{n\to\infty} d(x_{n+1}, fz) \leq \lim_{n\to\infty} H(fx_n, fz) = 0.$$

Therefore, $z \in fz$ and hence f has a fixed point. □

Note that the continuity of the mapping f in Theorem 2.3 can be dropped if we replace Condition(3) by a suitable one as in the following result.

Theorem 2.4. *Let (X, d, s) be a $b-$metric space with coefficient $s \geq 1$ and $f : X \to CL(X)$ be an $(\alpha, k, \theta)-$contractive multi-valued mapping. Assume that the following conditions hold:*

1. *f is a weak $(\alpha)-$admissible,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, y) \geq 1$ and $y \in fx_0$,*
3. *if $\{x_n\}$ is a sequence in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then we have,*

$$x, y \in X,\ \alpha(x, y) \geq 1 \Rightarrow\ \theta(H(fx_n, fx)) \leq \lambda\theta(M(x_n, x))^k,$$

where

$$M(x_n, x) = \max\{d(x_n, x), d(x_n, fx_n), d(x, fx), d(x, fx_n), \frac{1}{2}[d(x_n, fx)+d(x, fx_n)]\}$$

then f has a fixed point.

Proof. Following the proof of Theorem2.3, we know that $\{x_n\}$ is a Cauchy sequence in X such that $x_n \to z$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$.
Suppose that $d(z, fz) > 0$, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \Rightarrow \theta(d(x_{n+1}, fz)) \leq \theta(H(fx_n, fz)) \leq \lambda(\theta(M(x_n, z)))^k \leq (\theta(M(x_n, z)))^k, \tag{2.13}$$

where

$$\begin{aligned} M(x_n, z) &= \max\{d(x_n, z), d(x_n, fx_n), d(z, fz), \frac{1}{2}[d(x_n, fz) + d(z, fx_n)]\} \\ &= \max\{d(x_n, z), d(x_n, x_{n+1}), d(z, fz), \frac{1}{2}[d(x_n, fz) + d(z, x_{n+1})]\}, \end{aligned} \tag{2.14}$$

for all $n \in \mathbb{N}$. letting $n \to \infty$ in(2.14), we have

$$\theta(d(z, fz)) \leq \lambda(\theta(d(z, fz)))^k \leq (\theta(d(z, fz)))^k, \tag{2.15}$$

which is a contradiction. Therefore, we have $d(z, fz) = 0$, that is, $z \in fz$ and hence f has a fixed point. □

Corollary 2.5. *Let (X, d, s) be a $b-$metric space with coefficient $s \geq 1$ and $f : X \to CL(X)$ be a weak (α)-admissible mapping. We say that f is an $(\alpha, k, \theta)-$contractive multi-valued mapping if there exists $\alpha : X \times X \to [0, +\infty)$ and $\theta \in \Theta$ such that:*

$$\alpha(x,y)\theta(H(fx,fy)) \leq \lambda(\theta(\max\{d(x,y),d(x,fx),d(y,fy),\frac{1}{2}[d(x,fy)+d(y,fx)]\}))^k,$$

Assume that the following conditions hold:

1. *f is a weak $(\alpha)-$admissible,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, y) \geq 1$ and $y \in fx_0$,*
3. *f is a weak $(\alpha)-$continuous multi-valued mapping.*

then f has a fixed point.

Proof. Let $\alpha(x, y) \geq 1$ for every $x, y \in X$. Then by 3.1, we have:

$$\begin{aligned}\theta(H(fx,fy)) &\leq \alpha(x,y)\theta(H(fx,fy)) \\ &\leq (\theta(\max\{d(x,y),d(x,fx),d(y,fy),\frac{1}{2}[d(x,fy)+d(y,fx)]\}))^k \\ &\leq (\theta(\max\{d(x,y),d(x,fx),d(y,fy),\frac{1}{2}[d(x,fy)+d(y,fx)]\}))^k,\end{aligned}$$

this guides that f is a weak $(\alpha) - (k, \theta)$-admissible multi-valued mapping. So, by following the proof Theorem (2.3) we obtain the desired outcome. □

Corollary 2.6. *Let (X, d, s) be a $b-$metric space with coefficient $s \geq 1$ and $f : X \to CL(X)$ be a weak (α)-admissible mapping. We say that f is an $(\alpha, k, \theta)-$contractive multi-valued mappings if there exists $\alpha : X \times X \to [0, +\infty)$ and $\theta \in \Theta$. Assume that the following conditions hold:*

1. *f is a weak $(\alpha)-$admissible,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, y) \geq 1$ and $y \in fx_0$,*
3. *if $\{x_n\}$ is a sequence in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then we have,*
$$\alpha(x,y)\theta(H(fx_n,fx)) \leq \lambda\theta(M(x_n,x))^k,$$

where

$$M(x_n,x) = \max\{d(x_n,x),d(x_n,fx_n),d(x,fx),d(x,fx_n),\frac{1}{2}[d(x_n,fx)+d(x,fx_n)]\}$$

then f has a fixed point.

Proof. Let $\alpha(x,y) \geq 1$ for every $x, y \in X$. Then by 3.1, we have:

$$\begin{aligned}\theta(H(fx,fy)) &\leq \alpha(x,y)\theta(H(fx,fy))\\ &\leq \lambda(\theta(\max\{d(x,y),d(x,fx),d(y,fy),\frac{1}{2}[d(x,fy)+d(y,fx)]\}))^k\\ &\leq (\theta(\max\{d(x,y),d(x,fx),d(y,fy),\frac{1}{2}[d(x,fy)+d(y,fx)]\}))^k,\end{aligned}$$

this guides that f is an $(\alpha, k, \theta)-$contractive multi-valued mapping. So, by following the proof Theorem (2.4) we obtain the desired outcome. □

Example 2.7. *Let $x = [0,\infty)$ and $d : X \times X \to [0,\infty)$ be defined by $d(x,y) = (x-y)^2$ for all $x, y \in X$. Define $f : X \to X$ and $\alpha : X \to [0,\infty)$ by*

$$fx = \begin{cases} \{\frac{x}{9}, \frac{x}{4}\} & \text{if } x \in [0,1),\\ \{2x - \frac{1}{4}, 2x\} & \text{if } x \in [1,\infty)\end{cases}$$

$$\alpha(x,y) = \begin{cases} e^{|x-y|} & \text{if } x,y \in [0,1],\\ e^{|x+y|} & \text{otherwise}\end{cases}$$

It's easy to see that (X,d) is a $b-$metric space.
Now, we want to show that the Theorem 2.3 can be guarantee the existence of fixed point of f. Firstly, we will show that f is a weak $\alpha-$admissible mapping For $x, y, z \in X$, $y \in fx$ and $z \in fy$, we have
$\alpha(x,y) = \alpha(x, \frac{x}{4}) = e^{|x-\frac{x}{4}|} \geq 1$ then we have
$\alpha(y,z) = \alpha(\frac{x}{4}, \frac{x}{16}) = e^{|\frac{x}{4}-\frac{x}{16}|} \geq 1$,
and it is satisfied on the the remain intervals.

Secondly, we will prove that f is an $(\alpha, k, \theta)-$contractive multi-valued mappings with $k = \frac{1}{2}$. Define the function $\theta : [0,\infty) \to [0,\infty)$ by $\theta(t) = e^{\sqrt{te^t}}$. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n) \geq 1$ and $x_n \to x$ as $n \to \infty$. Then for $x, y \in [0,1)$ and $\lambda = \frac{1}{2}$, we have

$$\begin{aligned}\theta(H(fx,fy)) &= \theta(\frac{1}{4}(x-y)^2)\\ &= e^{\sqrt{\frac{1}{4}(x-y)^2 e^{\frac{1}{4}(x-y)^2}}}\\ &\leq \frac{1}{2}e^{\sqrt{(x-y)^2 e^{(x-y)^2}}}\\ &= \frac{1}{2}e^{\sqrt{d(x,y)e^{d(x,y)}}}\\ &\leq \frac{1}{2}e^{\sqrt{M(x,y)e^{M(x,y)}}}\\ &= \lambda(\theta(M(x,y)))^k.\end{aligned}$$

Thus all conditions of Theorem (2.3) are satisfied, which implying that f has fixed point.

3 Applications

Next, we will show that some results can be deduced easily from our Theorem (2.3).

3.1 Standard fixed point theorems

Letting $s = 1$ in Theorem (2.3), we may get the following fixed point theorem.

Corollary 3.1. *Let (X, d) be a metric space and $f : X \to CL(X)$ be an weak (α)-admissible mapping. We say that f is an $(\alpha, k, \theta)-$contractive multi-valued mapping if there exists $\alpha : X \times X \to [0, +\infty)$ and $\theta \in \Theta$ such that:*

$$\alpha(x,y)\theta(H(fx,fy)) \leq (\theta(\max\{d(x,y),d(x,fx),d(y,fy),\frac{1}{2}[d(x,fy)+d(y,fx)]\}))^k,$$

Assume that the following conditions hold:

1. *f is a weak $(\alpha)-$admissible,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, y) \geq 1$ and $y \in fx_0$,*
3. *f is a weak $(\alpha)-$continuous multi-valued mapping.*

then f has a fixed point.

Proof. Let $\alpha(x, y) = 1$ for every $x, y \in X$. Then, we have:

$$\begin{aligned}\theta(H(fx,fy)) &\leq \alpha(x,y)\theta(H(fx,fy)) \\ &\leq (\theta(\max\{d(x,y),d(x,fx),d(y,fy),\frac{1}{2}[d(x,fy)+d(y,fx)]\}))^k,\end{aligned}$$

this guides that f is an $(\alpha, k, \theta)-$contractive multi-valued mapping. So, by following the proof Theorem (2.3) we obtain the desired outcome. □

Corollary 3.2. *Let (X, d) be a metric space and $f : X \to CL(X)$ be a weak (α)-admissible mapping. We say that f is an $(\alpha, k, \theta)-$contractive multi-valued mapping if there exists $\alpha : X \times X \to [0, +\infty)$ and $\theta \in \Theta$. Assume that the following conditions hold:*

1. *f is a weak $(\alpha)-$admissible,*

2. *there exists $x_0 \in X$ such that $\alpha(x_0, y) \geq 1$ and $y \in fx_0$,*

3. *if $\{x_n\}$ is a sequence in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then we have,*

$$\alpha(x, y)\theta(H(fx_n, fx)) \leq \theta(M(x_n, x))^k,$$

where

$$M(x_n, x) = \max\{d(x_n, x), d(x_n, fx_n), d(x, fx), d(x, fx_n), \frac{1}{2}[d(x_n, fx)+d(x, fx_n)]\}$$

then f has a fixed point.

Proof. Let $\alpha(x, y) = 1$ for every $x, y \in X$. Then by 3.1, we have:

$$\begin{aligned} \theta(H(fx, fy)) &\leq \alpha(x, y)\theta(H(fx, fy)) \\ &\leq (\theta(\max\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\}))^k, \end{aligned}$$

this guides that f is an $(\alpha, k, \theta)-$contractive multi-valued mapping. So, by following the proof Theorem (2.4) we obtain the desired outcome.

□

3.2 Fixed point theorem on metric spaces endowed with a graph

In 2008, Jachymski [29] obtained fixed point theorems on a metric space with a graph and generalized simultaneously Banach contraction principle from metric and partially ordered metric spaces, we need to introduce some concepts.

Let (X, d) be a metric space and Δ be the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, $E(G)$ being the set of the edges of the graph. Assuming that G has no parallel edges, we will suppose that G can be identified with the $(V(G), E(G))$.

If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $(x_i)_{i=0}^k$ of vertices such that $x_0 = x$, $x_k = y$ and $(x_{i-1}, x_i) \in E(G)$, for $i \in \{1, 2, ..., k\}$. Let us denote by $\hat{G}$ the undirected graph obtained from G by ignoring the direction of edges. Notice that a graph G is connected if there is a path between any two vertices and it is weakly connected if $\hat{G}$ is connected.

Definition 3.3. [30] Let X be a nonempty set endowed with a graph G and $f : X \to N(X)$ be a multi-valued mapping, where X is a nonempty set. The mapping T preserves edges weakly if, for each $x \in X$ and $y \in fx$ with $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for all $z \in fy$.

Definition 3.4. [30] Let (X, d) be a metric space endowed with a graph G. The metric space X is said to be $E(G)-$complete if every Cauchy sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ converges in X.

Definition 3.5. [30] Let (X, d) be a metric space endowed with a graph G. A mapping $f : X \to CL(X)$ is called an $E(G)-$continuous mapping to $(CL(X), H)$ if, for any $x \in X$ and any sequence $\{x_n\}$ with $\lim_{n\to\infty} d(x_n, x) = 0$ and $(x_n, x_{n+1}) \in E(G$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} H(fx_n, fx) = 0$$

Definition 3.6. Let (X, d, s) be a $b-$metric space with coefficient $s \geq 1$ endowed with a graph G and $f : X \to CL(X)$ be a weak $(\alpha)-$admissible mapping. We say that f is an $((E(G), \alpha, k, \theta)$-admissible multi-valued mapping if there exists $\alpha : X \times X \to [0, +\infty)$, $\theta \in \Theta$, $k \in (0, 1)$ and constant $\lambda \in (0, \frac{1}{s}]$ such that:

$$x, y \in X,\ \alpha(x, y) \geq 1 \Rightarrow\ \theta(H(fx, fy)) \leq \lambda\theta(M(x, y))^k, \tag{3.1}$$

where

$$M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), d(y, fx), \frac{1}{2}[d(x, fy) + d(y, fx)]\}$$

for all $x, y \in E(G)$.

Theorem 3.7. *Let (X, d, s) be a $b-$metric space with coefficient $s \geq 1$ and G be a directed graph and $f : X \to CL(X)$ be an $((E(G), \alpha, k, \theta)$-admissible multi-valued mapping for all $x, y \in E(G)$. Assume that the following conditions hold:*

1. $(x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)$, *and* $(x, y) \in E(G), (y, z) \in E(G) \Rightarrow (x, z) \in E(G)$
2. f *is a weak* $(\alpha)-$*admissible,*

3. *there exists* $x_0 \in X$ *such that* $(x_0, y) \in E(G) \Rightarrow \alpha(x_0, y) \geq 1$ *and* $y \in fx_0$,

4. f *is a weak* $(\alpha)-$*continuous multi-valued mapping.*

then f *has a fixed point.*

Proof. Consider the mapping $\alpha : X \times X \to [0, +\infty)$ defined by

$$\alpha(x) = \begin{cases} 1 & \text{if } x, y \in [E(G)], \\ 0 & \text{otherwise} \end{cases}$$

which means all the hypotheses of Theorem (2.3) are satisfied. So we can deduce that T has a unique fixed point. □

Theorem 3.8. *Let* (X, d, s) *be a* $b-$*metric space with coefficient* $s \geq 1$ *and* G *be a directed graph and* $f : X \to CL(X)$ *be an* $((E(G), \alpha, k, \theta)$*-admissible multi-valued mapping for all* $x, y \in E(G)$*. Assume that the following conditions hold:*

1. $(x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)$, *and* $(x, y) \in E(G), (y, z) \in E(G) \Rightarrow (x, z) \in E(G)$

2. f *is a weak* $(\alpha)-$*admissible,*

3. *there exists* $x_0 \in X$ *such that* $(x_0, y) \in E(G) \Rightarrow \alpha(x_0, y) \geq 1$ *and* $y \in fx_0$,

4. *if* $\{x_n\}$ *is a sequence in* $E(G)$ *with* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *then we have,*

$$x, y \in E(G),\ \alpha(x, y) \geq 1 \Rightarrow\ \theta(H(fx_n, fx)) \leq \lambda\theta(M(x_n, x))^k,$$

where

$$M(x_n, x) = \max\{d(x_n, x), d(x_n, fx_n), d(x, fx), d(x, fx_n), \frac{1}{2}[d(x_n, fx) + d(x, fx_n)]\}$$

then f *has a fixed point.*

Proof. Consider the mapping $\alpha : X \times X \to [0, +\infty)$ defined by

$$\alpha(x) = \begin{cases} 1 & \text{if } x, y \in [E(G)], \\ 0 & \text{otherwise} \end{cases}$$

which means all the hypotheses of Theorem (2.4) are satisfied. So we can deduce that T has a unique fixed point. □

Acknowledgement. The authors would like to acknowledge the grant: UKM Grant DIP-2017-011 and Ministry of Education, Malaysia grant FRGS/1/2017/STG06/UKM/01/1 for financial support.

References

[1] M. U. Ali, T. Kamran, E. Karapinar, (α, ψ, ξ)- *contractive multivalued mappings*, fixed Point Theory and Appl., **2014** (7)(2014), 8 pages.

[2] S. Alizadeh, F. Moradlou, P. Salimi, *Some fixed point results for* $(\alpha, \beta)-(\psi, \phi)$*-contractive mappings*, Filomat, **28** (2014), 635-647.

[3] A. Pansuwan, W. Sintunavarat, V. Parvaneh, Y. Je Cho, *Some fixed point theorems for* $(\alpha, \theta, k)-$*contractive multi-valued mappings with some applications*, fixed Point Theory and Appl., **132,**(2015).

[4] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal., **30,** (1989), Unianowsk, Gos. Ped. Inst. 26-37.

[5] S. Czerwik, *Contraction mappings in b-metric spaces,*Acta Math. Inform. Univ. Ostrav., **1,** (1993), 5-11.

[6] D. Dukic, Z. Kadelburg, S. Radenovic, *Fixed point of Geraghty-type mappings in various generalized metric spaces*, Abstract Appl. Anal. (2011) 13 pages. Article ID 561245.

[7] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for a* $\alpha-\psi$*-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154-2165.

[8] W. Sintunavarat, *Nonlinear integral equations with new admissibility types in b-metric spaces*, J. Fixed Point Theory Appl., 18:397416, 2016.

[9] M. U. Ali, T. Kamran, E. Karapinar,(α, ψ, ξ)- *contractive multivalued mappings*, fixed Point Theory and Appl., **2014**(7) (2014), 8 pages.

[10] M. Boriceanu, *Fixed point theory for multivalued generalized contraction on a set with two* $b-$*metrics*,Studia Universitatis Babes-Bolyai,SeriesMathematica, **54,** no. 3, pp. 314, 2009

[11] S. Czerwick, K. Dlutek, and S. L. Singh, *Round-off stability of iteration procedures for set-valued operators in* $b-$*metric spaces*,journal of Natural and Physical Sciences, **11,** pp. 8794, 2007.

[12] S. Alizadeh, F. Moradlou, P. Salimi, *Some fixed point results for* $(\alpha, \beta)-(\psi, \phi)$*-contractive mappings*, Filomat, **28** (2014), 635-647.1, 2, 2.14.

[13] H. Aydi, M. Postolache, W. Shatanawi, *Coupled fixed point results for* (ψ, ϕ)*-weakly contractive mappings in ordered G-metric spaces*, Comput. Math. Appl., **63**(2012), 298-309.

[14] H. Aydi, W. Shatanawi, C. Vetro, *On generalized weak G-contraction mapping in G-metric spaces*, Comput. Math. Appl., **62,** (2011), 4223-4229.

[15] H. Qawaqneh, M. S. M. Noorani, W. Shatanawi, H. Alsamir, *Common fixed points for pairs of triangular* $(\alpha)-$*admissible mappings*, Journal of Nonlinear Sciences and Application, 10, 61926204 (2017).

[16] H. Qawaqneh, M. S. M. Noorani, W. Shatanawi, K. Abodayeh, H. Alsamir, *Common fixed points for pairs of triangular* $(\alpha)-$*admissible mappings*, Journal of Mathematical Analysis, 9(1), 3851, (2018).

[17] H. Qawaqneh, M.S.M. Noorani, W. Shatanawi, *Fixed Point Results for Geraghty Type Generalized* $F-$*contraction for Weak* α*-admissible Mapping in Metric-like Spaces*, Journal of Mathematical Analysis, **11,** (2018), 702-716.

[18] H. Qawaqneh, M.S.M. Noorani, W. Shatanawi, *Common Fixed Point Theorems for Generalized Geraghty* (α, ψ, ϕ)*-Quasi Contraction Type Mapping in Partially Ordered Metric-like Spaces*, Axioms, 7(4), 74 (2018).

[19] M. S. Khan, *A fixed point theorem for metric spaces*, Rend. Inst. Math. Univ. Trieste, **8,** (1976), 69-72.

[20] W. Shatanawi and M. Postolache, *common fixed point results for mappings under nonlinear contraction of cyclic form in ordered metric spaces*, Fixed Point Theory and Appl., **60,** (1), (2013), 1-13.

[21] M. Jleli, E. Karapnar, E. Samet, *Further generalizations of the Banach contraction principle*. J. Inequal. Appl. 2014, 439 (2014)

[22] N. Hussain, J. Ahmad, A. Azam, *Generalized fixed point theorems for multivalued* $\alpha-\psi$*-contractive mappings*, J. Ineq. Appl., **2014** (2014), 15 pages.

[23] N. Hussain, V. Parvaneh, S. J. Hoseini Ghoncheh, *Generalized contractive mappings and weakly α-admissible pairs in* G-*metric spaces*, The Scientific World J., **2014** (2014), 15 pages.

[24] M. Jleli, B. Samet, C. Vetro, F. Vetro, *Fixed Points for Multivalued Mappings in b-Metric Spaces*. Abstr. Appl. Anal., **2015**(2015), Article ID 718074, 7 pages.

[25] M. Jleli, B. Samet, *A new generalization of the Banach contraction principle*. J. Inequal. Appl., 2014 (2014), 8 pages. 1, 1.6, 1

[26] J. Ahmad, A. E. Al-Mazrooei, Y. J. Cho, Y.-O. Yang, *Fixed point results for generalized* $\Theta-$*contractions*, J. Nonlinear Sci. Appl., 10 (2017), 23502358. 1, 1.8, 1.9

[27] J. Nadler, *Multivalued contraction mappings*, Pacific J. of Math., **30**(1969), 475-488.

[28] A, Meir, E, Keeler, *A theorem on contraction mappings*. J. Math. Anal. Appl. **28**(1969), 326-329.

[29] J. Jachymski, *The contraction principle for mappings on a metric space with a graph* Proc.Amer. Math. Soc., 136 (2008), 1359-1373.

[30] M. A. Kutbi, W. Sintunavarat, *On new fixed point results for* $(\alpha,\psi)-$*contractive multi-valued mappings on complete metric spaces and their consequences.* Fixed Point Theory Appl. 2015, 2 (2015)

[31] A. Latif, M. E. Gordji, E. Karapinar, W. Sintunavarat, *Fixed point results for generalized* (α,ψ)*-Meir-Keeler contractive mappings and applications*, J. Ineq. Appl., **2014** (2014), 11 pages.1, 1.

[32] N. Redjel, A, Dehici, E. Karapinar, E, Erhan, *Fixed point theorems for* $(\alpha-\psi)$*-Meir-Keeler-Khan mappings.* J. Nonlinear Sci. Appl., **8**(2015), 955-964.

[33] E. Karapinar, P. Kumam, P. Salimi, *On* α,ψ*-Meir-Keeler contractive mappings*, Fixed Point Theory Appl. **2013**, 2013:12.

[34] E, Karapinar, B. Samet, *Generalized* $\alpha-\psi$*-contractive type mappings and related fixed point theorems with applications*, Abstr. Appl. Anal., **2012**(2012), Article ID 793486, 17 pages.