International Journal of Mathematics and
Computer Science, **14**(2019), no. 1, 121–133

Semi-Cartesian Squares and the Snake Lemma

Jean-Claude Raoult
20 avenue Jean Baptiste Marrou
F-63122 Ceyrat, France

email: jean-claude.raoult@sfr.fr

(Received December 17, 2017, Accepted August 3, 2018)

Abstract

The snake lemma is proved entirely within category theory (section 3) without the help of "points with value in…" à la Grothendieck nor pseudo-elements (Guglielmetti & Zaganidis [2009]) nor homology functor (C. Berger [2004]). Instead, we use consistently semi-cartesian squares (section 2), promoted by Chevalley. Section 1 is devoted to a few basic results on abelian categories, for further use.

1 Introduction

This paper is mainly intended to promote the semi-cartesian squares, introduced by Chevalley in a course given at the Henri Poincaré Institute (IHP), and is an example of their flexibility. The first two sections are extracted from this course. The third is a purely categorical proof of the snake lemma. Such a purely categorical proof has already been written by C. Berger [1] using a quite different method using a homological functor and by Rafael Guglielmetti & Dimitri Zaganidis [3] using another notion of point.
Categories are supposed to be known: objects and arrows between objects. Arrows are composed associatively and each object X has an identity arrow denoted 1_X. The dual or opposite category has same objects but arrows are reversed. The class of arrows from X to Y is denoted by $\mathrm{Hom}_{\mathbf{C}}(X,Y)$. In a small category, arrows form a set (objects also, why?). An initial (final)

Key words and phrases: Abelian categories, Snake lemma, Self-contained paper.
AMS (MOS) Subject Classifications: 18A35, 18E10.
ISSN 1814-0432, 2019, http://ijmcs.future-in-tech.net

object is an object with a unique arrow to (from) every object. By definition, *monomorphisms* are arrows that can be simplified from the left ($mu = mv$ implies $u = v$) and *epimorphisms* arrows that can be simplified from the right ($up = vp$ implies $u = v$). Categories will be denoted by bold upper case letters.

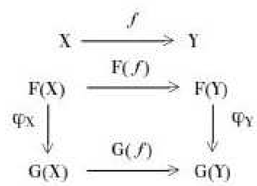


Figure 1.

A *functor* is a mapping $F : \mathbf{C} \to \mathbf{D}$ between categories which is compatible with the identities and the composition. Given two functors F and G from $\mathbf{C}$ to $\mathbf{D}$, a *natural* (or *functorial*) *morphism* $\varphi : F \to G$ is a family of arrows of $\mathbf{D}$ indexed by objects of $\mathbf{C}$ such that the squares of fig.1 are commutative for all arrows f of $\mathbf{C}$. If $\mathbf{C}$ is a small category, functors $\mathbf{C} \to \mathbf{D}$ are the objects of a category $F(C, D)$ the arrows of which are the natural morphisms.

2 Summary of abelian category

Given a small category $\mathbf{I}$ of indices, a functor $A : \mathbf{I} \to \mathbf{C}$ can be seen as a commutative diagram of type $\mathbf{I}$ in **C**Ẇith every object X of $\mathbf{C}$ is associated a constant diagram $(K_X)_i = X$ and $(K_X)_{(1_i)} = 1_X$. A projective (inductive) limit, $\lim\limits_{\leftarrow} A$ (resp. $\lim\limits_{\rightarrow} A$), of a functor $A : \mathbf{I} \to \mathbf{C}$ is a right (left) adjoint of functor $K : X \to K_X$ i.e.

$$Hom_{F(I,C)}(K_X, A) \approx Hom_{\mathbf{C}}(X, \lim_{\leftarrow} A)$$

resp.

$$Hom(A, K_X) \approx Hom(\lim_{\rightarrow} A, X).$$

Examples:

1. $\mathbf{I} = \emptyset$: the projective (inductive) limit of the empty set is the final (initial) object.

2. $\mathbf{I} = \{1, 2\}$: the projective limit of $A_1,\ A_2$ is the product $A1 \times A_2$ and the inductive limit is the sum $A_1 + A_2$.

3. $\mathbf{I} = 1 \rightrightarrows 2$: the projective limit of a double arrow $(u, v) : A_1 \rightrightarrows A_2$ is the kernel or equalizer of (u, v). The inductive limit is its cokernel or coequalizer.

4. $\mathbf{I} = 1 \to 0 \leftarrow 2$: the projective limit of $A_1 \to A_0 \leftarrow A_2$ is the cartesian square or fiber product built on these arrows (left square beneath). The inductive limit is got by reversing the arrows: it is a cocartesian square or amalgamated sum (right square beneath).

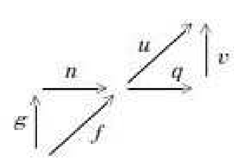
$$\begin{array}{ccc} P & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_0 \end{array} \qquad\qquad \begin{array}{ccc} A_0 & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & S \end{array}$$

Definition A category is *abelian* if
1. it contains a null object (i. e. initial and final) denoted by 0;
2. it accepts finite projective and inductive limits;
3. every monomorphism is a kernel and every epimorphism is a cokernel.

These axioms are due to P.Freyd [2]. They are preserved by duality.

The null arrow $A \to B$ is the composed arrow $A \to 0 \to B$; the kernel of a single arrow $u : A \to B$ is the kernel of $(u, 0)$. It is easy to show that every kernel is a monomorphism and that every cokernel is an epimorphism. Condition (3) shows that these two notions coincide and more precisely:

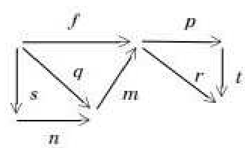
Lemma 2.1. *1) If n is a kernel of an epimorphism q, then q is a cokernel of n. 2) If q is a cokernel of a monomorphism n, then n is a kernel of q.*



Case 2 is dual of case 1. For case 1, q is a cokernel of an arrow $f : qf = 0$ and there exists therefore a unique arrow g such that $f = ng$ because n is a kernel of q. Suppose an arrow u such that $a = 0$. Then $ung = uf = 0$ and since q is a cokernel of f, there exists a unique arrow v such that $u = vq$, QED.

One can deduce the following decomposition of the arrows.

Proposition 2.1. *Every arrow can be decomposed into $f = mq$ where m is a monomorphism and q an epimorphism. This decomposition is unique up to a unique isomorphism.*



Let m the kernel of a cokernel p of f. Then $pf = 0 \Longleftrightarrow$ there exists a unique q such that $f = mq$. To show that q is an epimorphism, let us remark: a) f epimorphism $\Longleftrightarrow p = 0 \Longleftarrow m$ invertible (because $pm = 0 \Longleftarrow pmm^{-1} = p = 0$)

b) Decompose q like f: n is a kernel of a cokernel of q. Considering a), it is enough to show that n is invertible. Now let r satisfying $rmn = 0$. Then $rmns = rf = 0$. Since 0p is a cokernel of f, there exists a unique t such that $r = tp$. Hence p is a cokernel of mn and from lemma 2.1, mn is a kernel of p. But m is also a kernel of p, therefore n is invertible.

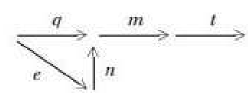
c) Let $f = m'q'$ be an other decomposition. Since p is a cokernel of $m'q'$ and q' is an epimorphism, p is a cokernel of m'. By lemma 2.1, m' is a kernel of p. Since m is another one, they are isomorphic, QED.

Corollary 2.1. *An arrow which is a monomorphism and an epimorphism is an isomorphism.*

The decomposition of f into an epimorphism followed by a monomorphism is unique up to isomorphisms. But f has two such decompositions: $1f = f1$. Therefore it is invertible, QED.

Proposition 2.2. *If q is an epimorphism, then mq and m have same cokernel. The converse is true if m is a monomorphism. If m is a monomorphism, then mq and m have same kernel. The converse is true is m is an epimorphism.*

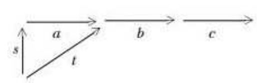
If q is an epimorphism, every arrow t satisfies $tm = 0tmq = 0$. Hence, m and mq have same cokernel. Conversely, decompose q into an epimorphism e



and a monomorphism n: The direct part shows that mne and mn have same cokernel. The hypothesis becomes: monomorphisms m and mn have same cokernel. They are therefore kernel of the same arrow; they are isomorphic and n is invertible: q is an epimorphism.

The second assertion is the dual of the first, QED.

Proposition 2.3. *If ba is a kernel of c and b is a monomorphism, then a is a kernel of cb.*

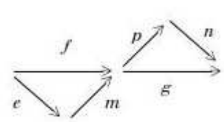


Let t be an arrow with $cbt = 0$. Since ba is a kernel of c, there exists a unique arrow s such that $bt = bas$. Since b is a monomorphism, $t = as$, QED.

The following notion is a generalisation of the notions of kernel and cokernel, which is well-known.

Definition 2.7. Let two successive morphisms $f = mq$ and $g = np$, decomposed into epimorphisms followed by monomorphisms. The sequence (f, g) is exact when m is a kernel of p.

Equivalently (proposition 1.5), one can require that m be a kernel of g, or that p be a cokernel of m, or that p be a cokernel of f.



Finally, recall that in an abelian category, there exists an isomorphism from the sum to the product and that the insertions $i : A \to A + B$ and $j : B \to A + B$, and the projections $p : A \times B \to A$ and $q : A \times B \to B$ satisfy

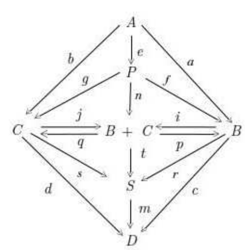
$$\begin{aligned} pi &= 1 \qquad qi = 0, \\ pj &= 0 \qquad qj = 1. \\ ip + jq &= 1 \end{aligned}$$

These equalities caracterize the direct sum of A and B, which will be denoted by A+B.

3 Semi-cartesian squares

What does one get by composing cartesian and cocartesian squares? Semi-cartesian squares in the following sense.

Proposition 3.1. *Let* $ca = db$ *be a commutative square as in Fig.2. Let B+C be the direct sum with insertions (i,j) and projections (p,q); construct the fiber product (P, f, g) of (c,d) with the kernel n of cpdq, and the amalgamated sum (S, r, s) of (a,c) with the cokernel t of ia + jb. Then there exist unique arrows* $e : A \to P$ *and* $m : S \to C$ *making commutative the obvious triangles of figure 2. Then the following conditions are equivalent:*
(i) e is an epimorphism,
(ii) m is a monomorphism,
(iii) the sequence $0 \to P \to^{n} B + C \to^{t} S \to 0$ *is exact,*
(iv) the sequence $A \to^{ia+jb} B + C \to^{cp-dq} D$ *is exact.*



When constructing P and S we defined

$$f = pn, \quad r = ti,$$
$$g = qn, \quad s = -tj.$$

So there exists a unique e with $a = fe$ and $b = ge$; and a unique m with $c = mr$ and $d = ms$. Composing on the right $1S = ip + jq$ with ne, one finds $ne = ia + jb$. Therefore t is a cokernel of ne.

(i) (iii): For the sequence in (iii) to be exact, it is necessary and sufficient that t be a cokernel of n, that is, that e be an epimorphism.

(iii) (ii): Condition (iii) is preserved by duality and is thus is equivalent to (ii) which is dual of(i).

(iii) (iv): because n is a monomorphism and q an epimorphism.

(iv) (iii): because the cokernel t of ne is cokernel of a kernel of $mt = cp - dq$, that is of n, hence(iii), QED.

Definition 2.2. A commutative square is called semi-cartesian when it satisfies the conditions of proposition 2.1.

For instance, a cartesian square (e is invertible), or a cocartesian square (m is invertible), is semi-cartesian. Hereafter is a partial converse in which notations are those of figure 2.

Proposition 3.2. *In a semi-cartesian square $ca = db$, if a is a monomorphism, then d is a monomorphism and the square is cartesian. If d is an epimorphism, then a is an epimorphism and the square is cocartesian.*

With the notations of Fig.2, since a is a monomorphism, e is also a monomorphism. Since it is an epimorphism, it is invertible and the given square is cartesian. Let $k : N \to C$ be a kernel of d and $0 : N \to B$ the null arrow. There exists a unique arrow $h : N \to A$ such that $k = bh$ and $0 = ah$. But a is a monomorphism, therefore $h = 0$, hence $k = 0$, QED.

Contrary to arrows, squares will be written in the same order as they are drawn.

Proposition 3.3. *1) Suppose* K *cocartesian. Then* KL *semi-cartesian* L *semi-cartesian.*

2) Suppose L *cartesian. Then* KL *semi-cartesian* K *semi-cartesian.*

3) K *and* L *semi-cartesians* KL *semi-cartesian.*

1) Let (r, s) be an amalgamated sum of (c, v) and m the unique arrow such that $w = mr$ and $d = ms$. The square $r(ca) = (sb)u$ is composed of cocartesian squares and therefore is cocartesian. then KL semi-cartesian m monomorphism L semi-cartesian (see Fig.3).

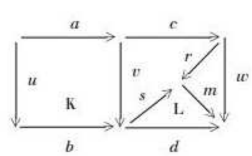


Figure 3.

2) Dual of (1).

3) Consider figure 4; set $S = B +_A C$, with a unique monomorphism $n : S \to D$; set $T = S +_B E$. The square $ACTE$ is then cocartesian (composition of cocartesian squares) hence a unique arrow $m : T \to F$ making the diagram commutative. Since L is semi-cartesian and $BSTE$ cocartesian, square $SDFT$ is semi-cartesian from (1) and m is a monomorphism from proposition 2.3, QED.

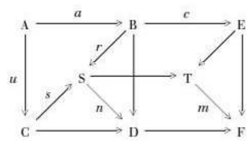


Figure 4.

Proposition 2.4 shows that a semi-cartesian square remains a semi-cartesian square when is removed a cartesian square on the right or a cocartesian square on the left; and also that semi-cartesian squares are got by composing cartesian and cocartesian squares. This is always the case, as shown by the corollary of the following proposition.

Proposition 3.4. *Let KL be a semi-cartesian square. If K is an epimorphism, then L is semi-cartesian. If L is a monomorphism, then K is semi-cartesian.*

Let us prove the first assertion; the second is dual.

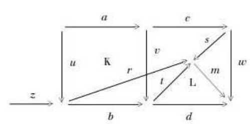


Figure 5.

Let (r, s) be an amalgamated sum of (u, ca); since KL is semi-cartesian, there exists a unique monomorphism m with $mr = db$. Since b is an epimorphism, it is a cokernel of some z, and $rz = 0$ (compose on the left with m monomorphism). This implies a unique t such that $r = tb$; and $tv = sc$ (compose on the right with epimorphism a). Le square $tv = sc$ is cocartesian: if $xc = yv$, a fortiori $xca = yva = ybu$ and since $(ru = s(ca))$ is a cocartesian square, there exists a unique n such that $x = ns$ and $yb = nr = ntb$. Since b is an epimorphism, one deduces $y = nt$. Since m is a monomorphism, square L is semi-cartesian, QED.

In other terms, in the class of semi-cartesian squares, one can simplify by epimorphisms on the left and by monomorphisms on the right. Beware that the converse is false: a semi-cartesian square (for instance the identity) preceded by an epimorphism is not necessarily semi-cartesian (there exists epimorphisms that are not semi-cartesian).

Corollary 3.1. *Semi-cartesian squares can be decomposed into a cocartesian epimorphism followed by a cartesian monomorphism.*

Decompose the square into an epimorphism followed by a monomorphism. Proposition 2.5 ensures that they are also semi-cartesian squares. From proposition 2.3, the first one is cocartesian and the second one is cartesian.

Proposition 3.5. *Consider two successive commutative squares* K *and* L *as in figure 6.*

1) Suppose that K *is a kernel of* L*. Then:*

a) w *monomorphism* $\implies$ K *cartesian.*

b) L *semi-cartesian* $\implies$ u *epimorphism.*

2) Dualy suppose that L *is a cokernel of* K*. Then:*

c) u *epimorphism* $\implies$ L *cocartesian.*

d) K *semi-cartesian* $\implies$ w *monomorphism.*

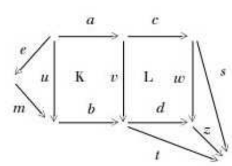


Figure 6.

Assertions (a),(b) are dual of (c),(d). Let us show (c) and consider figure 7, in which (s,t) verifies only $sc = tv$. Then $0 = sca = tva = tbu$ hence $tb = 0$ since u is an epimorphism. Since d is cokernel of b, there exists a unique arrow z such that $t = zd$. One checks $s = zw$ by composing with epimorphism c on the right.

Let us show (b). After decomposing L with the help of the above corollary, one may assume that L is a cocartesian epimorphism. Let $u = me$ be the decomposition of u into an epimorphism followed by a monomorphism. If one shows that d is a cokernel of bm, then bm will be a kernel of d (cf. Lemma 1.1) as is b, and therefore m will be invertible. Now let t be such that $tbm = 0$. It follows $tbme = 0 = tva$. Since c is a cokernel of a, there is a unique arrow s such that $sc = tv$. Since L is cocartesian, there exists a unique arrow z such that $t = zd$ (and $s = zw$), QED.

4 The snake lemma

The snake lemma constructs an exact sequence connecting kernels and cokernels.

Proposition 4.1. *Suppose two successive squares K and L, where L is semi-cartesian. If* (a, c) *is exact and* $db = 0$*, then* (b, d) *is exact. Dualy, supposing K semi-cartesian, then if* (b, d) *is exact and* $ca = 0$*, then* (a, c) *is exact.*

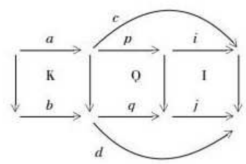


Figure 7.

Let Q be the cokernel of K so that L = QI (see Fig.7). Since $ca = 0$ and $db = 0$, there exist unique arrows i and j such that $c = ip$ and $d = jq$. Then I is semi-cartesian by proposition 2.5. Since the sequence (a, c) is exact, i is a monomorphism. From proposition 2.3, j also is a monomorphism (and I is cartesian): (b, d) is exact, QED.

For each arrow u one selects a kernel arrow of u and denotes its source by Ker(u). In this way, Ker(u) becomes a functor.

Proposition 4.2. *Kernel functors are left-exact; cokernel functors are right-exact.*

Kernels are (finite) projective limits. Therefore, they commute with projective limits. Dualy, cokernel functors Coker are right-exact.

The following lemma, called the snake lemma, connects these two functors. Decompose $a = me$ into an epimorphism e followed by a monomor-

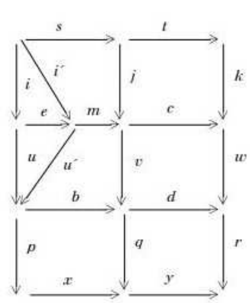


Figure 8.

phism m as in figure 7. There exists arrows $i\prime$ and $u\prime$ such that $js = mi\prime$ and $vm = bu\prime$, because s, m and b are the respective kernels of t, c and d. And since the functor Ker is left exact, $i\prime$ is a kernel of $u\prime$. In this way, changing notations, one may assume that a is a kernel of c and dually that d is a cokernel of b.

Lemma 4.1. *Given a diagram like Fig. 8, in which i, j, k are kernels of u, v, w, and p, q, r are their cokernels, in which c is a cokernel of a and b is a kernel of d, there exists an arrow d such that the following sequence is exact:*

$$Ker(u) \to^s Ker(v) \to^t Ker(w) \to^d Coker(u) \to^x Coker(v) \to^y Coker(w).$$

Construction of diagram 9. Let (m, f) be the fiber product of (k, c). The square $kf = cm$ is cartesian and since c is an epimorphism, so is f and the square is cocartesian (proposition 2.3). Let z be a kernel of f. Since $cmz = kfz = 0$ and a is a kernel of c, there exists a unique arrow l such that $al = mz$. The square thus built is a kernel of the square built over m and k; since k is a monomorphism, this square is cartesian (proposition 2.7).

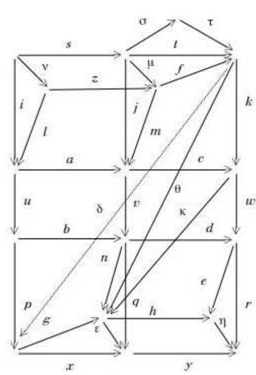


Figure 9.

Dually, one builds the amalgamated sum (n, g) of (p, b). This square is cocartesian and since b is a monomorphism, so is g and the square is cartesian (prop. 2.3). Similarly, one builds the cokernel h of g and one completes the square over h and d, which is a cokernel of the square built over g and b; since p is an epimorphism, this square is cocartesian (proposition 2.7(c)).

Arrow nvm satisfies $(nvm)z = g(pu)l = 0$ and since f is a cokernel of z, there exists a unique arrow θ such that $nvm = \theta f$. Now $h\theta$ is nul because $h\theta f = 0$ and f is an epimorphism. Therefore θ factorises through the kernel of h, that is g: there exists a unique arrow δ such that $\theta = g\delta$. This terminates the construction of δ. There remains to show that the sequence (t, δ) is exact or again, since g is a monomorphism, that (t, θ) is exact; by the duality property (δ, x) will also be exact. It is already clear that θt is nul: $\theta t = nvj = 0$. Let us show that the sequence (t, θ) is exact.

Step1: Notice that $nva = gpu = 0$ implies that nv factorizes through the cokernel c of a: $nv = \kappa c$ for a unique arrow κ. Moreover, $\kappa k f = \kappa c m = nvm = \theta f$, hence $\kappa k = \theta$ (because f is an epimorphism). Therefore, proving that (t, θ) is exact reduces to show that $(t, \kappa k)$ is exact or again, with the decomposition $t = \tau\sigma$ into an epimorphism followed by a monomorphism, that τ is a kernel of κk.

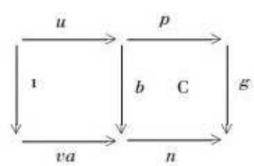


Figure 10.

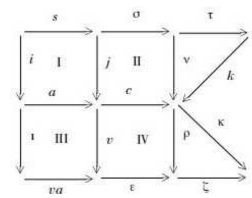


Figure 11 .

Step2: The sequence (va, n) is exact. Indeed, in figure 10, the sequence (u, p) is exact, $n(va) = gpu = 0$ and square C is cocartesian by construction. Proposition 3.1 ensures that the sequence (va, n) is exact. Since (s, t) is exact, σ is a cokernel of s. Since c is a cokernel of a, there exists a unique arrow ν such that $\nu\sigma = cj$. And $\nu = k\tau$ is a monomorphism, since k and τ are two monomorphisms.

Decompose n into an epimorphism ϵ followed by a monomorphism ζ. Since (va, n) is an exact sequence, ϵ is a cokernel of va; since c is a cokernel of a, there exists a unique arrow ρ such that $\rho c = \epsilon v$ (see Fig.11).

Step 3: Now, square IV in fig.11 is a cokernel of square III, and since 1 is an epimorphism, IV is cocartesian (proposition 2.7(c)). Further, the sequence (j, v) is exact and $\rho\nu = 0$, as can be checked if we precede it with the epimorphism σ: $\rho\nu\sigma = \epsilon v j = 0$. From proposition 3.1, (ν, ρ) is exact. Since ν is a monomorphism, it is a kernel of ρ, and also of $\zeta\rho = \kappa$ since ζ is a monomorphism. Proposition 1.6 terminates the proof: $\nu = k\tau$ is a kernel of κ, therefore t is a kernel of $\kappa k = \theta$, QED.

References

[1] Clemens Berger, *Structures algébriques et algébre homologique,* cours de DEA, Nice, 2004-2007.

[2] Peter J. Freyd, *Abelian categories,* Harper & Row, 1964.

[3] Rafael Guglielmetti, Dimitri Zaganidis, *Introduction á la théorie des catégories et aux lemmes de diagrammes,* mémoire of l' École Polytechnique Fédérale de Lausanne, 2009.