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Properties of ω-Order-Preserving Partial Contraction Mapping and its Relation to C_0-Semigroup

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Abstract

This paper consists of fixed point result on $\omega-$order preserving partial contraction mapping $(\omega - OCP_n)$ in semigroup of linear operator.

1 Introduction

Let X be a Banach space, $(T(t))_{t\geq 0}$ the C_0-semigroup which is strongly continuous one parameter semigroup of bounded linear operator in X and $L(X)$ be a bounded linear operator in X. Also, let $X_n \subseteq X$ be a finite set, CP_n semigroup of partial contraction mapping, OCP_n the semigroup of all order-preserving partial contraction mapping, $\omega - OCP_n \subseteq OCP_n$ be ω-order-preserving partial contraction mapping and $Mm(\mathbb{N})$ a matrix. Problem concerning the existence of fixed point in the study of semigroup theory has been of considerable interest in the study of linear and bounded operator.

Fixed point of linear and bounded operator is an important research branch of a linear analysis and has been applied in the study of semigroup theory. This paper will focus on fixed point results on $\omega - OCP_n$ on a

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Banach space and in relation to a special class of linear semigroup called C_0-Semigroup.

In 1993, Higgins proved some combinatorial results for Semigroup of Order-preserving Mappings. Saleh and Khamsi (2013) established, among others, fixed point results for asymptotic pointwise non-expansive semigroup in Metric Spaces. Adeshola (2013) in his Ph.D. thesis obtained several semigroups of full contraction mappings of a finite chain. Furthermore, Alata *et. al.* (2015) provided some results on common fixed point for generalized f-contraction Mapping. For relevant work on non-linear and one-parameter semigroups, see [Bellem-Morante and Mc Bride (1998), Brezis (2011) & Engel and Nagel (1999)].

2 Preliminaries

Definition 2.1: (Semigroup)
A semigroup is a pair $(S, *)$ in which is a non empty set and $*$ is a binary associative operation on S. That is the equation

$$(x * y) * z = x * (y * z) \tag{2.1}$$

holds for all $x, y, z \in S$
Definition 2.2: (C_0-Semigroup)
A C_0-semigroup is a strongly continuous one parameter semigroup of bounded linear operator on a Banach space.
Definition 2.3: (ω-OCP_n)
A transformation $\alpha \in P_n$ is called ω-order-preserving partial contraction mapping if $\forall x, y \in$Dom$\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise $T(0) = I$.

2.1 Properties of C_0-semigroup

A C_0-Semigroup is a family, $T = \{T(t) : t \in \mathbb{R}_+\}$ of a bounded linear operator from X to X satisfying

i. $T(t+s) = T(t)T(s) \quad \forall\ t, s \in \mathbb{R}_+$

ii. $e^{(t+s)A} = e^{tA}e^{sA} \quad \forall\ t, s > 0$, where A is a generator

iii. $T(0) = I$, the identity operator on X

iv. $\lim_{t\to 0^+} \|T(t)_{x_0} - x_0\| = 0$, for each $x_0 \in X$, (strongly continuous)

v. $\lim_{t\to 0} \|T(t) - I\| = 0$, (uniformly continuous)

Definition 2.3 Infinitesimal Generator
Let $T(t)$ be a C_0-semigroup, the infinitesimal generator of $T(t)$ (denoted by A) is given by the equation:

$$\lim_{t\to 0} A_t x_0 = \lim_{t\to 0} \frac{T(t)_{x_0} - x_0}{t}, \qquad \forall x_0 \in X \tag{2.2}$$

Example Some of the simplest example of $\omega - OCP_n$ in C_0-semigroup is a 2×2 matrix
Suppose

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

and $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^t & e^{2t} \\ e^{2t} & e^{2t} \end{pmatrix}$$

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3 Main Results

In this section, results on fixed point theorems on ω-OCP_n in Banach space and on C_0-Semigruop are considered.
Theorem 3.1 Let X be a Banach space and $T : X \to X$ be an ω-order preserving partial contraction mapping for which $a \in (0,1)$ and for all $x, y \in X$ whenever $\|T_x - T_y\| \leq \ a\|x - y\|$. Then, T has a fixed point $u \in X$. Furthermore for any $x \in X$ there exists a limit such that

$$\|T^n(x) - u\| \leq \frac{a^n}{1-a}\|x - T(x)\| \tag{3.1}$$

Proof Suppose a mapping T that maps a complete normed space onto a complete normed space is an ω-order preserving partial contraction mapping for which $a \in (0,1)$ and for all $x \in X$ whenever $\|T_x - T_y\| \leq \ a\|x - y\|$, then

$$\|T^n(x) - T^{n+1}(x)\| \leq a\|T^{n-1}(x) - T^n(x)\| \leq \cdots \leq a^n\|x - T(x)\| \tag{3.2}$$

Thus for $m > n$ where $n \in \mathbb{N}\ U\ \{0\}$

$$\|T^n(x)-T^m(x)\| \leq \|T^n(x)-T^{n+1}(x)\|+\|T^{n+1}(x)-T^{n+2}(x)\|+\cdots+\|T^{m-1}(x)-T^m(x)\|$$

$$\begin{aligned} &\leq a^n\|x-T(x)\| + \cdots + a^{m-1}\|x-T(x)\| \\ &\leq a^n\|x-T(x)\|[1+a+a^2+\cdots] \\ &= \frac{a^n}{1-a}\|x-T(x)\| \end{aligned}$$

That is for $m > n$, $n \in (0, 1, \cdots)$, we have

$$\|T^n(x) - T^m(x)\| \leq \frac{a^n}{1-a}\|x - T(x)\| \tag{3.3}$$

Suppose there exists $x, y \epsilon X$ with $x = T(x)$ and $y = T(y)$, then by contraction mapping, we have

$$\|T_x - T_y\| \leq \ a\|x - y\|$$

and

$$\|T_x - T_y\| = \|x - y\|$$

therefore $\|x - y\| = 0$, then,

$$x = y. \tag{3.4}$$

Since X is complete, then there exist $u \in X$ such

$$\lim_{n\to\infty} T^n(x) = u \tag{3.5}$$

Moreover, the continuity of T yields

$$u = \lim_{n\to\infty} T^{n+1}(x) = \lim_{n\to\infty} T(T^n(x)) = T(u) \tag{3.6}$$

therefore u is fixed point of T.
Finally, letting $m \to \infty$ in (3.3), then

$$\|T^n(x) - u\| \leq \frac{a^n}{1-a}\|x - T(x)\| \tag{3.7}$$

Hence the proof.

Proposition 3.2 Let X Banach space. A mapping $T(t) : \mathbb{R}_+ \to e^{tA} \in \omega - OCP_n$ is continuous for any giving $A \in \omega - OCP_n$ such that $\omega - OCP_n \in Mm(\mathbb{N})$ and satisfies the functional equation $e^{(t+s)A} = e^{tA}e^{sA}$ if $t, s > 0$ and equals to an identity I whenever t and s vanish.

Proof First, we show that the functional equation is Cauchy product series, so that

$$\left(\sum_{m=0}^{\infty} a_m\right) \cdot \left(\sum_{m=0}^{\infty} b_m\right) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} a_k b_{m-k} \tag{3.8}$$

and

$$(x+y)^m = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} x^{m-k} y^k \tag{3.9}$$

we have

$$e^{(t+s)A} = e^{tA}e^{sA}$$

$$\Longrightarrow \quad \sum_{k=0}^{\infty} \frac{t^k \|A\|^k}{k!} < \infty \tag{3.10}$$

Since the series (3.10) converges, we show that it is a Cauchy product for series. Then we have

$$\left(\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}\right) \cdot \left(\sum_{k=0}^{\infty} \frac{s^k A^k}{k!}\right) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{t^{m-k}A^{m-k}}{(m-k)!} \cdot \frac{s^k A^k}{k!}$$

$$= \sum_{m=0}^{\infty} \frac{(t+s)^m A^m}{m!} = e^{(t+s)A} \tag{3.11}$$

We want to show that $T(t)$ is continuous. Therefore, by the property of the functional equation, we let $e^{(t+h)A} \to \; e^{tA} \; = \; e^{tA}(e^{hA} - I), \quad \forall\, t, h \in \; \mathbb{N}$

$$\Longrightarrow \quad e^{tA}e^{hA} \; - \; e^{tA}$$

then, it sufficient to show that

$$\lim_{h \to 0} e^{hA} = I$$

hence,

$$\|e^{hA} - I\| = \left\| \sum_{k=1}^{\infty} \frac{h^k A^k}{k!} \right\|$$

$$\leq \sum_{k=1}^{\infty} \frac{|h^k| \|A^k\|}{k!}$$
$$= e^{|h|\|A\|} - 1 \tag{3.12}$$

$e^{|h|\|A\|} - 1 \to 0$ as $h \to 0$. Hence the proof.

ω-OCP_n on Semigroup of Linear Operator

3.1 Proposition

Let $T(t)$ be an exponentially bounded one parameter semigroup generated by matrix $Mm(N)$ for some $A \in \omega - OCP_n$ where $\omega - OCP_n \in Mm(N)$, then the function $T(\cdot) : \mathbb{R} \to Mm(N)$ is differentiable if it satisfies the differential equation:

$$\begin{cases} \frac{d}{dt} T(t) = AT(t) & t \geq 0 \\ T(0) = I \end{cases} \tag{3.13}$$

Proof Let $T(t) = e^{tA}$ for some $A \in \omega - OCP_n$, so that

$$\frac{d}{dt}(T(t)) = Ae^{tA}$$
$$\implies \quad T'(t) = AT(t)$$

at $t = 0$, clearly we have

$$A = T'(0)$$

we need to show that $T(\cdot)$ satisfies the differential equation, by C_0-semigroup properties, we have

$$\frac{T(t+h) - T(t)}{h} = \frac{T(h) - 1}{h} T(t) \ \ \forall \ t, h \in \mathbb{N}$$

so that

$$\lim_{h \to 0} \frac{T(h) - 1}{h} = A$$

with respect to norm, we have

$$\left\| \frac{T(h) - 1}{h} - A \right\| \leq \sum_{k=2}^{\infty} \frac{|h|^{k-1} \|A\|^k}{k!}$$
$$\leq \left(\frac{e^{|h|\|A\|}}{h} - \frac{1}{h} \right) - \|A\|$$

as $h \to 0$

$$\left(\frac{e^{|h|\|A\|}}{h} - \frac{1}{h}\right) \to \|A\|$$

and

$$\frac{T(h)-1}{h} \to \quad A$$

so its differentiable. Hence the proof.

3.2 Proposition

Let $A \in \omega - OCP_n$ such that $\omega - OCP_n \in L(X)$ where $L(X)$ is a bounded linear operator from X to X. Then $T(t) = e^{tA}$: $t \in \mathbb{N}$ is uniformly continuous semigroup.

Proof Suppose A is bounded linear operator from X onto X, since $\|A\| < \infty$ and thus $\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$ converges for $t \geq 0$ to the bounded linear operator $T(t)$ and

$$\left(\sum_{i=0}^{\infty} \frac{(t)^i}{i!}\right)\left(\sum_{j=0}^{\infty} \frac{(s)^j}{j!}\right) = \sum_{k=0}^{\infty} \frac{(t+s)^k}{k!}$$

$T(0) = I$ is clear.

We need to show that $T(t)$ is a uniformly continuous semigroup, but

$$\begin{aligned}\|T(t) - I\| &= \left\|\sum_{k=1}^{\infty} \frac{t^k A^k}{k!}\right\| \\ &\leq \sum_{k=1}^{\infty} \frac{t^k \|A\|^k}{k!} \\ &= e^{t\|A\|} - 1\end{aligned}$$

$$e^{t\|A\|} - 1 \ \to 0 \ \text{ as } \ t \to 0$$

Hence $T(t)$ is uniformly continuous.

Conclusion In this paper, it has been shown that $\omega - OCP_n$ has a fixed point and satisfies the properties of a semigroup of linear operator which is C_0-Semigroup.

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