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Convergence and Stability of Some Iterative Sequences in G-metric space

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Abstract

The purpose of this article is to prove the convergence and stability of Mann and Ishikawa iterative procedures by applying $\lambda-$generalized, Geraghty and Ciric contractive conditions in a complete G-metric space.

1 Introduction

Stability of a fixed point iteration procedure occurs if small modifications in the initial data or in the data that are involved in the system will produce a small influence on the result of the fixed point. Huang and Changfeng (2014) presented a fixed-point iterative method for solving systems of nonlinear equations. The convergence theorem of their method was proved under suitable conditions. Katchang and Kumam (2011) introduced a new modified Ishikawa iterative process for computing fixed points of an infinite family of nonexpansive mapping in the framework of Banach spaces. The work established the strong convergence theorem of the proposed iterative scheme under some mild conditions to solve a variational inequality. Rafiq (2006) established a general theorem to approximate fixed points of z-operators on a

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normed space through the Mann iteration process with error. In the early sixties, the notion of 2-metric space was introduced by Gahler which he claimed is a generalization of ordinary metric space, satisfying some properties.
In 1992, Dhage introduced a new structure of a generalized metric space which he called D-metric space. Since then, there have been a lot of generalizations of metric spaces. Mustafa and Sims (2006) proved a new generalization of a metric space which they called G-metric space after proving that most of the desired properties do not hold in the D-metric space. For related works we refer the reader to Banach (1922), Abbas et.al. (2011, 2012), Mustafa and Obiedat (2010) and Rhoade (1977).

A. Some useful iterative conditions

Definition 1.1 [Mann (1953)]: For $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad n = 0, 1, 2, \dots,$$

where $\{x_n\}_{n=0}^{\infty} \subset [0,1]$, is called the Mann iterative process.

Definition 1.2 [Ishikawa (1974)]: The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned}$$

where $n \geq 0$ is called the Ishikawa iterative sequence, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequence in $[0,1]$.

B. Contractive conditions to be considered are:

Definition 1.3 [Agarwal et.al. (2015)]: A mapping $T : X \to X$ is said to be a λ-generalised contraction if for every $x, y, z \in X$ there exist non-negative numbers $q(x,y,z), r(x,y,z), s(x,y,z)$ and $t(x,y,z)$ such that

$$\sup_{x,y,z \in X} \{q(x,y,z) + r(x,y,z) + s(x,y,z) + 2t(x,y,z)\} := \lambda < 1$$

and

$$\begin{aligned} G(Tx,Ty,Tz) \leq\ & q(x,y,z)G(x,y,z) + r(x,y,z)G(x,Tx,Tx) \\ & + s(x,y,z)G(y,Ty,Ty) + (x,y,z)\{G(x,Ty,Ty) + G(y,Tx,Tx)\} \end{aligned}$$

Definition 1.4 [Geraghty (1973)]: A mapping $T : X \to X$ is said to be a Geraghty contraction if there exists $\beta \in \digamma$ such that for every $x, y, z \in X$

$$G(Tx, Ty, Tz) \leq \beta[(G(x, y, z))]G(x, y, z),$$

where the class $\digamma$ denotes those functions $\beta : [0, \infty] \to [0, \infty]$ satisfying $\beta(t_n) \to 0$ as $n \to \infty$.

Definition 1.5 [Ciric (1974)]: A mapping $T : X \to X$ is said to be a Ciric contraction if there exists a constant h, with $0 \leq h < 1$, such that for every $x, y, z \in X$,

$$G(Tx, Ty, Tz) \leq h \max\{G(x, y, z), \frac{1}{2}[G(x, Tx, Tx) + G(y, Ty, Ty)], \frac{1}{2}[G(x, Tx, Tx) + G(y, Tx, Tx)]\}$$

Lemma 1 [Mustafa and Sims (2006)]: If (X, G) is a G-metric, then

$$G(x, y, y) \leq 2G(y, x, x) \qquad for\, all\, x, y \in X$$

Lemma 2 [Mustafa and Sims (2006)]: Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers and $0 \leq q < 1$ such that

$$\alpha_{n+1} \leq q\alpha_n + \beta_n, for\, all\, n \geq 0.$$

1. If $\lim_{n\to\infty} \beta_n = 0$, then $\lim_{n\to\infty} \alpha_n = 0$
2. If $\sum_0^{\infty} \alpha_n < \infty$, then $\sum_0^{\infty} \beta_n < \infty$

.

Definition 1.6 [Rauf et. al. (2017)]: Define a fixed point iteration procedure by a general relation of the form

$$x_{n+1} = f(T, x_n) \qquad \forall n = 0, 1, 2, \ldots$$

Let (X, G) be a complete G-metric space, $T : X \to X$, $x_0 \in X$ and assume that the iterative procedure is defined as $x_{n+1} = f(T, x_n) \quad \forall n = 0, 1, 2, \ldots$ that is, the sequence converges to a fixed point ω of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

$$\epsilon_n = G(y_{n+1}, f(T, y_n), f(T, y_n)), \quad for \quad n = 0, 1, 2, \ldots$$

The fixed point iterative procedure as defined above is T-stable (or stable) with respect to T if and only if

$$\lim_{n\to 0} \epsilon_n = 0 \qquad \Longrightarrow \; y_n = \omega.$$

This paper establishes new results on convergence and stability of some iterative procedures via $\lambda-$generalized, Geraghty and Ciric contractive conditions in a complete G-metric space.

2 Main Results

Throughout this section, $1-\alpha_n$ is assumed positive (otherwise one can take its absolute value instead).

2.1 Convergence Results

Theorem 2.1: Let (X,G) be a complete G-metric space and $T : X \to X$ be a self-map with a fixed point ω satisfying λ- generalized contraction for which $x, y \in X$. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann Iterative scheme. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to ω.

Proof. Assuming $x_{n+1} \neq \omega$, since ω is a fixed point, we have

$$\begin{aligned} G(x_{n+1},\omega,\omega) =& G(Tx_n,\omega,\omega) \\ &= G((1-\alpha_n)x_n + \alpha_n Tx_n, \omega,\omega) \\ &\leq G((1-\alpha_n)x_n,\omega,\omega) + G(\alpha_n Tx_n,\omega,\omega) \\ &\leq (1-\alpha_n)G(x_n,\omega,\omega) + \alpha_n G(Tx_n,\omega,\omega) \\ &= (1-\alpha_n)G(x_n,\omega,\omega) + \alpha_n G(Tx_n,T\omega,T\omega) \end{aligned} \tag{2.1}$$

If $y = z = \omega$ in our λ generalized contractive condition, substituting into equation (2.1) gives

$$\begin{aligned} G(x_{n+1},\omega,\omega) \leq& (1-\alpha_n)G(x_n,\omega,\omega) + \alpha_n[q(x,\omega,\omega)G(x_n,\omega,\omega) \\ &+ r(x,\omega,\omega)G(x_n,Tx_n,Tx_n) + s(x,\omega,\omega)G(\omega,T\omega,T\omega) \\ &+ t(x,\omega,\omega)[G(x_n,\omega,\omega) + G(\omega,Tx_n,Tx_n)]] \\ &= (1-\alpha_n)G(x_n,\omega,\omega) + \alpha_n[[q(x,\omega,\omega)G(x_n,\omega,\omega) \\ &+ t(x,\omega,\omega)[G(x_n,\omega,\omega) + G(\omega,Tx_n,Tx_n)]] \end{aligned}$$

By Lemma 1

$$\begin{aligned} G(x_{n+1},\omega,\omega) \leq &[(1-\alpha_n)+\alpha_n q(x,\omega,\omega)+t(x,\omega,\omega)]G(x_n,\omega,\omega) \\ &+ 2t(x,\omega,\omega)G(x_n,\omega,\omega.) \end{aligned}$$

Hence

$$\begin{aligned} G(x_{n+1},\omega,\omega) &\leq [(1-\alpha_n)+\alpha_n q(x,\omega,\omega)+t(x,\omega,\omega)]G(x_n,\omega,\omega) \\ &+ 2t(x,\omega,\omega)G(x_n,\omega,\omega) \\ &= [(1-\alpha_n)+\alpha_n q(x,\omega,\omega)+3t(x,\omega,\omega)]G(x_n,\omega,\omega) \end{aligned}$$

since $0 \leq q(x,\omega,\omega)+r(x,\omega,\omega)+s(x,\omega,\omega)+2t(x,\omega,\omega) < 1$ and $\sum \alpha_n = \infty$. Then

$$G(x_{n+1},\omega,\omega) \leq G(x_n,\omega,\omega)$$

$$\lim_{n\to\infty} G(x_n,\omega,\omega) = 0$$

We conclude that the sequence $\{x_n\}$ converges strongly to the fixed point ω.

Theorem 2.2: Let (X,G) be a complete G-metric space and $T: X \to X$ be a self-map with a fixed point ω satisfying Geraghty contraction for which $x, y \in X$. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann Iterative scheme. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to ω.

Proof. Assuming $x_{n+1} \neq \omega$, since ω is a fixed point, we have

$$\begin{aligned} G(x_{n+1},\omega,\omega) =& G(Tx_n,T\omega,T\omega) \\ &= G((1-\alpha_n)x_n+\alpha_n Tx_n,\omega,\omega) \\ &\leq G((1-\alpha_n)x_n,\omega,\omega)+G(\alpha_n Tx_n,\omega,\omega) \\ &\leq (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_n G(Tx_n,\omega,\omega) \\ &= (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_n G(Tx_n,T\omega,T\omega) \\ &\leq (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_n[\beta[G(x_n,\omega,\omega)]G(x_n,\omega,\omega)] \\ &= (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_n\beta G(x_n,\omega,\omega)G(x_n,\omega,\omega) \\ &= [(1-\alpha_n)+\alpha_n\beta G(x_n,\omega,\omega)]G(x_n,\omega,\omega) \end{aligned}$$

while $0 \leq \beta < 1$ and $\sum \alpha_n = \infty$. Then $G(x_{n+1},\omega,\omega) \leq G(x_n,\omega,\omega)$ and $\lim_{n\to\infty} G(x_n,\omega,\omega) = 0$. Hence the sequence converges strongly to the fixed point ω.

Theorem 2.3: Let (X, G) be a complete G-metric space and $T : X \to X$ be a self-map with a fixed point ω satisfying Ciric contraction for which $x, y \in X$. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann Iterative scheme. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to ω.

Proof. Assuming $x_{n+1} \neq \omega$, since ω is a fixed point, we have

$$
\begin{aligned}
G(x_{n+1}, \omega, \omega) =& G(Tx_n, T\omega, T\omega) \\
&= G((1-\alpha_n)x_n + \alpha_n Tx_n, \omega, \omega) \\
&\leq G((1-\alpha_n)x_n, \omega, \omega) + G(\alpha_n Tx_n, \omega, \omega) \\
&\leq (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n G(Tx_n, \omega, \omega) \\
&= (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n G(Tx_n, T\omega, T\omega) \\
&\leq (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n[h \max\{G(x_n, \omega, \omega), \\
&\frac{1}{2}[G(x_n, Tx_n, Tx_n) + G(\omega, T\omega, T\omega)] \\
&+ \frac{1}{2}[G(x_n, Tx_n, Tx_n) + G(\omega, Tx_n, Tx_n)]\}] \\
&\leq (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n[h \max\{G(x_n, \omega, \omega), \frac{1}{2}G(\omega, x_n, x_n\}]
\end{aligned}
$$

By Lemma 1

$$
\begin{aligned}
&G(x_{n+1}, \omega, \omega) \leq (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n[h \max\{G(x_n, \omega, \omega), G(x_n, \omega, \omega\}] \\
&= (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n h G(x_n, \omega, \omega) \\
&= [(1-\alpha_n) + \alpha_n h]G(x_n, \omega, \omega)
\end{aligned}
$$

since $0 \leq h < 1$ and $\sum \alpha_n = \infty$. Then $G(x_{n+1}, \omega, \omega) \leq G(x_n, \omega, \omega)$ and $\lim_{n\to\infty} G(x_n, \omega, \omega) = 0$. Hence we claim that sequence sequence converges strongly to ω.

Theorem 2.4: Let (X, G) be a complete G-metric space and $T : X \to X$ be a self-map with a fixed point ω satisfying λ-generalized contraction for which $x, y \in X$. Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa Iterative scheme. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to ω.

Proof. Assuming $x_{n+1} \neq \omega$, since ω is a fixed point, we have

$$\begin{aligned}
G(x_{n+1}, \omega, \omega) =& G(Tx_n, T\omega, T\omega) \\
&= G((1-\alpha_n)x_n + \alpha_n Ty_n, \omega, \omega) \\
&\leq G((1-\alpha_n)x_n, \omega, \omega) + G(\alpha_n Ty_n, \omega, \omega) \\
&\leq (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n G(Ty_n, \omega, \omega) \\
&= (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n G(Ty_n, T\omega, T\omega) \\
&\leq (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n[q(x, \omega, \omega)G(y_n, \omega, \omega) \\
&+ r(x, \omega, \omega)G(y_n, Ty_n, Ty_n) + s(x, \omega, \omega)G(\omega, T\omega, T\omega) \\
&+ t(x, \omega, \omega)[G(y_n, \omega, \omega) + G(\omega, Ty_n, Ty_n)]] \\
&\leq (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n[[q(x, \omega, \omega)G(y_n, \omega, \omega) \\
&+ t(x, \omega, \omega)[G(y_n, \omega, \omega) + G(\omega, Ty_n, Ty_n).]]
\end{aligned}$$

Hence

$$\begin{aligned}
G(x_{n+1}, \omega, \omega) \leq & (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n[q(x, \omega, \omega)G(y_n, \omega, \omega) + t(x, \omega, \omega)[G(y_n, \omega, \omega) \\
&+ 2G(y_n, \omega, \omega)]
\end{aligned}$$

But $y_n = (1-\gamma_n)x_n + Tx_n$

$$\begin{aligned}
\Rightarrow G(x_{n+1}, \omega, \omega) \leq & (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n[q(x, \omega, \omega)G((1-\gamma_n)x_n \\
&+ \gamma_n Tx_n, \omega, \omega) + t(x, \omega, \omega)[G((1-\gamma_n)x_n \\
&+ \gamma_n Tx_n, \omega, \omega) + 2G((1-\gamma_n)x_n + \gamma_n Tx_n, \omega, \omega)]] \\
\leq & (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n[q(x, \omega, \omega)[(1-\gamma_n)G(x_n, \omega, \omega) + \gamma_n G(x_n, \omega, \omega)] \\
&+ t(x, \omega, \omega)[(1-\gamma_n)G(x_n, \omega, \omega) + \gamma_n G(x_n, \omega, \omega)] + [2(1-\gamma_n)G(x_n, \omega, \omega) \\
&+ 2\gamma_n G(x_n, \omega, \omega)]]
\end{aligned}$$

leading to

$$\begin{aligned}
G(x_{n+1}, \omega, \omega) \leq & [(1-\alpha_n) + \alpha_n(1-\gamma_n)q(x, \omega, \omega) + \alpha_n\gamma_n q(x, \omega, \omega) + \alpha_n(1-\gamma_n)t(x, \omega, \omega) \\
&+ 2\alpha_n(1-\gamma_n)t(x, \omega, \omega) + \alpha_n\gamma_n t(x, \omega, \omega) + 2\alpha_n\gamma_n t(x, \omega, \omega)]G(x_n, \omega, \omega)
\end{aligned}$$

Since $0 \leq q(x, \omega, \omega) + r(x, \omega, \omega) + s(x, \omega, \omega) + 2t(x, \omega, \omega) < 1$ and $\sum \alpha_n, \gamma_n = \infty$, $G(x_{n+1}, \omega, \omega) \leq G(x_n, \omega, \omega)$ and $\lim_{n\to\infty} G(x_n, \omega, \omega) = 0$. Hence the sequence converges strongly to ω.

Theorem 2.5: Let (X, G) be a complete G-metric space and $T : X \to X$ be a self-map with a fixed point ω satisfying Geraghty generalized contraction for which $x, y \in X$. Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa Iterative scheme. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to ω.

Proof. Assuming $x_{n+1} \neq \omega$, since ω is the fixed point, we have

$$
\begin{aligned}
G(x_{n+1},\omega,\omega) =& G(Tx_n, T\omega, T\omega)\\
&= G((1-\alpha_n)x_n + \alpha_n Ty_n, \omega, \omega)\\
&\leq G((1-\alpha_n)x_n, \omega, \omega) + G(\alpha_n Ty_n, \omega, \omega)\\
&\leq (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n G(Ty_n, \omega, \omega)\\
&= (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n G(Ty_n, T\omega, T\omega)\\
&\leq (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n[\beta[G(y_n, \omega, \omega)]G(y_n, \omega, \omega)]\\
&\leq (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n\beta[G((1-\gamma_n)x_n + \gamma_n Tx_n, \omega, \omega)G((1-\gamma_n)x_n\\
&+ \gamma_n Tx_n, \omega, \omega)]\\
&\leq (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n\beta[[(1-\gamma_n)G(x_n, \omega, \omega)\\
&+ \gamma_n G(x_n, \omega, \omega)][(1-\gamma_n)G(x_n, \omega, \omega) + \gamma_n G(x_n, \omega, \omega)]\\
&= (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n\beta[(1-\gamma_n)^2 G(x_n, \omega, \omega)^2\\
&+ 2\gamma_n(1-\gamma_n)G(x_n, \omega, \omega)^2 + \gamma_n^2 G(x_n, \omega, \omega)^2]\\
&= (1-\alpha_n)G(x_n, \omega, \omega) + \alpha_n\beta[(1-\gamma_n)^2 + 2\gamma_n(1-\gamma_n) + \gamma_n^2]G(x_n, \omega, \omega)^2\\
&= [(1-\alpha_n) + \alpha_n\beta[(1-\gamma_n)^2 + 2\gamma_n(1-\gamma_n) + \gamma_n^2]G(x_n, \omega, \omega)]G(x_n, \omega, \omega)
\end{aligned}
$$

Since $0 \leq \beta < 1$ and $\sum \alpha_n, \gamma_n = \infty$, $G(x_{n+1}, \omega, \omega) \leq G(x_n, \omega, \omega)$ and $\lim_{n\to\infty} G(x_n, \omega, \omega) = 0$. Hence, the sequence converges strongly to the fixed point.

Theorem 2.6: Let (X,G) be a complete G-metric space and $T : X \to X$ be a self-map with a fixed point ω satisfying Ciric generalized contraction for which $x, y \in X$. Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa Iterative scheme. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to ω.

Proof. Assuming $x_{n+1} \neq \omega$, since ω is a fixed point, we have

$$
\begin{aligned}
G(x_{n+1},\omega,\omega) =& G(Tx_n,T\omega,T\omega)\\
&= G((1-\alpha_n)x_n+\alpha_nTy_n,\omega,\omega)\\
&\leq G((1-\alpha_n)x_n,\omega,\omega)+G(\alpha_nTy_n,\omega,\omega)\\
&\leq (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_nG(Ty_n,\omega,\omega)\\
&= (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_nG(Ty_n,T\omega,T\omega)\\
&\leq (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_n[h\max[G(y_n,\omega,\omega),\frac{1}{2}[G(y_n,Ty_n,Ty_n)\\
&+ G(\omega,T\omega,T\omega)],\frac{1}{2}[G(y_n,Ty_n,Ty_n)+G(\omega,y_n,y_n)]]]\\
&= (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_n[h\max[G(y_n,\omega,\omega),\frac{1}{2}G(\omega,y_n,y_n)]]\\
&\leq (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_n[h\max[G(y_n,\omega,\omega),G(y_n,\omega,\omega)]\\
&= (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_nhG(y_n,\omega,\omega)
\end{aligned}
$$

But $y_n=(1-\gamma_n)x_n+\gamma_nTx_n$. Hence

$$
\begin{aligned}
G(x_{n+1},\omega,\omega) &\leq (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_nhG((1-\gamma_n)x_n+\gamma_nTx_n,\omega,\omega)\\
&\leq (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_nh[G((1-\gamma_n)x_n,\omega,\omega)+\gamma_nG(x_n,\omega,\omega)]\\
&= (1-\alpha_n)G(x_n,\omega,\omega)+\alpha_nh[(1-\gamma_n)G(x_n,\omega,\omega)+\gamma_nG(x_n,\omega,\omega)]\\
&= [(1-\alpha_n)+\alpha_nh[(1-\gamma_n)+\gamma_n]]G(x_n,\omega,\omega)
\end{aligned}
$$

Since $0\leq h<1$ and $\sum\alpha_n,\gamma_n=\infty$,

$$G(x_{n+1},\omega,\omega)\leq G(x_n,\omega,\omega)$$

$$\lim_{n\to\infty}G(x_n,\omega,\omega)=0$$

Hence the sequence converges strongly ω.

2.2 Stability results

Theorem 2.7: Let (X,G) be a complete metric G-space and $T:X\to X$ be a self map with a fixed point ω satisfying λ generalised contraction for which $x,y,z\in X$, $0\leq q(x,\omega,\omega)+r(x,\omega,\omega)+s(x,\omega,\omega)+2t(x,\omega,\omega)<1$. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann Iterative scheme. Then the sequence $\{x_n\}_{n=0}^{\infty}$ is T-Stable (or stable) with respect to T.

Proof. Since ω is a fixed point, assuming $y_{n+1} \neq \omega$ and by Lemma 1, we define ϵ_n as

$$\epsilon_n = G(y_{n+1}, (1-\alpha_n)y_n + \alpha_n Ty_n, (1-\alpha_n)y_n + \alpha_n Ty_n)$$

$$\begin{aligned}
G(y_{n+1}, \omega, \omega) &\leq G(y_{n+1}, (1-\alpha_n)y_n + \alpha_n Ty_n, (1-\alpha_n)y_n + \alpha_n Ty_n) \\
&+ G((1-\alpha_n)y_n + \alpha_n Ty_n, \omega, \omega) \\
&= G((1-\alpha_n)y_n + \alpha_n Ty_n, \omega, \omega) + \epsilon_n \\
&\leq G((1-\alpha_n)y_n, \omega, \omega) + G(\alpha_n Ty_n, \omega, \omega) + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n G(Ty_n, \omega, \omega) + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[q(y, \omega, \omega)G(y_n, \omega, \omega) \\
&+ r(y, \omega, \omega)G(y_n, Ty_n, Ty_n) + s(y, \omega, \omega)G(\omega, T\omega, T\omega) \\
&+ t(y, \omega, \omega)[G(y_n, \omega, \omega) + G(\omega, Ty_n, Ty_n)]] + \epsilon_n \\
&= (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[q(y, \omega, \omega)G(y_n, \omega, \omega) + 3t(y, \omega, \omega)G(y_n, \omega, \omega)] + \epsilon_n \\
&= [(1-\alpha_n) + \alpha_n[q(y, \omega, \omega) + 3t(y, \omega, \omega)]]G(y_n, \omega, \omega) + \epsilon_n.
\end{aligned}$$

Suppose $\lim_{n\to\infty} \epsilon_n = 0$. Since $0 \leq q(x, \omega, \omega) + r(x, \omega, \omega) + s(x, \omega, \omega) + 2t(x, \omega, \omega) < 1$ and $\alpha_n > 0$, it suffices to show, using Lemma 2 and the fact that $\sup_{x,y,z\in X}\{q(x, y, z) + r(x, y, z) + s(x, y, z) + 2t(x, y, z)\} := \lambda < 1$, that $\lim_{n\to\infty} y_n = \omega$. Since $G(x_{n+1}, (1-\alpha_n)x_n + \alpha_n Tx_n, (1-\alpha_n)x_n + \alpha_n Tx_n) = 0$, $\lim_{n\to\infty} x_n = \omega$.
Therefore, the sequence $\{x_n\}$ is stable with respect to T.

Theorem 2.8: Let (X, G) be a complete metric G-space and $T : X \to X$ be a self map with a fixed point ω satisfying Geraghty generalised contraction for which $x, y, z \in X$, $0 \leq \beta < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann Iterative scheme. Then the sequence $\{x_n\}_{n=0}^{\infty}$ is T-Stable (or stable) with respect to T.

Proof. Assuming $y_{n+1} \neq \omega$, and by Lemma 1, since ω is a fixed point, we define ϵ_n as $\epsilon_n = G(y_{n+1}, (1-\alpha_n)y_n + \alpha_n Ty_n, (1-\alpha_n)y_n + \alpha_n Ty_n)$. Then

$$\begin{aligned}
G(y_{n+1}, \omega, \omega) &\leq G(y_{n+1}, (1-\alpha_n)y_n + \alpha_n Ty_n, (1-\alpha_n)y_n + \alpha_n Ty_n) \\
&+ G((1-\alpha_n)y_n + \alpha_n Ty_n, \omega, \omega) \\
&= G((1-\alpha_n)y_n + \alpha_n Ty_n, \omega, \omega) + \epsilon_n \\
&\leq G((1-\alpha_n)y_n, \omega, \omega) + G(\alpha_n Ty_n, \omega, \omega) + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n G(Ty_n, \omega, \omega) + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[\beta[G(y_n, \omega, \omega)]G(y_n, \omega, \omega)] + \epsilon_n \\
&= (1-\alpha_n) + \alpha_n[\beta G(y_n, \omega, \omega)]G(y_n, \omega, \omega) + \epsilon_n
\end{aligned}$$

Suppose $\lim_{n\to\infty} \epsilon_n = 0$. Since $0 \leq \beta < 1$ and $\alpha_n > 0$, it suffice to show, using Lemma 2, that $\lim_{n\to\infty} y_n = \omega$. Since $G(x_{n+1}, (1-\alpha_n)x_n + \alpha_n Tx_n, (1-\alpha_n)x_n + \alpha_n Tx_n) = 0$, $\lim_{n\to\infty} x_n = \omega$. Hence the sequence $\{x_n\}$ is stable with respect to T.

Theorem 2.9: Let (X, G) be a complete metric G-space and $T : X \to X$ be a self map with a fixed point ω satisfying Ciric generalised contraction for which $x, y, z \in X$, $0 \leq h < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann Iterative scheme. Then the sequence $\{x_n\}_{n=0}^{\infty}$ is T-Stable (or stable) with respect to T.

Proof. Assuming $y_{n+1} \neq \omega$ and by Lamma 1, since ω is a fixed point, we define ϵ_n as $\epsilon_n = G(y_{n+1}, (1-\alpha_n)y_n + \alpha_n Ty_n, (1-\alpha_n)y_n + \alpha_n Ty_n)$. Then

$$\begin{aligned}
G(y_{n+1}, \omega, \omega) &\leq G(y_{n+1}, (1-\alpha_n)y_n + \alpha_n Ty_n, (1-\alpha_n)y_n + \alpha_n Ty_n) \\
&\quad + G((1-\alpha_n)y_n + \alpha_n Ty_n, \omega, \omega) \\
&= G((1-\alpha_n)y_n + \alpha_n Ty_n, \omega, \omega) + \epsilon_n \\
&\leq G((1-\alpha_n)y_n, \omega, \omega) + G(\alpha_n Ty_n, \omega, \omega) + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n G(Ty_n, \omega, \omega) + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[h \max\{G(y_n, \omega, \omega), \frac{1}{2}[G(y_n, Ty_n, Ty_n) \\
&\quad + G(\omega, T\omega, T\omega)], +\frac{1}{2}[G(y_n, Ty_n, Ty_n) + G(\omega, Ty_n, Ty_n)]\}] + \epsilon_n \\
&= (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[h \max\{G(y_n, \omega, \omega), G(y_n, \omega, \omega)\}] + \epsilon_n \\
&= (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n h G(y_n, \omega, \omega) + \epsilon_n \\
&= [(1-\alpha_n) + \alpha_n h]G(y_n, \omega, \omega) + \epsilon_n
\end{aligned}$$

Suppose $\lim_{n\to\infty} \epsilon_n = 0$, since $0 \leq h < 1$ and $\alpha_n > 0$, it suffices to show, using Lemma 2, that $\lim_{n\to\infty} y_n = \omega$. As $G(x_{n+1}, (1-\alpha_n)x_n + \alpha_n Tx_n, (1-\alpha_n)x_n + \alpha_n Tx_n) = 0$, $\lim_{n\to\infty} x_n = \omega$ and hence the sequence is stable with respect to T.

The next results are on stability of two-step iteration.

Theorem 2.10: Let (X, G) be a complete metric G-space and $T : X \to X$ be a self map with a fixed point ω satisfying λ generalised contraction for which $x, y, z \in X$, $0 \leq q(x, \omega, \omega) + r(x, \omega, \omega) + s(x, \omega, \omega) + 2t(x, \omega, \omega) < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa Iterative scheme. Then the sequence $\{x_n\}_{n=0}^{\infty}$ is T-Stable (or stable) with respect to T.

Proof. Assuming $y_{n+1} \neq \omega$, since ω is a fixed point, we define ϵ_n as $\epsilon_n = G(y_{n+1}, (1-\alpha_n)y_n+\alpha_n T s_n, (1-\alpha_n)y_n+\alpha_n T s_n)$ where $s_n = (1-\gamma_n)y_n+\gamma_n T y_n$. Hence

$$\begin{aligned}
G(y_{n+1}, \omega, \omega) &\leq G(y_{n+1}, (1-\alpha_n)y_n + \alpha_n T s_n, (1-\alpha_n)y_n + \alpha_n T s_n) \\
&+ G((1-\alpha_n)y_n + \alpha_n T s_n, \omega, \omega) \\
&= G((1-\alpha_n)y_n + \alpha_n T s_n, \omega, \omega) + \epsilon_n \\
&\leq G((1-\alpha_n)y_n, \omega, \omega) + G(\alpha_n T s_n, \omega, \omega) + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n G(T s_n, \omega, \omega) + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[q(y, \omega, \omega)G(s_n, \omega, \omega) \\
&+ r(y, \omega, \omega)G(s_n, T s_n, T s_n) + s(y, \omega, \omega)G(\omega, T\omega, T\omega) \\
&+ t(y, \omega, \omega)[G(s_n, \omega, \omega) + G(\omega, T s_n, T s_n)]] + \epsilon_n \\
&= (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[q(y, \omega, \omega)G(s_n, \omega, \omega) + t(y, \omega, \omega)[G(s_n, \omega, \omega) \\
&+ 2G(s_n, \omega, \omega)]] + \epsilon_n.
\end{aligned}$$

But $s_n = (1-\gamma_n)y_n + \gamma_n T y_n$. Thus

$$\begin{aligned}
G(y_{n+1}, \omega, \omega) &\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[q(y, \omega, \omega)G(s_n, \omega, \omega) \\
&+ t(y, \omega, \omega)[G((1-\gamma_n)y_n + \gamma_n T y_n, \omega, \omega) \\
&+ 2tG((1-\gamma_n)y_n + \gamma_n T y_n, \omega, \omega)]] + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[q(y, \omega, \omega)[(1-\gamma_n)G(y_n, \omega, \omega) \\
&+ \gamma_n G(y_n, \omega, \omega)] + 3t(y, \omega, \omega)[(1-\gamma_n)G(y_n, \omega, \omega) + \gamma_n G(y_n, \omega, \omega)] + \epsilon_n \\
&= [(1-\alpha_n) + \alpha_n q(y, \omega, \omega)[(1-\gamma_n) + \gamma_n] + 3t(y, \omega, \omega)[(1-\gamma_n) \\
&+ \gamma_n]G(y_n, \omega, \omega)] + \epsilon_n
\end{aligned}$$

If $\lim_{n\to\infty} \epsilon_n = 0$ and since $0 \leq q(x, \omega, \omega)+r(x, \omega, \omega)+s(x, \omega, \omega)+2t(x, \omega, \omega) < 1$ and $\alpha_n > 0$, it suffice to show, using Lemma 2 and the fact that

$$\sup_{x,y,z\in X}\{q(x, y, z) + r(x, y, z) + s(x, y, z) + 2t(x, y, z)\} := \lambda < 1$$

we have

$$\lim_{n\to\infty} y_n = \omega$$

Since $G(x_{n+1}, (1-\alpha_n)x_n + \alpha_n T z_n, (1-\alpha_n)x_n + \alpha_n T z_n) = 0$ Then

$$\lim_{n\to\infty} x_n = \omega.$$

Hence the sequence is stable with respect to T.

Theorem 2.11: Let (X, G) be a complete metric G-space and $T : X \to X$ be a self map with a fixed point ω satisfying Geraghty generalised contraction for which $x, y, z \in X$, $0 \leq \beta < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa Iterative scheme. Then the sequence $\{x_n\}_{n=0}^{\infty}$ is T-Stable (or stable) with respect to T.

Proof. Assuming $y_{n+1} \neq \omega$, by Lemma 1, since ω is a fixed point, we define ϵ_n as $\epsilon_n = G(y_{n+1}, (1-\alpha_n)y_n + \alpha_n Ts_n, (1-\alpha_n)y_n + \alpha_n Ts_n)$. Then

$$\begin{aligned} G(y_{n+1}, \omega, \omega) &\leq G(y_{n+1}, (1-\alpha_n)y_n + \alpha_n Ts_n, (1-\alpha_n)y_n + \alpha_n Ts_n) \\ &\quad + G((1-\alpha_n)y_n + \alpha_n Ts_n, \omega, \omega) \\ &= G((1-\alpha_n)y_n + \alpha_n Ts_n, \omega, \omega) + \epsilon_n \\ &\leq G((1-\alpha_n)y_n, \omega, \omega) + G(\alpha_n Ts_n, \omega, \omega) + \epsilon_n \\ &\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n G(Ts_n, \omega, \omega) + \epsilon_n \\ &\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[\beta[G(s_n, \omega, \omega)]G(s_n, \omega, \omega)] + \epsilon_n. \end{aligned}$$

But $s_n = (1-\gamma_n)y_n + \gamma_n Ty_n$. Then

$$\begin{aligned} G(y_{n+1}, \omega, \omega) &\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[\beta[G((1-\gamma_n)y_n + \gamma_n Ty_n, \omega, \omega)]G((1-\gamma_n)y_n \\ &\quad + \gamma_n Ty_n, \omega, \omega)] + \epsilon_n \\ &\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[\beta[(1-\gamma_n)G(y_n, \omega, \omega) + \gamma_n G(y_n, \omega, \omega)]^2] + \epsilon_n \\ &= [(1-\alpha_n) + \gamma_n[(1-\gamma_n) + \gamma_n]^2 G(y_n, \omega, \omega)]G(y_n, \omega, \omega) + \epsilon_n \end{aligned}$$

If $\lim_{n\to\infty} \epsilon_n = 0$ and since $0 \leq \beta < 1$, $\beta(t_n) \to 0$ and $\alpha_n, \gamma_n > 0$, it suffices from Lemma2 that $\lim_{n\to\infty} y_n = \omega$. Also, $G(x_{n+1}, (1-\alpha_n)x_n + \alpha_n Tz_n, (1-\alpha_n)x_n + \alpha_n Tz_n) = 0$. Then $\lim_{n\to\infty} x_n = \omega$ and hence we conclude that the sequence is stable with respect to T.

Theorem 2.12: Let (X, G) be a complete metric G-space and $T : X \to X$ be a self map with a fixed point ω satisfying Ciric generalised contraction for which $x, y, z \in X$, $0 \leq h < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa Iterative scheme. Then the sequence $\{x_n\}_{n=0}^{\infty}$ is T-Stable (or stable) with respect to T.

Proof. Assuming $y_{n+1} \neq \omega$, by Lemma 1, since ω is a fixed point, we define ϵ_n as $\epsilon_n = G(y_{n+1}, (1-\alpha_n)y_n + \alpha_n Ty_n, (1-\alpha_n)y_n + \alpha_n Ty_n)$

$$\begin{aligned}
G(y_{n+1}, \omega, \omega) &\leq G(y_{n+1}, (1-\alpha_n)y_n + \alpha_n Ts_n, (1-\alpha_n)y_n + \alpha_n Ts_n) \\
&+ G((1-\alpha_n)y_n + \alpha_n Ts_n, \omega, \omega) \\
&= G((1-\alpha_n)y_n + \alpha_n Ts_n, \omega, \omega) + \epsilon_n \\
&\leq G((1-\alpha_n)y_n, \omega, \omega) + G(\alpha_n Ts_n, \omega, \omega) + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n G(Ts_n, \omega, \omega) + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[h \max\{G(s_n, \omega, \omega), \frac{1}{2}[G(s_n, Ts_n, Ts_n) \\
&+ G(\omega, T\omega, T\omega)], + \frac{1}{2}[G(s_n, Ts_n, Ts_n) + G(\omega, Ts_n, Ts_n)]\}] + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n[h \max\{G(s_n, \omega, \omega), G(s_n, \omega, \omega)\}] + \epsilon_n \\
&= (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n hG(s_n, \omega, \omega) + \epsilon_n.
\end{aligned}$$

But $s_n = (1-\gamma_n)y_n + \gamma_n Ty_n$. Then

$$\begin{aligned}
G(y_{n+1}, \omega, \omega) &\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n hG((1-\gamma_n)y_n + \gamma_n Ty_n, \omega, \omega) + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n h[(1-\gamma_n)G(y_n, \omega, \omega) + \gamma_n G(Ty_n, \omega, \omega)] + \epsilon_n \\
&\leq (1-\alpha_n)G(y_n, \omega, \omega) + \alpha_n h[(1-\gamma_n)G(y_n, \omega, \omega) + 2\gamma_n G(y_n, \omega, \omega)] + \epsilon_n \\
&= [(1-\alpha_n) + \alpha_n h[(1-\gamma_n) + 2\gamma_n]]G(y_n, \omega, \omega)] + \epsilon_n
\end{aligned}$$

Suppose $\lim_{n\to\infty} \epsilon_n = 0$. Since $0 \leq h < 1$ and $\alpha_n, \gamma_n > 0$, it suffices to show, using Lemma 2, that $\lim_{n\to\infty} y_n = \omega$. As $G(x_{n+1}, (1-\alpha_n)x_n + \alpha_n Tz_n, (1-\alpha_n)x_n + \alpha_n Tz_n) = 0$, $\lim_{n\to\infty} x_n = \omega$ and $\{x_n\}$ is stable with respect to T.

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