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On Some properties of o-anti fuzzy subgroups

Umer Shuaib, Muhammad Shaheryar

Department of Mathematics
Government College University
Faisalabad, 38000, Pakistan

email: mumershuaib@gcuf.edu.pk, shaheryar3088@gmail.com

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Abstract

In this paper, we initiate the study of o-anti fuzzy subgroup and prove that every anti fuzzy subgroup is o-anti fuzzy subgroup. We introduce the notion of o-anti fuzzy cosets and establish their algebraic properties. We also define o-anti fuzzy normal subgroup and quotient group with respect to this particular group and prove some of it's various group theoretic properties.

1 Introduction

Zadeh [13] initiated the study of fuzzy set and since then there has been a fabulous concentration in this particular branch of mathematics due to its various applications ranging from computer science and engineering to the study of social and economic behaviors. For instance, the fuzzy sets expected for the progression of fuzzy signal controllers, and their individual participation capacities, are acknowledged as essential aspects practically equivalent to controller activities. Be that as it may, the effect of their definitions on movement signal work has not been enough dissected in the article. Our plan to add to vanquish this issue. It introduces a request of the finish of little changes in accordance with some fuzzy sets system for fuzzy signal controllers. Rosenfeld [6] initiated the study of algebraic

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structures on fuzzy sets by defining fuzzy groups. The recent developments about the applications of fuzzy sets in different algebraic structures may be viewed in [1,3,4,6,9,10,11,12,14,15]. Biswas [2] introduced the notion of anti fuzzy subgroup and established some fundamental features of this phenomenon. Onasanya [5] established many important anti fuzzy properties of fuzzy groups. Sharma [7] studied the normality of anti fuzzy subgroups

This paper is in continuation of the idea of omicron fuzzy subgroups defined in [8]. In this paper, we define a fuzzy set with respect to s-operator. We use this fuzzy subset to define o-anti fuzzy subgroup and establish o-anti fuzzy versions of some basic results of group theory. We also introduce the concepts of o-anti fuzzy cosets and o-anti fuzzy normal subgroups and establish the isomorphism between the quotient group with respect to o-anti fuzzy normal subgroup and quotient group with respect to the normal subgroup G_{A^o}. Moreover, we prove the homomorphic image (preimage) of an o-anti fuzzy subgroup is an o-anti fuzzy subgroup by using the classical homomorphism.

2 Preliminaries

In this section, we study some fundamental characterizations of anti fuzzy subgroup which play a key role in obtaining the basic group theoretic results in terms of their respective anti fuzzy versions. Some details of these concepts are given below which are very essential for our further discussion.

Definition (2.1) [13]: Let X be a nonempty set. A mapping

$$A : X \longrightarrow [0,\ 1]$$

is called a fuzzy subset of X.

Definition (2.2) [3]: Let A be a fuzzy subset of a universe X and $\delta \in [0, 1]$. The set $A^\delta = \{x \in X : A(x) \geq \delta\}$ is called level subset of a fuzzy set A.

Definition (2.3) [2]: Let A be a fuzzy subset of a group G. Then A is called a Anti fuzzy subgroup if

1. $A(xy) \leq \max\{A(x), A(y)\}$
2. $A(x^{-1}) \leq A(x)$, for all $x, y \in G$

It is easy to show that an Anti fuzzy subgroup of a group G satisfies $A(x)\geq A(e)$ and $A(x^{-1}) = A(x)$, for all $x\in G$, where e is the identity element of G.

Proposition (2.4) [2]: A fuzzy subset A of a group G is an Anti fuzzy subgroup of a group G if and only if $A(xy^{-1})\leq \max\{A(x), A(y)\}$, for all $x, y\in G$.

Theorem (2.5) [2]: Let G be a group and A be a fuzzy subset of G, then A is Anti fuzzy subgroup if and only if the level subset A^{δ} for $\delta\in$ [0, 1], $A(e)\leq\delta$, is a subgroup of G, where e is an identity of G.

Definition (2.6) [7]: An Anti fuzzy subgroup A of a group G is called Anti fuzzy normal subgroup if $A(xy) = A(yx)$, for all $x, y\in G$.

Definition (2.7) [7]: Let A be an Anti fuzzy subgroup of a group G. For any $x\in G$, the fuzzy set xA defined by $(xA)(y) = A(x^{-1}y)$, for all $y\in G$ is called a left Anti fuzzy coset of A. The right Anti fuzzy coset of A may be defined in the same way.

Definition (2.8) [7]: Let $f : G_1\longrightarrow G_2$ be a homomorphism from a group G_1 into a group G_2. Let A and B be fuzzy subsets of G_1 and G_2 respectively. Then $f(A)$ and $f^{-1}(B)$ are the image of fuzzy set A and the inverse image of fuzzy set B respectively for every $y\in G_2$ defined as

$$f(A)(y) = \begin{cases} \sup(A(x) : x\in f^{-1}(y)), & \text{if } f^{-1}(y)\neq\phi \\ 1, & \text{if } f^{-1}(y) = \phi \end{cases}$$

for every $x\in G_1, f^{-1}(B)(x) = Bf(x)$.

Remark (2.9) [7]: It is quite evident that a group homomorphism f admits the following characteristics:

1. $f\,(A)\,f\,(x) \geq A\,(x)$, for every element $x \in G_1$
2. If f is bijective map, then $f\,(A)\,f\,(x) = A\,(x)$, for all $x \in G_1$.

Definition (2.10) [11]: A function $s : [0, 1] \times [0, 1]\longrightarrow[0, 1]$ is said to be a t -conorm if and only if s admits following properties for all a, b, c, d in [0,1]

1. $s(a, b) = s(b, a)$
2. $s(a, s(b, c)) = s(s(a, b), c)$

3. $s(a,0) = s(0,a) = a$

4. If $a \leq c$ and $b \leq d$ then $s(a,b) \leq s(c,d)$

Definition (2.11) [11]: Let s_b: [0,1] $\times$ [0,1] $\rightarrow$ [0,1] be the bounded sum conorm defined by

$$s_b(a,b) = \min(a+b,1) \quad , \quad 0 \leq a \leq 1 \; , \; 0 \leq b \leq 1.$$

Clearly bounded sum conorm satisfies all the axioms of t-conorm.

3 o-Anti fuzzy subsets and their properties

Definition (3.1) Let A be a fuzzy subset of a set X and $\delta \in$ [0,1]. The fuzzy set A^o of X is called the o-Anti fuzzy subset of X (w.r.t fuzzy set A) and is defined as

$$A^o(x) = s_b(A(x), 1-\delta), \text{ for all } x \in X.$$

Remark (3.2): It is important to note that one can obtain the classical fuzzy subset $A(x)$ by choosing the value of $\delta = 1$ in above definition whereas the case become crisp for the choice of $\delta = 0$. These algebraic facts lead to note that the case illustrates the o-Anti fuzzy version with respect to any fuzzy subset for the value of δ, when $\delta \in$ (0, 1).

Theorem (3.3): Let A and B be any two fuzzy subsets of X. Then $(A \cup B)^o = A^o \cup B^o$.

Proof: In view of definition (3.1), we have

$$(A \cup B)^o(x) = s_b((A \cup B)(x), 1-\delta)$$

$$= s_b(\max(A(x), B(x)), 1-\delta)$$

$$= \max(s_b(A(x), B(x)), 1-\delta)$$

$$= \max(s_b(A(x), 1-\delta), s_b(B(x), 1-\delta)$$

$$= \max(A^o(x), B^o(x))$$

This implies that $(A \cup B)^o = A^o \cup B^o$.

4 o-anti fuzzy subgroups

In this section, we define the notion of o-anti fuzzy subgroup and o-anti fuzzy normal subgroup. We prove that every anti fuzzy subgroup (anti fuzzy normal subgroup) is also o-anti fuzzy subgroup (o-anti fuzzy normal subgroup) but converse need not to be true. The notion of o-anti fuzzy coset has also been defined and discussed deeply in this section. Moreover, in view of o-anti fuzzy normal subgroup, we apply this idea to introduce the concept of quotient group with respect to this particular anti fuzzy normal subgroup. This leads us to establish a natural homomorphism from a group G to its quotient group with respect to o-anti fuzzy normal subgroup. We also obtain the homomorphic image and pre-image of o-anti fuzzy subgroup (o-anti fuzzy normal subgroup). We conclude this section by establishing an isomorphism between the quotient group with respect to o-anti fuzzy normal subgroup and quotient group with respect to the normal subgroup G_{A^o}.

Definition (4.1): Let A be a fuzzy subset of a group G and $\delta \in [0, 1]$. Then A is called o-anti fuzzy subgroup of G.In other words A is o-anti fuzzy subgroup if A^o admits the following

1. $A^o(xy) \leq \max\{A^o(x), A^o(y)\}$
2. $A^o(x^{-1}) \leq A^o(x)$, for all $x, y \in G$.

Proposition (4.2): If A:$G \longrightarrow$ [0, 1] is an o-anti fuzzy subgroup of a group G, then

1. $A^o(x) \geq A^o(e)$, for all $x \in G$, where e is the identity element of G.
2. $A^o(xy^{-1}) = A^o(e)$ which implies that $A^o(x) = A^o(y)$, for all $x, y \in G$.

Proof: (i) $A^o(e) = A^o(xx^{-1}) \leq \max(A^o(x), A^o(x^{-1})) = \max(A^o(x), A^o(x)) = A^o(x)$.
Hence $A^o(e) \leq A^o(x)$, for all $x \in G$.
(ii) $A^o(x) = A^o(xy^{-1}y) \leq \max(A^o(xy^{-1}), A^o(y)) = \max(A^o(e), A^o(y)) = A^o(y)$
Hence $A^o(x) \leq A^o(y)$
Similarly $A^o(y) \leq A^o(x)$
This implies that $A^o(x) = A^o(y)$, for all $x, y \in G$.

In the following result, we establish a condition under which an o-anti fuzzy subset of a group G is an o-anti fuzzy subgroup.

Theorem (4.3): Let A^o be an o-anti fuzzy subset of a group G. Then A^o is o-anti fuzzy subgroup of G if and only if $A_{\mathrm{o}}^{\mathrm{t}}$ is subgroup of G for all $t \geq A^o(e)$.

Proof: It is quite obvious that A^o is non-empty. Since A^o be o-anti fuzzy subgroup of a group G, which implies that $A^o(x) \geq A^o(e)$, for all $x \in G$.

Let $x, y \in A_{\mathrm{o}}^{\mathrm{t}}$. Then $A^o(x) \leq t$ and $A^o(y) \leq t$.

Now $A^o(xy^{-1}) \leq \max(A^o(x), A^(y^{-1})) = \max(A^o(x), A^o(y)) \leq \max(t, t) = t$.
This implies that $xy^{-1} \in A_{\mathrm{o}}^{\mathrm{t}}$.
Hence $A_{\mathrm{o}}^{\mathrm{t}}$ is subgroup of G.
Conversely, suppose $A_{\mathrm{o}}^{\mathrm{t}}$ is subgroup of G, for all $t \geq A^o(e)$.
Let $x, y \in G$ and let $A^o(x) = a$, $A^o(y) = b$, where $a, b \in$ [0,1]
Let $c = \max(a, b)$, then $x, y \in A_{\mathrm{o}}^{\mathrm{c}}$, where $c \geq A^o(e)$
So, by assumption, $A_{\mathrm{o}}^{\mathrm{c}}$ is subgroup of G.
This implies that $xy^{-1} \in A_{\mathrm{o}}^{\mathrm{c}}$ and hence $A^o(xy^{-1}) \leq \max(A^o(x), A^o(y))$
Consequently, A^o is fuzzy subgroup of G. The following result leads to note that every anti fuzzy subgroup of a group G is an o-anti fuzzy subgroup of G.

Proposition (4.4): Every anti fuzzy subgroup of a group G is an o-anti fuzzy Subgroup of G.

Proof: Let A be anti fuzzy subgroup of a group G and let x, y be any two elements in G. Consider,

$$A^o(xy) = s_b(A(xy), 1 - \delta)$$

$$\leq s_b(\max(A(x), A(y)), 1 - \delta)$$

$$= \max(s_b(A(x), A(y)), 1 - \delta)$$

$$= \max(s_b(A(x), 1 - \delta), s_b(A(y), 1 - \delta))$$

$$A^o(xy) \leq \max(A^o(x), A^o(y))$$

Further $A^o(x^{-1}) = s_b(A(x^{-1}), 1 - \delta) = s_b(A(x), 1 - \delta) = A^o(x)$.
Consequently, A is o-anti fuzzy subgroup of G.

Remark (4.5): The converse of above proposition need not to be true.

Example (4.6): Let $G = \{e, a, b, ab\}$, where $a^2 = b^2 = e$ and $ab = ba$ be the Klein four group. Let the fuzzy set A of G be defined as

A=$\{< e, 0.1 >, < a, 0.3 >, < b, 0.4 >, < ab, 0.5 >\}$
Take $\delta = 0.05$ then
$A^o(x) = s_b(A(x), 1-\delta)$=min$(A(x)+1-\delta, 1)$=min$(A(x)+1-0.05, 1)$
$A^o(x) = 1$, for all $x \in G$.
This implies that
$A^o(xy) \leq \max(A^o(x), A^o(y))$
Further, we have $a^{-1} = a, b^{-1} = b$ and $(ab)^{-1} = ab$.
Hence we have $A^o(x^{-1}) = A^o(x)$, for all $x \in G$.
This implies that A is o-anti fuzzy subgroup of G.But clearly A is not anti fuzzy subgroup of G.

Proposition (4.7): The Union of two o-anti fuzzy subgroups of a group G is also o-anti fuzzy subgroup.

Proof: Let A and B be two o-anti fuzzy subgroups of a group G.
Consider, for all $x, y \in G$,

$$\begin{aligned}(A \cup B)^o(xy) &= (A^o \cup B^o)(xy)\\ &= \max(A^o(xy), B^o(xy))\\ &\leq \max(\max(A^o(x), A^o(y)), \max(B^o(x), B^o(y)))\\ &= \max(\max(A^o(x), B^o(x)), \max(A^o(y), B^o(y)))\\ &= \max((A \cup B)^o(x), (A \cup B)^o(y)).\end{aligned}$$

Thus $(A \cup B)^o(xy) \leq \max((A \cup B)^o(x), (A \cup B)^o(y))$
Moreover,

$$\begin{aligned}(A \cup B)^o(x^{-1}) &= (A^o \cup B^o)(x^{-1})\\ &= \max(A^o(x^{-1}), B^o(x^{-1}))\\ &= \max(A^o(x), B^o(x))\\ (A \cup B)^o(x^{-1}) &= (A \cup B)^o(x).\end{aligned}$$

Consequently, $(A \cup B)$ is o-anti fuzzy subgroup of G.

Corollary (4.8): The union of any finite number of o-anti fuzzy subgroups of a group G is also o-anti fuzzy subgroup of G.

Remark (4.9): The intersection of two o-anti fuzzy subgroups of a group G need not be o-anti fuzzy subgroup of G as the following example shows:

Example (4.10): Consider the group of integers Z. Define the two Fuzzy subsets A and B of Z as follows

$$A(x) = \begin{cases} 0.5, & \text{if } x = 3Z \\ 1, & \text{otherwise} \end{cases}$$

and

$$B(x) = \begin{cases} 0.8, & \text{if } x = 2Z \\ 0.83, & \text{otherwise} \end{cases}$$

It can be easily verified that A and B are o-anti fuzzy subgroups of Z.
Now, $(A \cap B)(x) = \min(A(x), B(x))$

Therefore, $(A \cap B)(x) = \begin{cases} 0.5 & \text{if } x \in 3z \\ 0.8 & \text{if } x \in 2z - 3z \\ 0.83 & \text{otherwise} \end{cases}$

take $x = 9$ and $y = 4$.
Then, $(A \cap B)(x) = 0.5$ and $(A \cap B)(y) = 0.8$.
But $(A \cap B)$ $(x - y)$=$(A \cap B)$ (9-4)=$(A \cap B)$ (5)= 0.83
And max $((A \cap B)(x), (A \cap B)(y))$ = max $(0.5, 0.8) = 0.8$
Clearly, $(A \cap B)$ $(x - y) >$ max $((A \cap B)(x), (A \cap B)(y))$ Consequently, $A \cap B$ is not o-anti fuzzy subgroup of G.
Hence, we see that, the intersection of two o-anti fuzzy subgroups of G need not be o-anti fuzzy Subgroup of G.

Definition (4.11): Let A be an o-anti fuzzy subgroup of a group G and $\delta \in$ [0,1]. The right o-anti fuzzy coset of A in G is denoted by $A^o x$ and is defined as $A^o x(g) = s_b(A(gx^{-1}), 1 - \delta)$, for all $x, y \in G$
Similarly, we define the o-anti fuzzy left coset xA^o of G as follows
$xA^o(g) = s_b(A(x^{-1}g), 1 - \delta)$, for all $x, y \in G$.

Definition (4.12): Let A be an o-anti fuzzy subgroup of a group G and $\delta \in$ [0,1]. Then A is called o-anti fuzzy normal subgroup of G if and only if $xA^o = A^o x$, for all $x \in G$.
The following result leads to note that every anti fuzzy normal subgroup of a group G is an o-anti fuzzy normal Subgroup of G.

Proposition (4.13): Every anti fuzzy normal subgroup of a group G is an o-anti fuzzy normal subgroup of G.

Proof: Let A be anti fuzzy normal subgroup of a group G. Then for any $x \in G$, we have $xA = Ax$, which implies that $xA(g) = Ax(g)$, for any $g \in G$.

Then we have
$A(x^{-1}g) = A(gx^{-1})$, which implies that $s_b(A(x^{-1}g), 1-\delta) = s_b(A(gx^{-1}), 1-\delta)$ Hence, $xA^o = A^o x$, for all $x \in G$.
Consequently, A is o-anti fuzzy normal subgroup of G. The converse of the above result need not to be true.

Example (4.14): Consider the dihedral group of degree 3 with finite presentation
$G = D_3 = < a, b : a^3 = b^2 = e, ba = a^2 b >$. Define the anti fuzzy subgroup of D_3 by

$$A(x) = \begin{cases} 0.1, & \text{if } x \in < b > \\ 0.2, & \text{otherwise} \end{cases}$$

Take $\delta = 0.6$, we have
$xA^o(g) = s_b(A(x^{-1}g), 1-\delta) = s_b(A(x^{-1}g), 0.6) = A^o x$
This shows that A is o-anti fuzzy normal subgroup of G.
$A(a^2(ab)) = A(a^3 b) = A(b) = 0.1$
$A((ab)a^2) = A(a(ba)a) = A(a(a^2 b)a) = A(a^3 ba) = A(ba) = 0.2$
This implies that A is not anti fuzzy normal subgroup of G.

Proposition (4.15): Let A be an o-anti fuzzy normal subgroup of a group G. Then $A^o(y^{-1}xy) = A^o(x)$ or equivalently, $A^o(xy) = A^o(yx)$, hold for all $x, y \in G$.

Proof: Since A be an o-anti fuzzy normal subgroup of a group G.
Therefore, $xA^o = A^o x$, holds for all $x \in G$
This implies that

$$xA^o(y^{-1}) = A^o x(y^{-1}), y \in G$$

.
In view of definition (4.10), the above relation becomes
$s_b(A(x^{-1}y^{-1}), 1-\delta) = s_b(A(y^{-1}x^{-1}), 1-\delta)$, which implies that, $A^o((yx)^{-1}) = A^o((xy)^{-1})$.
Consequently, $A^o(xy) = A^o(yx)$.

Definition (4.16): Let A be an o-anti fuzzy normal subgroup of a group G. We define a set
$G_{A^o} = \{x \in G : A^o(x) = A^o(e)\}$, where e is the identity element of G.

The following result illustrate that the set G_{A^o} is infect a normal subgroup of G.

Proposition (4.17): Let A be an o-anti fuzzy normal subgroup of a group G. Then G_{A^o} is a normal subgroup of G.

Proof: Obviously, $G_{A^o}{\neq}\phi$, for $e{\in}G_{A^o}$
Let $x, y{\in}G_{A^o}$ be any element. Then we have
$A^o(xy^{-1}){\leq}\max(A^o(x), A^o(y)){=}\max(A^o(e), A^o(e)) = A^o(e)$
This implies that
$A^o(xy^{-1}){\leq}A^o(e)$, but $A^o(xy^{-1}){\geq}A^o(e)$
Therefore $A^o(xy^{-1}) = A^o(e)$, which implies that $xy^{-1}{\in}G_{A^o}$
Hence G_{A^o} is a subgroup of G. Further,, let $x{\in}G_{A^o}$ and $y{\in}G$, we have
$A^o(y^{-1}xy) = A^o(x) = A^o(e)$.
This implies that $y^{-1}xy{\in}G_{A^o}$.
Consequently, G_{A^o} is normal subgroup of G.

Proposition (4.18): Let A be an o-anti fuzzy normal subgroup of G, then

1. $xA^o = yA^o$ if and only if $x^{-1}y{\in}G_{A^o}$
2. $A^ox = A^oy$ if and only if $xy^{-1}{\in}G_{A^o}$.

Proof: (i) suppose that $xA^o = yA^o$, for $x, y{\in}G$.In view of definition (3.1), the above relation yields that

$$A^o(x^{-1}y) = s_b(A(x^{-1}y), 1-\delta)$$
$$= (xA^o)(y)$$
$$= (yA^o)(y)$$
$$= s_b(A(y^{-1}y), 1-\delta)$$
$$= s_b(A(e), 1-\delta)$$
$$= A^o(e).$$

This implies that $x^{-1}y{\in}G_{A^o}$
Conversely, let $x^{-1}y{\in}G_{A^o}$, which implies that $A^o(x^{-1}y) = A^o(e)$.
For any element $z{\in}G_{A^o}$

$$(xA^o)(z) = s_b(A(x^{-1}z), 1-\delta)$$

$$
\begin{aligned}
&= A^o(x^{-1}z) \\
&= A^o((x^{-1}y)(y^{-1}z)) \\
&\leq \max(A^o((x^{-1}y)), A^o(y^{-1}z)) \\
&= \max(A^o(e), A^o(y^{-1}z)) \\
&= A^o(y^{-1}z) \\
&= (yA^o)(z)
\end{aligned}
$$

Interchanging the roles of x and y, we get $(xA^o)(z) = (yA^o)(z)$, for all $z \in G$.
Consequently, $(xA^o) = (yA^o)$.
(ii) One can prove this part analogous to (i).

Proposition (4.19): Let A be an o-anti fuzzy normal subgroup of a group G and x, y, u, v be any element in G. If$xA^o = uA^o$ and $yA^o = vA^o$ then $xyA^o = uvA^o$.

Proof: Given that $xA^o = uA^o$ and $yA^o = vA^o$, which implies that $x^{-1}u$ and $y^{-1}v \in G_{A^o}$
Consider, $(xy)^{-1}uv = y^{-1}(x^{-1}u)(yy^{-1})v = [y^{-1}(x^{-1}u)y](y^{-1})v) \in G_{A^o}$
This implies that $(xy)^{-1}uv \in G_{A^o}$
Consequently, $xyA^o = uvA^o$.

Definition (4.20): Let A be an o-anti fuzzy normal subgroup of a group G. The set of all o-anti fuzzy cosets of A denoted by G/A^o forms a group under the binary operation * defined as follow.
Let $xA^o, yA^o \in G/A^o, xA^o * yA^o = (x * y)A^o)$, for $x, y \in G$. This group is called the factor group or the quotient group of G with respect to o-anti fuzzy normal subgroup A^o.

Theorem (4.21): The set G/A^o defined in definition (4.18) forms a group under the above stated binary operation *.

Proof: Let $A^o x_1 = A^o x_2$ and $A^o y_1 = A^o y_2$, for $x_1, x_2, y_1, y_2 \in G$.
Let $g \in G$ be any element of G.
$[A^o x_1 * A^o y_1](g) = (A^o x_1 y_1)(g) = s_b(A(g(x_1 y_1)^{-1}), 1-\delta) = s_b(A(gy_1^{-1}x_1^{-1}), 1-\delta)$
$= s_b(A(gy_1^{-1})x_1^{-1}, 1-\delta) = A^o x_1(gy_1^{-1}) = A^o x_2(gy_1^{-1}) = s_b(A(gy_1^{-1})x_2^{-1}, 1-\delta)$
$= s_b(A(x_2^{-1}g)y_1^{-1}, 1-\delta) = A^o y_1(x_2^{-1}g) = A^o y_2(x_2^{-1}g) = s_b(A(x_2^{-1}g)y_2^{-1}, 1-\delta)$
$= s_b(A(y_2^{-1}x_2^{-1})g, 1-\delta) = s_b(A(x_2 y_2)^{-1}g, 1-\delta) = s_b(Ag(x_2 y_2)^{-1}, 1-\delta) = (A^o x_2 y_2)(g)$.

This implies that * is well defined.
Obviously, the set G/A^o admits closure and associative properties with respect to the binary operation $*$.
Moreover, $A^o * xA^o = eA^o * xA^o) = (e * x)A^o = xA^o$, which implies that A^o is identity of G/A^o.
It is easy to note that inverse of each element of G/A^o exist as if for $xA^o \in G/A^o$, there exist $x^{-1}A^o \in G/A^o$ such that
$(x^{-1}A^o) * (xA^o) = (x^{-1} * x)A^o = A^o$.
Consequently, (G/A^o) is a group under *.

Theorem (4.22): Let A^o be an o-anti fuzzy normal subgroup of a group G. Then there exists a natural epimorphism between G and G/A^o which may be defined as $x \longmapsto A^o x, x \in G$, where G_{A^o} is the kernel of this homomorphism.

Proof: f is homomorphism as if for $x, y \in G$, we have
$f(xy) = A^o xy = A^o x A^o y = f(x)f(y)$.
Obviously f is surjective as well.
Consequently, f is an epimorphism from G to G/A^o.
Further, $kerf = \{x \in G : f(x) = A^o e\} = \{x \in G : A^o x = A^o e\} = \{x \in G : xe^{-1} \in G_{A^o}\} = \{x \in G : x \in G_{A^o}\} = G_{A^o}$.

Theorem (4.23): Let A^o be an o-anti fuzzy normal subgroup of a group G. Then show that $G/A^o \cong G/G_{A^o}$.

Proof: In view of definition (4.15), G/G_{A^o} is well defined.
Define a map $f : G/A^o \rightarrow G/G_{A^o}$ by the rule

$$f(xA^o) = xG_{A^o}, x \in G$$

f is well defined because if $xA^o = yA^o$, which implies that $xG_{A^o} = yG_{A^o}$.
This implies that $f(xA^o) = f(yA^o)$
f is injective as if $f(xA^o) = f(yA^o)$, which implies that $xG_{A^o} = yG_{A^o}$. Hence, $xA^o = yA^o$.f is surjective as for each $xG_{A^o} \in G/G_{A^o}$, there exist $xA^o \in G/A^o$ such that $f(xA^o) = xG_{A^o}$
f is homomorphism as for each $xA^o, yA^o \in G/A^o$
$f(xA^o yA^o) = f((xy)A^o) = xyG_{A^o} = xG_{A^o} yG_{A^o} = f(xA^o)f(yA^o)$.
Consequently, there is an isomorphism between G/A^o and G/G_{A^o}.

5 Homomorphism of o-fuzzy subgroups

Theorem (5.1): Let $f{:}G_1{\longrightarrow}G_2$ be a bijective homomorphism from a group G_1 to a group G_2 and B be an o-anti fuzzy subgroup of group G_2. Then $f^{-1}(B)$ is an o-anti fuzzy subgroup of group G_1.

Proof: Given that B is an o-anti fuzzy subgroup of group G_2. Let $x_1, x_2{\in}G_1$ be any element. Then

$$\begin{aligned}
(f^{-1}(B))^o(x_1x_2) &= f^{-1}(B^o)(x_1x_2)\\
&= B^o(f(x_1x_2))\\
&= B^o(f(x_1)f(x_2))\\
&\leq \max\{B^o(f(x_1)), B^o(f(x_2))\}\\
&= \max\{f^{-1}(B^o)(x_1), f^{-1}(B^o)(x_2)\}\\
&= \max\{(f^{-1}(B))^o(x_1), (f^{-1}(B))^o(x_2)\}
\end{aligned}$$

Thus,

$$(f^{-1}(B))^o(x_1x_2) \leq \max\{(f^{-1}(B))^o(x_1), (f^{-1}(B))^o(x_2)\}$$

Also,

$$\begin{aligned}
(f^{-1}(B))^o(x^{-1}) &= f^{-1}(B^o)(x^{-1})\\
&= B^o(f(x^{-1}))\\
&= B^o(f(x)^{-1})\\
&= B^o(f(x))\\
&= f^{-1}(B^o)(x)
\end{aligned}$$

Consequently, $f^{-1}(B)$ is o-anti fuzzy subgroup of group G_1.

Theorem (5.2): Let $f{:}G_1{\longrightarrow}G_2$ be a isomorphism from a group G_1 to a group G_2 and B be an o-anti fuzzy normal subgroup of group G_2.Then $f^{-1}(B)$ is an o-anti fuzzy normal subgroup of group G_1.
Proof: Given that B is an o-anti fuzzy normal subgroup of group G_2. Let $x_1, x_2{\in}G_1$ be any element. Then

$$\begin{aligned}
(f^{-1}(B))^o(x_1x_2) &= f^{-1}(B^o)(x_1x_2)\\
&= B^o(f(x_1x_2))
\end{aligned}$$

$$= B^o(f(x_1)f(x_2))$$

$$= B^o(f(x_2)f(x_1))$$

$$= B^o(f(x_2x_1))$$

$$(f^{-1}(B))^o(x_1x_2) = f^{-1}(B^o)(x_2x_1)$$

Consequently, $f^{-1}(B)$ is o-anti fuzzy normal subgroup of group G_1.

Theorem (5.3): Let $f{:}G_1{\longrightarrow}G_2$ be a bijective homomorphism from a group G_1 to a group G_2 and A be an o-anti fuzzy subgroup of group G_1. Then $f(A)$ is an o-anti fuzzy subgroup of group G_2.

Proof: Given that A is an o-anti fuzzy subgroup of group G_1. Let $y_1, y_2{\in}G_2$ be any element. Then there exists unique element $x_1, x_2{\in}G_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$

consider,

$$(f(A))^o(y_1y_2) = s_b(f(A)(y_1y_2), 1 - \delta)$$

$$= s_b(f(A)f(x_1)f(x_2), 1 - \delta)$$

$$= s_b(f(A)f(x_1x_2), 1 - \delta)$$

$$= s_b(A(x_1x_2), 1 - \delta)$$

$$= A^o(x_1x_2)$$

$$\leq \max(A^o(x_1), A^o(x_2), \text{for all} x_1, x_2{\in}G_1$$

$$\leq \max(\min\{A^o(x_1) : f(x_1) = y_1\}, \min\{A^o(x_2) : f(x_2) = y_2\})$$

$$= \max(f(A^o)(y_1), f(A^o)(y_2))$$

$$= \max((f(A))^o(y_1), (f(A))^o(y_2))$$

Further,

$$(f(A))^o(y^{-1}) = f(A^o)(y^{-1})$$

$$= \min\{A^o(x^{-1}) : f(x^{-1}) = y^{-1}\}$$

$$= \min\{A^o(x) : f(x) = y\}$$

$$= (f(A))^o(y)$$

Consequently, $f(A)$ is o-anti fuzzy subgroup of G_2.

Theorem (5.4): Let $f{:}G_1{\longrightarrow}G_2$ be a bijective homomorphism from a group G_1 to a group G_2 and A be an o-anti fuzzy normal subgroup of group G_1.Then $f(A)$ is an o-anti fuzzy normal subgroup of group G_2.

Proof: In view of theorem (5.3), it is sufficient to show that $f(A^o)$ is anti fuzzy normal in G_2.
Given that A is o-anti fuzzy normal subgroup of group G_1. Let $y_1, y_2{\in}G_2$ be any element.
Then there exists unique elements $x_1, x_2{\in}G_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$
consider,

$$(f(A))^o(y_1y_2) = s_b(f(A)(y_1y_2), 1-\delta)$$
$$= s_b(f(A)f(x_1)f(x_2), 1-\delta)$$
$$= s_b(f(A)f(x_1x_2), 1-\delta)$$
$$= s_b(A(x_1x_2), 1-\delta)$$
$$= A^o(x_1x_2) = A^o(x_2x_1)$$
$$= s_b(A(x_2x_1), 1-\delta)$$
$$= s_b(f(A)f(x_2x_1), 1-\delta)$$
$$= s_b(f(A)f(x_2)f(x_1), 1-\delta)$$
$$= s_b(f(A)(y_2y_1), 1-\delta)$$
$$= (f(A))^o(y_2y_1).$$

Consequently, $f(A)$ is o-anti fuzzy normal subgroup of G_2.

6 Conclusion

In this paper, we have introduced the concept of o-anti fuzzy subgroup and o-anti fuzzy coset of a given group and have used them to introduce the concept of o-anti fuzzy normal subgroup and have discussed various related properties. We have also studied the effect on the image and inverse image of o-anti fuzzy subgroup (normal subgroup) under group homomorphism.

In subsequent studies, we shall extend this idea to intuitionistic fuzzy sets and will investigate its various algebraic properties.

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