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Friends of 15 Live Far Away

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Abstract

Any positive integer n other than 15 with abundancy index 8/5 must have at least five distinct prime factors, the smallest factor necessarily being 5. Further, any prime factor of n that is congruent to 2 modulo 3 must be raised to an even power in n and any prime factor of n that is congruent to 1 modulo 3 must be raised to a power congruent to 0 or 1 modulo 3.

Friendly numbers and the necessary conditions for n to have a friend have been a topic of study in the past by students at the Auburn REU in Mathematics. Their work has led to publications, in particular [1] on friends of 10 and [4] on friends of 12, while this paper investigates friends of 15 and introduces new methods to the study of this problem based on exponent congruences.

1 The Abundancy Index

Definition 2. Let n be a positive integer. The abundancy index $I(n) = \frac{\sigma(n)}{n}$ is the quotient of the divisor function, $\sigma(n) = \sum_{d>0, d|n} d$, and n.

The abundancy index has several properties which facilitate our study of it (see [1] for more background and [2] and [3] for proofs):

Let m and n be positive integers and consider only positive primes p.

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1. $I(n) \geq 1$, with equality if and only if $n = 1$.
2. If $m \mid n$ then $I(m) \leq I(n)$ with equality if and only if $m = n$.
3. If $p_1, \ldots, p_k$ are distinct primes and $e_1, \ldots, e_k$ are positive integers then
$$I\left(\prod_{j=1}^{k} p_j^{e_j}\right) = \prod_{j=1}^{k}\left(\sum_{i=0}^{e_j} p_j^{-i}\right) = \prod_{j=1}^{k} \frac{p_j^{e_j+1}-1}{p_j^{e_j}(p_j-1)}.$$
This follows from the analogous relation for σ:
$$\sigma\left(\prod_{j=1}^{k} p_j^{e_j}\right) = \prod_{j=1}^{k}\left(\sum_{i=0}^{e_j} p_j^{i}\right) = \prod_{j=1}^{k} \frac{p_j^{e_j+1}-1}{p_j-1}.$$
Note that the factor $p_j - 1$ is used entirely for notational convenience, and as such, $p_j - 1$ is not actually a factor of $\sigma(p_j^{e_j})$.
Property (3) implies a property of I shared by σ:
4. I is weakly multiplicative (i.e., if $\gcd(m, n) = 1$, then $I(mn) = I(m)I(n)$).
5. Suppose that $p_1, \ldots, p_k$ are distinct primes, $q_1, \ldots, q_k$ are distinct primes, $e_1, \ldots, e_k$ are positive integers, and for all $j \in \{1, \ldots, k\}$, $p_j \leq q_j$. Then
$$I\left(\prod_{j=1}^{k} p_j^{e_j}\right) \geq I\left(\prod_{j=1}^{k} q_j^{e_j}\right)$$
with equality if and only if $p_j = q_j$, for all $j \in \{1, \ldots, k\}$. This follows from (3) and the observation that if $e \geq 1$, then $\frac{x^{e+1}-1}{x^e(x-1)}$ is a decreasing function of x on $(1, \infty)$.
6. If the distinct prime factors of n are $p_1, \ldots, p_k$, then
$$\prod_{j=1}^{k} \frac{p_j+1}{p_j} \leq I(n) < \prod_{j=1}^{k} \frac{p_j}{p_j-1}.$$
The first inequality follows from (3) with all powers set equal to 1. Although related to (5), the latter half of this property is most easily seen by applying (3) and the observation that for $p > 1$,
$$\frac{p^{e+1}-1}{p^{e+1}-p^e} = \frac{p - \frac{1}{p^e}}{p-1}$$
increases to $\frac{p}{p-1}$ as $e \to \infty$.

Definition 3. Positive integers m and n are friends if $m \neq n$ and $I(m) = I(n)$.

The following lemma will prove useful later:

Lemma 3.1. *Since all elements of $(\mathbb{Z}/p\mathbb{Z})^\times$ are roots of unity modulo p, for all elements $x \in \mathbb{Z}/p\mathbb{Z}$ there exists some minimal k such that*

$$\sum_{j=0}^{k-1} x^j \equiv 0 \pmod{p}.$$

Thus for all primes $q \neq p$, $\sigma(q^{k-1}) = \sum_{j=0}^{k-1} q^j \equiv 0 \pmod{p}$ for some k.

Proof. Lemma 3.1 is a well-known property of roots of unity modulo n. It follows from the observation that

$$(x-1)\sum_{j=0}^{p-1} x^j = x^p - 1$$

and for $x \not\equiv 1 \pmod{p}$ this implies that $\sum_{j=0}^{p-1} x^j \equiv 0 \pmod{p}$, while for $x \equiv 1 \pmod{p}$, $\sum_{j=0}^{p-1} x^j \equiv 0 \pmod{p}$ as well.

So for any distinct primes p, q, for some power k, $p \mid \sigma(q^{k-1})$. In fact, since

$$\sigma(q^{k\ell-1}) = \sum_{i=0}^{k\ell-1} q^i = \left(\sum_{i=0}^{\ell-1} q^{ki}\right)\left(\sum_{j=0}^{k-1} q^j\right), \tag{3.1}$$

the corollary below follows:

Corollary 3.1. *If $p \mid \sigma(q^{k-1})$, then $p \mid \sigma(q^{k\ell-1})$ for all positive ℓ (here, $k-1$ and $k\ell-1$ are used as upper limits of the sums so that we sum a total of k and $k\ell$ terms respectively).*

This means that if, in an attempt to cancel a factor of p from the denominator of $I(n)$ using $p \mid \sigma(q^{k-1})$, we introduce a factor of pr to the numerator, then we must also account for the factor of r, even if we choose to use a higher power of q such as $\sigma(q^{k\ell-1})$ instead.

Cancellation between factors in the numerator and the denominator of $I(n)$ is necessary for a number to have friends:

Lemma 3.2. *If* $\gcd(\sigma(n), n) = 1$ *then* n *has no friends.*

Proof. Assuming that n has a friend m, then $\frac{\sigma(n)}{n} = I(n) = I(m) = \frac{\sigma(m)}{m}$ implies that $n\sigma(m) = m\sigma(n)$. But $m, n, \sigma(m), \sigma(n)$ are positive integers, and $\gcd(n, \sigma(n)) = 1$. Since $\sigma(n) \geq n$, this implies that $n \mid m$. By Property (2), $n \mid m$ implies that $I(m) \geq I(n)$ with equality if and only if $n = m$. So there is no other number $m \neq n$ with $I(m) = I(n)$, so n has no friends.

4 Friends of 15

Theorem 5. *If* n *is a friend of 15, then* n *has at least* $k \geq 5$ *distinct prime factors, the smallest being 5.*

Proof. Note that $I(15) = \frac{1+3+5+15}{15} = \frac{8}{5}$. Let n be a friend of 15, and write $n = \prod_{i=1}^{k} p_i^{e_i}$ for some distinct primes $p_1, \ldots, p_k$ and some positive $e_1, \ldots, e_k$.

Claim 5.1. n is a multiple of 5 but not of 2 or of 3.

Proof of Claim 5.1. Since $I(15) = \frac{1+3+5+15}{15} = \frac{8}{5}, 5\sigma(n) = 8n$ and so $5 \mid n$. If 2 divided n, then by Property 4, $I(n) \geq I(10) = \frac{9}{5} > I(15) = I(n)$, which is a contradiction. Similarly, if $3, 5 \mid n$, then $I(n) \geq I(15)$ with equality only when $n = 15$, so 3 does not divide n.

Claim 5.2. 3 does not divide $\sigma(n)$.

Proof of Claim 5.2. Since $5\sigma(n) = 8n$ and $n, \sigma(n)$ are integers, the fact that $3 \nmid n$ implies that $3 \nmid \sigma(n)$.

Claim 5.3. $k > 2$.

Proof of Claim 5.3. Suppose that n has at most two distinct prime factors. Then by Claim 7.1, $n = 5^i p^j$ for a prime $p > 5$ and $i, j \geq 0$. Then Properties (5) and (6) imply that $I(n) = I(5^i)I(p^j) \leq I(5^i)I(7^j) < \frac{5}{5-1} \cdot \frac{7}{7-1} = \frac{5}{4} \cdot \frac{7}{6} < \frac{8}{5}$.

Claim 5.4. If $p_i \equiv 2 \pmod 3$ then $2 \mid e_i$. If $p_i \equiv 1 \pmod 3$ then $e_i \not\equiv 2 \pmod 3$. Also, n has at least one prime factor $p_i \equiv 1 \pmod 3$ such that $e_i \equiv 1 \pmod 2$. Furthermore, since $8 = 2^3$, at least one such p_i, but no more than three such p_i, must have an odd power in the factorization of n.

Proof of Claim 5.4. Let $p_i \equiv 2 \mod 3$. Suppose that e_i is odd. Then

$$\sigma(p_i^{e_i}) = \sum_{\ell=0}^{e_i} p_i^{\ell} \equiv 1 + 2 + 1 + 2 + \cdots + 1 + 2 \mod 3 \equiv 0.$$

That would imply that $3 \mid \sigma(p_i^{e_i})$, which contradicts Claim 7.2. So necessarily $2 \mid e_i$.

Similarly, if $p_i \equiv 1 \mod 3$, then $e_i \not\equiv 2 \mod 3$.

Note that if $p_i \equiv 2 \mod 3$, then $2 \mid e_i$, which implies that $\sigma(p_i^{e_i})$ is odd. So since $8 \mid \sigma(n)$, in order for $\sigma(p_i^{e_i})$ to be even n must have at least one prime factor $p_i \equiv 1 \mod 3$ which is raised to an odd power e_i in the factorization of n. In order to prevent $16 \mid \sigma(n)$, no more than three prime factors may have e_i odd.

Claim 5.5. $k > 3$.

Proof of Claim 5.5. By Claim 7.3 we know that $k \geq 3$. Suppose that $k = 3$ and write $n = 5^{e_5} p^{e_p} q^{e_q}$ for some primes $5 < p < q$ and positive powers $e_5, e_p, e_q > 0$. By Properties (4), (5), and (6), if $q > 11$ then

$$I(n) \leq I(5^{e_5} p^{e_p} q^{e_q}) \leq I(5^{e_5} 7^{e_7} 13^{e_{13}}) < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{13}{12} < \frac{8}{5},$$

which contradicts the hypothesis that $I(n) = \frac{8}{5}$. Thus necessarily $q = 11$ and $p = 7$, so we have $n = 5^{e_5} 7^{e_7} 11^{e_{11}}$. What's more, since 5 and 11 are congruent to $2 \mod 3$, by Claim 7.4, both e_5 and e_{11} are even. So by Claim 7.3 in particular, $e_5, e_{11} \geq 2$. By Properties (5) and (6), $I(5^2 7^{e_7} 11^{e_{11}}) < \frac{31}{25} \cdot \frac{7}{6} \cdot \frac{11}{10} < \frac{8}{5}$, so necessarily $e_5 \geq 4$. Similarly, $I(5^{e_5} 7^1 11^{e_{11}}) < \frac{5}{4} \cdot \frac{8}{7} \cdot \frac{11}{10} < \frac{8}{5}$ implies that $e_7 > 1$. Finally, since $\sigma(5^{e_5})$ and $\sigma(11^{e_{11}})$ are both odd, Claim 7.4 implies that $\sigma(7^{e_7})$ must be a multiple of 8 but not of 16 since $8 \mid \sigma(n)$ but $16 \nmid \sigma(n)$. Recall that e_7 must be odd. However, if $e_7 \equiv 3 \pmod 4$ then for some $k \geq 1$

$$\sigma(7^{e_7}) = \sum_{i=0}^{4k-1} 7^i \equiv (1 + 7 + 1 + 7)k \pmod{16} \equiv 0 \pmod{16}.$$

So $e_7 \not\equiv \{2, 3, 4\} \pmod 4$, which leaves only $e_7 \equiv 1 \pmod 4$. Recall that $e_7 \not\equiv 2 \pmod 3$, and the congruence $e_7 \pmod{12} \equiv \{1, 9\}$ follows. Since $e_7 \neq 1$, this implies that $e_7 \geq 9$ and so $5^4 7^9 11^2 \mid n$, but $I(5^4 7^9 11^2) = \frac{3876 \cdot (7^{10} - 1) \cdot 133}{3125 \cdot 6 \cdot 7^9 \cdot 121} > \frac{8}{5}$. So $k = 3$ is impossible.

Claim 5.6. If $p_3 \neq 11$, then $k > 4$.

Proof of Claim 5.6. If $k = 4$ then $n = 5^{e_5} \cdot \prod_{i=2}^{4} p_i^{e_i}$. Note that if

$$p_2 \geq 11 \text{ then by Properties (4), (5), and (6), } I(n) < \frac{5}{4} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} < \frac{8}{5}.$$

This implies that $p_2 = 7$, so n is of the form $5^{e_5} 7^{e_7} p^{e_p} q^{e_q}$ such that $I(n) = \frac{8}{5}$. Also, since if $p \geq 23$, then $I(n) \leq I(5^{e_5} 7^{e_7} 23^{e_{23}} 29^{e_{29}}) < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{23}{22} \cdot \frac{29}{28} < \frac{8}{5}$, Properties (5) and (6) imply that $p \in \{11, 13, 17, 19\}$. The case $p_3 = 11$ will be left for claim 7.7. Let us consider the finitely many pairs of primes p, q with $11 < p < q$ such that $I(5^{e_5}7^{e_7}p^{e_p}q^{e_q}) \geq 8/5$ for sufficiently large values of e_5, e_7, e_p, e_q. Recall from the previous claim that $p \in \{13, 17, 19\}$. We find that these pairs using a simple loop written in Sage that finds all primes q such that

$$I\left(5^{e_5}7^{e_7}p^{e_p}q^{e_q}\right) < \frac{5}{5-1} \cdot \frac{7}{7-1} \cdot \frac{p}{p-1} \cdot \frac{q}{q-1} \not< 8/5. \tag{5.2}$$

Using Property (6), this implies that if $n = 5^{e_5}7^{e_7}p^{e_p}q^{e_q}$ then it is possible that $I(n) \not< \frac{8}{5}$, which is certainly necessary for $I(n) = \frac{8}{5}$ as desired. The loop generated the contents of the following table:

$p = 13$	$q \in \{17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79\}$
$p = 17$	$q \in \{19, 23, 29, 31\}$
$p = 19$	$q = 23$

We check that for $q > p$ not given in the lists above,

$$I(5^a7^b13^cq^d) \leq I(5^a7^b13^c83^d) < \frac{5 \cdot 7 \cdot 13 \cdot 83}{4 \cdot 6 \cdot 12 \cdot 82} = \frac{37765}{23616} < 8/5$$
$$I(5^a7^b17^cq^d) \leq I(5^a7^b17^c37^d) < \frac{5 \cdot 7 \cdot 17 \cdot 37}{4 \cdot 6 \cdot 16 \cdot 36} = \frac{22015}{13824} < 8/5$$
$$I(5^a7^b19^cq^d) \leq I(5^a7^b19^c29^d) < \frac{5 \cdot 7 \cdot 19 \cdot 29}{4 \cdot 6 \cdot 18 \cdot 28} = \frac{2755}{1728} < 8/5$$

This confirms that our loop found all of the finitely many pairs of primes $\{p, q\}$ with $q > p > 11$ such that $n = 5^{e_5}7^{e_7}p^{e_p}q^{e_q}$ can possibly achieve abundancy index at least 8/5, and that the 21 pairs listed above are an exhaustive list of such pairs.

Now we find the minimal a_i such that

$$I(p_i^{a_i}) \cdot \prod_{j \neq i} \frac{p_j}{p_j - 1} > 8/5 \tag{5.3}$$

which is a necessary condition to have $I(n) = 8/5$. That is, we find the minimum powers a_5, a_7, a_p, a_q that each individual prime $5, 7, p, q$ needs, in order to have $I(5^{e_5}7^{e_7}p^{e_p}q^{e_q}) \geq 8/5$ with arbitrarily high powers of the other three primes. This makes

$$I(5^{a_5}7^{a_7}p^{a_p}q^{a_q}) > 8/5$$

for 11 of the pairs p, q. This eliminates all but the following pairs, which are given in the chart below along with the minimum exponents a_5, a_7, a_p, a_q of each prime, as sought above:

p	13	13	13	13	13	13	13	13	17	17
q	23	29	37	41	43	47	53	79	19	31
a_5	2	2	2	2	2	2	4	6	2	4
a_7	1	1	3	3	3	3	3	4	1	3
a_p	1	1	1	1	1	1	1	3	2	2
a_q	2	2	1	2	1	2	2	3	1	3

But for most of these pairs if we increase a_5 by 2 then $I(5^{a_5+2}7^{a_7}p^{a_p}q^{a_q}) > 8/5$. Recall that a_5 must be even, so 2 is the minimum amount that can be added. Since $I(5^2) = \frac{31}{25}$ and $I(5^4) = \frac{781}{625} = \frac{71 \cdot 11}{625}$, this would imply that (if $a_5 = 2$) 31 must be a factor of n or that (if $a_5 = 4$) 11 and 71 must be factors of n, a contradiction. In fact, this occurs for all of the above pairs except for $\{13, 29\}, \{13, 79\}, \{17, 19\}$.

At this point, I introduce a new technique that will prove useful later. The technique is based on Lemma 3.1 and Corollary 3.1.

Remark 7. Lemma 3.1 says that $p_j \mid \sigma(p_i^{k-1})$ for some positive k, so we know that for any $i \neq j$, each prime p_i can contribute a factor p_j to the numerator of $I(n)$. Furthermore, since we are assuming that $I(n) = \frac{8}{5}$, for all $p_j \mid n$ (including 5, since $e_5 \geq 2$) we must have at least one factor of p_j in the numerator of $I(n)$, i.e. $p_j \mid \sigma(n)$, to cancel out the superfluous factor(s) of p_j in n. So at least one power of the largest prime involved, q, must divide $\sigma(n)$. However, recalling the discussion following Corollary 3.1, any factor of $\sigma(p'^{k-1})$ is also a factor of $\sigma(p'^{k\ell-1})$. So, if we must introduce another prime factor q' to $\sigma(n)$ in order to cancel the factor of q in n, then we have derived a contradiction.

The reader may verify the following facts:

$$29 \mid (5^{14} - 1); 29 \mid (7^7 - 1); 29 \mid (13^{14} - 1);$$
$$79 \mid (5^{39} - 1); 79 \mid (7^{78} - 1); 79 \mid (13^{39} - 1);$$
$$19 \mid (5^9 - 1); 19 \mid (7^3 - 1); 19 \mid (17^9 - 1).$$

An equivalent statement of these facts is

$$5^{14} \equiv 7^7 \equiv 13^{14} \equiv 1 \pmod{29};$$
$$5^{39} \equiv 7^{78} \equiv 13^{39} \equiv 1 \pmod{79};$$
$$5^9 \equiv 7^3 \equiv 17^9 \equiv 1 \pmod{19}.$$

Furthermore, the reader may verify that the exponents above are the minimal

ℓ for which these primes are ℓ-th roots of unity. Now, note that

$$3 \mid (5^{14}-1); 3 \mid (7^{7}-1); 3 \mid (13^{14}-1);$$
$$31 \mid (5^{39}-1); 3 \mid (7^{78}-1); 3 \mid (13^{39}-1);$$
$$31 \mid (5^{9}-1); 3 \mid (7^{3}-1); 307 \mid (17^{9}-1).$$

None of the primes above (3, 31, and 307) are among the factors $p' \in \{5, 7, p, q\}$ for $\{p, q\} \in \{\{13, 29\}, \{13, 79\}, \{17, 19\}\}$. So for every prime $p' \in \{5, 7, p, q\}$, if $q \mid \sigma(p'^e)$ then by Corollary 3.1 there is some $q' \notin \{5, 7, p, q\}$ such that $q' \mid \sigma(p'^e)$ as well, which contradicts the assumption that for $n = 5^{e_5}7^{e_7}p^{e_p}q^{e_q}$, $I(n) = \frac{8}{5}$. Thus if n has 4 distinct prime factors, then one of those 4 primes must be 11.

Claim 7.7. $k > 4$. It only remains to address the case that $p = 11$.

Proof of Claim 7.7. First, note that by Claim 5.4 and Property (6), $n = 5^{e_5}7^{e_7}11^{e_{11}}q^{e_q}$ has abundancy index bounded by

$$\frac{31}{25} \cdot \frac{8}{7} \cdot \frac{133}{121} \cdot \frac{(q^{e_q+1}-1)}{q^{e_q}(q-1)} \leq I(n) < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{q}{q-1}.$$

The former inequality indicates that $q \geq 37$ since if $q \leq 31$, then $\frac{31}{25} \cdot \frac{8}{7} \cdot \frac{133}{121} \cdot \frac{32}{31} > \frac{8}{5}$. However, $q = 37$ cannot lead to $I(n) = \frac{8}{5}$ because the minimal values, given the congruences modulo 3 established in Claim 5.4, $e_5 = e_{11} = 2, e_7 = e_{37} = 1$, would make $I(n) > \frac{8}{5}$. Thus $q \geq 41$. To proceed further, we take advantage of Corollary 3.1 and Remark 7:

Note that $p \geq 41$ was shown above. Thus if $e_5 = 2$ then we have introduced a factor of 31, a contradiction. Corollary 3.1 implies the following:

$7, 31 \mid (5^6 - 1) \Rightarrow 7 \nmid \sigma(5^{e_5})$ since this would introduce a factor of 31.
$11, 71 \mid (5^5 - 1) \Rightarrow \sigma(5^{e_5})$ introduces at least one prime factor other than 7 or 11.
$5, 16 \mid (7^4 - 1) \Rightarrow 5 \nmid \sigma(7^{e_7})$ since this would introduce a factor of 16.
$11, 191, 2801 \mid (7^{10} - 1) \Rightarrow 11 \nmid \sigma(7^{e_7})$ since this would introduce two new prime factors.
$7, 19 \mid (11^3 - 1) \Rightarrow 7 \nmid \sigma(11^{e_{11}})$ since this would introduce a factor of 19.
$5, 3221 \mid (11^5 - 1) \Rightarrow \sigma(11^{e_{11}})$ introduces at least one prime factor other than 5, or 7.

Together, these imply that $\sigma(n)$ has at least two prime factors other than $2, 5, 7, 11$, and thus that n must have five or more prime factors for $I(n) = \frac{8}{5}$ to be possible. Otherwise the factors introduced by $\sigma(n) = \sigma(5^{e_5}) \cdot \sigma(7^{e_7}) \cdot \sigma(11^{e_{11}}) \cdot \sigma(p^{e_p})$ cannot be canceled in the denominator of $I(n)$ by the factors of n since $\sigma(n)$ has at least six prime factors, while the numerator of $I(n)$ has only one, namely 2^3, and n was assumed to have only four factors.

This establishes that $k \geq 5$, completing the proof. This work allows us to derive a lower bound on friends of 15 through theory rather than computation.

Corollary 7.8. *If* 15 *has a friend* n*, then* $n > 3813775$.

Proof of Corollary 7.8. We already know that n has at least 5 prime factors, the smallest of which is 5, and that the power of 5 must be even. The next four primes are $7, 11, 13, 19$, but we have already shown in Claim 7.7 that

$$I(5^{e_5}7^{e_7}11^{e_{11}}13^{e_{13}}) > \frac{8}{5}.$$

Also, we have already shown in Claim 5.4 that 11 and any other $p \equiv 2 \pmod 3$ must have an even power (and thus if 11 divides n, then $n \geq 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 19$). However, $5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 19 > 5^2 \cdot 7 \cdot 13 \cdot 19 \cdot 31$ and thus to find a lower bound, we will choose factors $p_i \equiv 1 \pmod 3$. Note that $I(5^2 \cdot 7 \cdot 13 \cdot 19 \cdot 31) = \frac{31}{25} \cdot \frac{8}{7} \cdot \frac{14}{13} \cdot \frac{20}{19} \cdot \frac{32}{31} > \frac{8}{5}$, so the lowest value we are still considering for n is $5^2 \cdot 7 \cdot 13 \cdot 19 \cdot 37 = 3813775$. Here, $I(5^2 \cdot 7 \cdot 13 \cdot 19 \cdot 37) = \frac{31}{25} \cdot \frac{8}{7} \cdot \frac{14}{13} \cdot \frac{20}{19} \cdot \frac{38}{37} < \frac{8}{5}$, which shows that the lower bound is not sharp. Not only is this expected, since having the first quantity with 5 prime factors we tried give a friend of 15 would be quite the coincidence, this is the best lower bound that we can determine using theory alone, without resorting to computation.

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