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Classification of Nicely Edge-Distance-Balanced Graphs

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Abstract

A nonempty graph G is called nicely edge distance-balanced (as brief NEDB), whenever there exists a positive integer γ'_G, such that for any edge say $e = uv$ we have: $m_u^G(e) = m_v^G(e) = \gamma'_G$. Which $m_u^G(e)$ denotes the number of edges laying closer to the vertex u than vertex v and $m_v^G(e)$ is defined analogously. In this paper, we study on NEDB graph and it's basic properties and some operations. Also, we try to classify some families of graphs with related $\gamma'_G \leq 2$.

1 Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For any two vertices $u, v \in V(G)$, the distance $d_G(u, v)$ is defined as the length of the shortest path connecting u and v. Suppose $e = uv, e' = u'v'$ are arbitrary edges of G. The distance between x and e is defined as : $d_G(x', e) = min\{d_G(x', u), d_G(x', v)\}$ and the distance between e and e' is shown by:

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$d_G(e,e') = min\{d_G(u,e'), d_G(v,e')\} = min\{d_G(u,u'), d_G(u,v'), d_G(v,u'), d_G(v,v')\}$.
Set $N_u(e) = \{x \in V(G) \mid d_G(u,x) < d_G(v,x)\}$ and $N_v(e) = \{x \in V(G) \mid d_G(v,x) < d_G(u,x)\}$. The quantities $n_0(e)$, $n_u(e)$ and $n_v(e)$ are defined as follows:

$n_0(e) =\mid \{x \in V(G) \mid d(x,u) = d(x,v)\} \mid$,
$n_u(e) =\mid \{x \in V(G) \mid d(x,u) < d(x,v)\} \mid$,
$n_v(e) =\mid \{x \in V(G) \mid d(x,v) < d(x,u)\} \mid$.

Similarly, the quantities $m_0(e)$, $m_u(e)$ and $m_v(e)$ are defined as follows:
$m_0(e) =\mid \{e' \in E(G) \mid d(e',u) = d(e',v)\} \mid$,
$m_u(e) =\mid \{e' \in E(G) \mid d(e',u) < d(e',v)\} \mid$,
$m_v(e) =\mid \{e' \in E(G) \mid d(e',v) < d(e',u)\} \mid$.

Let $e = uv$ be an arbitrary edge of G. Then for any two non-negative integer i, j, we define:
$D'^i_j(e) = \{e' \in E(G) \mid d_G(e',u) = i, d_G(e',v) = j\}$.

The triangle inequality implies that only the sets $D'^{i-1}_i(e)$, $D'^i_i(e)$, $D'^i_{i-1}(e)$, for each $(1 \leq i \leq d)$ can be nonempty (d is the diameter of the graph G). Also, $D'^0_0(e) = \phi$.

The transmission, $T(u)$ and edge-transmission $T'(u)$ of a vertex $u \in V(G)$ are defined as $T(u) = \Sigma_{x\in V(G)} d(u,x)$ and $T'(u) = \Sigma_{e\in E(G)} d(e,u)$, respectively [13].

Let us explain some fundamental definition which are needed in this paper:

The graph G is called distance-balanced (as brief DB), if for any arbitrary edge $e = uv$ of G, the number of vertices are lying closer to u than to v is equal to the number of vertices which are lying closer to v than to u, [5, 7, 10].

The simple connected G is called strongly distance-balanced (SDB), if for any edge $e = uv$ in G and any positive integer t, the number of vertices at distance t from u and at distance $t+1$ from v is equal to the number of vertices at distance $t+1$ from u and at distance t from v, [1, 9].

A nonempty graph G is called nicely distance-balanced (NDB), whenever there exists a positive integer γ_G, such that for any two adjacent vertices u and v in G, there are exactly γ_G vertices of G which are closer to u than to v, and exactly γ_G vertices of G which are closer to v than to u, see [11].

Edge distance-balanced graphs (as brief EDB), are such graphs in which for every edge $e = uv$ the number of edges closer to vertex u than to vertex v is equal to the number of edges closer to v than to u, [13]. In the other hand, one can easily find a graph G as an EDB graph, if and only if:

$$m^G_u(e) = m^G_v(e), \text{ for any edge } e = uv \in E(G).$$

Suppose G is a simple connected graph. We define the graphs S(G) and R(G) as below by following Yan et al, see [15]:

i) S(G) is the graph, which is obtained by adding one additional vertex for each edge of G. For simplicity, dividing every edge into two parts by adding one new vertex.

ii) R(G) is the graph, which is obtained by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.

In the next section of this paper, we tried to introduce some basic properties of NEDB graphs and show that under which conditions, we have NEDB graphs. Also, in third section, we discuss for finding a formula for γ'_G in some families of graphs, and in last section we try to classify some families of graphs with respect to related $\gamma'_G \leq 2$.

2 Some basic properties of NEDB graphs

In this section, we discuss some basic properties of NEDB (Nicely Edge Distance-Balanced) graphs regarding their diameter d. Let start with the following observation:

Proposition 2.1. *Let G be a connected NEDB graph, $|E(G)| \geq 3$ edges and with diameter d. Then for any edge e, this equality holds:*

$$\textstyle\sum_{i=2}^{d+1} |D'^{i}_{i}(e)| = |E(G)| - (2\gamma'_G + 1).$$

Proof. Let $|E(G)| = m$ and $e = uv$ be any edge in G, which is fixed after selection. Number of edges e' except e can be divided into three distinct sets. First set, some edges which are closer to u than v, so they belong to $D'^{i-1}_{i}(e)$. Second set, some edges which are closer to v than u, so they belong to $D'^{i}_{i-1}(e)$. The last one, some edges that are equidistant to u as well as v. With the triangle inequality the only nonempty sets are $D'^{i-1}_{i}(e), D'^{i}_{i-1}(e)$ and $D'^{i}_{i}(e)$, for $(1 \leqslant i \leqslant d)$. Because G is NEDB graph the first and second sets are equal to γ'_G. □

It is necessary to mention that, if G is a NEDB graph, then G is a EDB graph. But the converse is not true, for example, the generalized Petersen graph GP(7,2) is a EDB graph which is not a NEDB graph. Which we will discuss in this paper.

Proposition 2.2. *Let G be a NEDB graph with diameter d. Then $d - 1 \leq \gamma'_G$.*

Proof. Consider a path $x_0, x_1, x_2, \cdots, x_{d-1}, x_d$. Also, we have $e_1, e_2, e_3, \cdots, e_d$ as a sequence of edges, such that $d_G(e_1, e_d) = d$. Now, let $e_1 = x_0x_1$ be any edge in G, which is fixed after selection. Then $|\{e_2, ..., e_d\}| = d - 1 \leq m_{x_1}(e) = \gamma'_G$. Hence the result follows. □

Proposition 2.3. *If G is a NEDB graph, then G has no pendant vertex.*

Proof. Let $e = ax$ is an edge in G such that x is a pendant vertex in G, so $deg(x) = 1$ but $deg(a) > 1$. So $0 = m_x(e) \neq m_a(e)$. So, but G is NEDB graph, which is a contradiction. □

Corollary 2.1. *Trees are not NEDB.*

Let u and v be any arbitrary adjacent vertices in G and $e = uv$. Consider $M_i(u) = |\{e' \in E(G)|d(e', u) = i\}|$ and $M_i[u] = |\{e' \in E(G)|d(e', u) \leq i\}|$. Obviously, if $i = 1$, then $M(u) = M[u]$.

Theorem 2.1. *A graph G with diameter d is EDB if and only if*

$$M[u] \setminus M[v] + \Sigma_{i=1}^{d} M_i(u) \setminus M_{i-1}(v) = M[v] \setminus M[u] + \Sigma_{i=1}^{d} M_i(v) \setminus M_{i-1}(u),$$

holds for any $u, v \in V(G)$, which are adjacent.

Proof. Suppose that G is a EDB graph. Then $m_u(e) = m_v(e)$, where $e = uv$. Also, $e' = u'v'$ any other edge in G which $e \neq e'$ we can write:

$$m_u(e) = |\{e'\}| + \Sigma_{i=1}^{d} D'^{i}_{i-1}(e).$$

Note that for any $i \geq 1$, $D'^{i}_{i-1}(e) = (M_i(u) \setminus M_i(v))$ and observe that $M(u) \setminus M[v] = m[u] \setminus M[v]$. For $i \geq 2$ by further computing $D'^{i-1}_{i}(e) = (M_i(u) \setminus M_i[v]) = (M_i(u) \setminus M_{i-1}(v)) \setminus (M_i(u) \cap M_i(v))$. Hence,

$$m_u(e) = |\{e\}| + |(M[u] \setminus M[v])| + \Sigma_{i=1}^{d}([M_i(u) \setminus M_{i-1}(v)] \setminus [M_i(u) \cap M_i(v)]).$$

Also, by the same argument for $m_v(e)$, we have:

$$m_v(e) = |\{e\}| + |(M[v] \setminus M[u])| + \Sigma_{i=1}^{d}([M_i(v) \setminus M_{i-1}(u)] \setminus [M_i(u) \cap M_i(v)]).$$

Since $[M_i(u) \cap M_i(v)]$ is a subset of both $M_i(u) \setminus M_{i-1}(v)$ and $M_i(v) \setminus M_{i-1}(v)$. Then $m_u(e) = m_v(e)$, if and only if

$$|M[u] \setminus M[v]| + \Sigma_{i=1}^{d} |M_i(u) \setminus M_{i-1}(v)| = |M[v] \setminus M[u]| + \Sigma_{i=1}^{d} |M_i(v) \setminus M_{i-1}(u)|.$$

Hence the proof is over. □

Corollary 2.2. *Let G be a regular graph of diameter d. Then G is EDB if and only if*

$$\Sigma_{i=1}^{d}|M_i(u)\setminus M_{i-1}(v)| = \Sigma_{i=1}^{d}|M_i(v)\setminus M_{i-1}(u)|,$$

holds for any edge $e = uv$.

Corollary 2.3. *Let G be a graph with $d = 2$. Then G is EDB if and only if G is regular.*

Proposition 2.4. *If G is a NEDB graph and $k-$regular then $k - 1 \leq \gamma'_G$.*

Proof. Assume that G is a NEDB as well as EDB. Thus by Corollary 2.3, G is regular. For the rest, without loss of generality, we can divide this proof into two part.
Part 1: Let $d = 1$ and $e = uv$ and $e' = u'v'$. We have the only sets $D_1'^0(e)$ and $D_0'^1(e)$. Since G is a regular and NEDB graph, $|D_1'^0(e)| = |D_0'^1(e)| = deg(u) - 1$. The equality holds.
Part 2: Let $d \geq 2$. We have $m_u(e) = |D_0'^1(e)| + |D_1'^2(e)| = \gamma'_G$, which $|D_{i-1}'^i(e)| > 0$, for each $1 \leq i \leq d$. Clearly the inequality holds. □

Note that the converse of the above proposition is not true. The counter examples are Generalized Petersen graphs $GP(6, 2)$ and $GP(9, 3)$ which are both regular but not NEDB, which will be discussed in the next section of this paper.

Proposition 2.5. *Let G be a connected and NEDB graph that has at least two edges, and G^c be its complement. If $e \in E(G)$ and $e' \in E(G^c)$, then $G - e$ and $G + e'$ are not NEDB graphs.*

Proof. As G is a connected NEDB graph, by Proposition 2.4. G is regular. If we omit an edge from G, then we have two vertices which have one degree less than the others. So G is not regular graph. With the same argument by adding one edge from G^c, therefore two vertices which have one degree more than rest of vertices. Again it is irregular, contrapositive to Proposition 2.4. Hence the result follows. □

3 NEDB in Some Families of Graphs

In this section, we tried to finding an exact formula for a positive integer γ'_G, in some families of graphs which mentioned as below. Next, we tried to find an answer to the question that; if G is NEDB, whether the $S(G)$ and $R(G)$ are NEDB or not?

First, we consider the complete graph, K_n. Then we have:

Lemma 3.1. *If G is a complete graph K_n, then G is a NEDB graph and $\gamma'_G = n-2$. Also:*

$$\sum_{i=1}^{d} \mid D'^{i}_{i}(e) \mid = (n-3)(n-2)/2.$$

Proof. If $G = K_n$, then $d = 1$ and each vertex has degree $n-1$. Let $e = uv$ as an arbitrary edge in G can be lying closer to u or closer to v or equidistant. Since G is a regular graph. Then $deg(u) - 1 = |D'^{0}_{1}(e)| = |D'^{1}_{0}(e)| = deg(v) - 1 = n-2 = \gamma'_G$.
By substitution in Proposition 2.1, the number of edges which are equidistant from u and v is equal to $(n-3)(n-2)/2$. □

In the second step, we study on complete bipartite graph family, $K_{n,n}$:

Lemma 3.2. *Let G be a complete bipartite graph $K_{n,n}$. Then G is NEDB with $\gamma'_G = n-1$. Moreover,*

$$\sum_{i=1}^{d} \mid D'^{i}_{i}(e) \mid = (n-1)^2.$$

Proof. Consider $G = K_{n,n}$, then G is a n- regular graph with $|V(G)| = 2n$, $|E(G)| = n^2$ and the diameter of G is $d = 2$. Let $e = uv$ be any arbitrary edge in G. Then u and v must belong to two disjoint sets in a complete bipartite graph. So $|D'^{0}_{1}(e)| = |D'^{1}_{0}(e)| = \gamma'_G = n-1$. Again by substitution in Proposition 2.1, the number of edges which are equidistant is $(n-1)^2$. □

Now, consider the cycle graphs on n vertices, C_n:

Lemma 3.3. *Let G be the n-cycle graph C_n with diameter d. Then G is NEDB with $\gamma'_G = d-1$, and $\Sigma_{i=1}^{d} \mid D'^{i}_{i}(e) \mid = 0$ or 1.*

Proof. Let $G = C_n$ be a cycle graph on n vertices as well as n edges. Let $e = uv$ be any arbitrary edge in G. Then we have 2 cases:
Case 1: Let $n = 2d-1$ be any odd number. Then, except e we have $2d-2$ edges in G . Clearly, we can see we have $(2d-2)/2$ edges near to u and $d-1$ edges near to v. Then $m_u^G(e) = m_v^G(e) = d-1$. Therefore, there are no edges equidistant to u as well as v. Hence $\Sigma_{i=1}^{d} \mid D'^{i}_{i}(e) \mid = 0$.
Case 2: Let $n = 2d$ be any even number. Then, we have an odd number $(2d-1)$ edges except e which can not be divided into equal number. Therefore $\Sigma_{i=1}^{d} |D'^{i}_{i}(e)| = 1$. The number of the rest of the edges are $2d-2$ which can be divided by 2 equal sets. Hence $\gamma'_G = d-1$. □

Next, we study on complete multipartite graph, $K_{p\times q}$, which is a complete graph which set of vertices decomposed into p disjoint sets such that there are 2 vertices in each p-disjoint sets and also, no two graphs vertices within the same set are adjacent:

Lemma 3.4. *Let G be the complete multipartite graph $k_{p\times q}$. Then it is a NEDB graph with $\gamma'_G = q(p-1)-1$. Moreover,*

$$\Sigma_{i=1}^{d}|D_i'^i(e)| = \Sigma_{m=1}^{pq/2} q(pq-2m) - 2q(p-1) - 3.$$

Proof. Prove by induction on q.
Let $q = 2$. Then $G = K_{p\times 2}$ is a p-partite, $(2p-2)$-regular graph with diameter 2.
Further, G is NEDB, so for any arbitrary vertices u and v which are adjacent, we have:

$$deg(u) - 1 = |D_1'^0(e)| = \gamma'_G = |D_0'^1(e)| = deg(v) - 1.$$

Hence, $\gamma'(G) = 2p-3$. Since $d = 2$, the rest of edges are equidistant which belong to $\Sigma_{i=2}^{d+1} D_i'^i(e)$ and , for each $i \geq 3$. As we know, number of edges in G is $2p(p-1)$, by Proposition 2.1, $\Sigma_{i=1}^{d} \mid D_i'^i(e) \mid = 2p^2 - 6p + 5$.
Now, let $q = 3$. Then $G = K_{p\times 3}$ is a complete p-partite $(3p-3)$-regular graph with diameter 2, which in every disjoint set have 3 vertices. Therefore, $\gamma'_G = 3p-4$.
Also, $|E(G)| = 9p(p-1)/2$, by substitution in Proposition 2.1. The result follows.
Now, let us assume that the result is true for $q-1$. We want to show that if we take $G = K_{p\times q}$, then we have $q(p-1)$-regular graph which in each p-disjoint sets, there are q vertices.
Suppose that $e = uv$ is any arbitrary edge in G. Then u is adjacent to all vertices in G except $q-1$ vertices in the set which u belongs to. So $\gamma'_G = q(p-1)-1$ and the rest of edges in G are in $D_0'^0(e)$. By Proposition 2.1, proof is over. □

Here, to state the next result, first we recall the definition of the generalized Petersen graphs. Let us take $n \geq 3$ denote a positive integer and let $k \in \{1, 2, \cdots, n-1\} \setminus \{n/2\}$. The generalized Petersen graph GP(n,k) is defined to have vertex set and edge set as below:

$$V(GP(n,k)) = \{u_i | i \in Z_n\} \cup \{v_i | i \in Z_n\},$$
$$E(GP(n,k)) = \{u_i u_{i+1} | i \in Z_n\} \cup \{v_i v_{i+k} | i \in Z_n\} \cup \{u_i v_i | i \in Z_n\}.$$

Lemma 3.5. *The generalized Petersen graph GP(n,1) is a NEDB graph if and only if* $n \in \{3, 4\}$.

Proof. If $n = 3$ or $n = 4$, then obviously we can see those graphs are NEDB. Conversely, we consider $G = GP(n, 1)$. By [2], we have three types of edges: 1) outer-cycle edges, 2) inner -cycle edges, 3) bridge, which the first and the last are isomorphic to each other. So, continue the proof by taking two cases for these two types of the edges.

Case 1: Let $n \neq 1$ be any odd number and assume that $e = uv$ be any arbitrary edges in outer (inner)-cycle. Then the only two edges are equidistant to u and v, which one of them belongs to inner (outer)-cycle and the other belongs to the bridge edges. Thus always $\Sigma_{i=1}^{d}|D_i'^i(e)| = 2$. By substitution in Proposition 2.1, $\gamma_G' = (3n-3)/2$.

Now, assume that $e = uv$ is a bridge edge. Then all the bridges in G except e are equidistant with u and v. So, $\Sigma_{i=1}^{d}|D_i'^i(e)| = n - 1$. Again by using Proposition 2.1. We get $\gamma_G' = n$. Since G is NEDB, these two amount for γ_G' must be equal, therefore $n = 3$.

Case 2: $n \neq 2$ be any even number. If $e = uv$ belongs to outer (inner)-cycle, then two edges in inner (outer)-cycle and one belongs to outer (inner)-cycle are equidistant to u and v. So always $\Sigma_{i=1}^{d+1}|D_i'^i(e)| = 3$. Using Proposition 2.1, we find $\gamma_G' = (3n-4)/2$.

Next, consider $e = uv$ as a bridge edges, all of the bridge edges except e are equidistant to u and v. So $\Sigma_{i=1}^{d+1}|D_i'^i(e)| = n - 1$. Again by Proposition 2.1, we have $\gamma_G' = n$. Since G is NEDB, these two different amount of γ_G' must be equal, so $n = 4$. $\square$

Proposition 3.1. *Let G and H be two simple connected graphs which both are NEDB as well as NDB. Then the cartesian product of G and H, say $G \times H$ is NEDB.*

Proof. Let G and H be any two connected and NEDB as well as NDB graphs. Assume that two partitions are:

$A = \{(a, u)(b, v) \in E(G \times H) | ab \in E(G), u = v\}$,

$B = \{(a, u)(b, v) \in E(G \times H) | a = b, uv \in E(H)\}$.

If $e = uv$ is any edge in A, then by [11, theorem 2.1],

$m_{(a,u)}^{G\times H}(e) = m_a^G(e)|V(H)| + n_a^G(e)|E(H)|$. (*)

Also we have:

$m_{(b,v)}^{G\times H}(e) = m_b^G(e)|V(H)| + n_b^G(e)|E(H)$

Since G and H both are NDB as well as NEDB, we have

$n_a^G(e) = n_b^G(e) = \gamma(G)$,

$m_a^G(e) = m_b^G(e) = \gamma'(G)$.

By (*) and (**), we have $m_{(a,u)}^{G\times H}(e) = m_{(b,v)}^{G\times H}(e)$. Hence $G \times H$ is a NEDB graph.

□

NDB is a necessary condition, as a counter example, suppose $G = C_4$ which is a NDB as well as NEDB graph and $H = GP(3,1)$ which is NEDB, but not NDB. One can observe that $G \times H$ is not regular, therefore it is not NEDB.

Proposition 3.2. *Let G be a NEDB graph. Then $S(G)$ is a NEDB graph if and only if G is a cycle graph on n vertices. Also, $\gamma'_G = \gamma'_{S(G)} = k-1$.*

Proof. If G is a NEDB graph on n vertices and m edges, then by Proposition 2.4, G is regular. By definition of $S(G)$, we have $n+m$ vertices and $2m$ edges, which all new added vertices have degree 2. By the regularity, the rest of vertices must have degree two. But this condition will happen only for cycle graph on n vertices.

For the next part, we have already introduced $\gamma'_{C_n} = k-1$, (for $k \geq 2$). It is known that, if G is a cycle graph on n vertices, then $|V(S(C_n))| = 2n$ and $|E(S(C_n))| = 2m$, which $m = n$. So number of edges and vertices in $S(G)$ is always an even number, say $2k$, for each $k \geq 3$.

Again, it is in form of a cycle graph on $2k$ edges, which $\gamma'_{S(G)} = k-1$. Hence the result follows.

□

Proposition 3.3. *If G is a nontrivial and connected NEDB graph, then $R(G)$ is a NEDB graph if and only if $|E(G)| = 1$.*

Proof. One way is clear, if $|E(G)| = 1$ so G has a path of length 1,then $R(G)$ is a NEDB graph.

For converse part, suppose R(G) is a NEDB graph, by contrary we assume that $\mid E(G) \mid\geq 2$, by the definition of $R(G)$, related to each edge we add a new vertex, which is adjacent to primary vertices. Hence, we can find at least one vertex with degree at least 4 and the rest are of degree 2, which is contrapositive to Proposition 2.4. It shows G is not NEDB, it is contradiction.

□

Corollary 3.1. *If G is any NEDB graph with $|E(G)| > 2$, then $R(G)$ is not a NEDB graph.*

4 Classification

In this section, our aim is to classify the graphs with respect to their γ'_G. Although most of the mentioned graphs are isomorphic to each other but to list all of them feel necessary.

Theorem 4.1. *A graph G is a NEDB graph with $\gamma'_G = 1$ if and only if it is one of the following:*
i) the complete graph $K_{\cong}$ Johnson graph $J(3,1)$,
ii) the 4-cycle C_4.

Proof. Let $d \neq 0$ denote the diameter of G. By Proposition 2.2, we have $d - 1 \leq \gamma'_G = 1$. Thus these cases may occur: $d = 1$, $d = 2$.
Case 1: If $d = 1$, then G is a complete graph. Since $\gamma'_G = 1$, for any edge $e = uv$ in G, we can assume $|D'^2_1(e)| = |D'^1_2(e)| = 1$. It means $\exists e', e'' \in E(G)$, e' and e'' are the only members in those sets which are adjacent. So, G is a C_3 which is isomorphic to K_3 and $J(3,1)$.
Case 2: Let $d = 2$ and e', e'' are not adjacent. Then, there is another edge say $e_1 \in E(G)$ which is adjacent to both of them simultaneously, so we get a cycle on 4 vertices. Hence the proof is over.

□

Theorem 4.2. *A graph G is NEDB graph with $\gamma'_G = 2$ if and only if it is one of the following graphs:*
i) the Johnson graph $J(4,1)$($\cong$ complete graph K_4),
ii) the 5-cycle C_5($\cong$ 5-paley graph),
iii) complete bipartite graph $K_{3,3}$,
iv) the 6-cycle C_6.

Proof. Let consider possible cases when $\gamma'_G = 2$. If $d \neq 0$ is a diameter of G, then by Proposition 2.2, $d - 1 \leq \gamma'_G = 2$. Therefore, d can be equal to 1, 2 or 3. Now, consider different cases as below:
If $d = 1$, then G is a complete graph K_n and by Lemma 3.1, $\gamma'_G = n - 2$. On the other side, $\gamma'_G = 2$, so $n = 4$. Which means G is $K_4 \cong J(4,1)$.
If $d = 2$, then we can consider two cases:
Case 1: $D'^2_2(e) = \phi$.
Subcase 1: We can assume that $D'^3_3(e) = \phi$. Then $\Sigma_{i=2}^{d+1}|D'^i_i(e)| = 0$. By using Proposition 2.1, the number of edges in G must be equal to 5.
Since $\gamma'_G = 2$, we can consider two assumption, which may occur:
First assumption, $|D'^2_1(e)| = |D'^1_2(e)| = 2$ and the remaining sets are empty. Then we have a tree on 6 vertices, by Corollary 2.1, G is not NEDB.

Second assumption, $|D_1'^2(e)| = |D_2'^1(e)| = 1$ and $|D_3'^2(e) vert = |D_2'^3(e)| = 1$. Then each edge in $D_2'^3(e)$ with $D_3'^2(e)$ must be adjacent. So we get a cycle on 5 nodes which is C_5 as well as 5-paley graph.
Subcase 2: Let $D_3'^3(e) = 1$. Then by the above conditions, we get C_6, which has $\gamma'_G = 2$, but $d = 3$ which is contradiction to $d = 2$. (We accept this part in case $d = 3$).
Subcase 3: Let $D_3'^3(e) \geq 2$. Then we have $|D_1'^2(e)| = |D_2'^1(e)| = 1$ and $|D_2'^3(e)| = |D_3'^2(e)| = 1$, we get multiple edges and that is a contradiction.
Case 2: $D_2'^2(e) \neq \phi$.
Subcase 1: Let $D_3'^3(e) = \phi$ and $D_2'^2(e) = 1$.
First assumption, $|D_1'^2(e)| = |D_2'^1| = 2$, the graph is irregular. So it is a contradiction.
Second assumption, $|D_1'^2(e)| = |D_2'^1(e)| = 1$ and $|D_2'^3(e)| = |D_3'^2(e)| = 1$, the graph is not regular.
Subcase 2: Consider $|D_2'^2(e)| = 2$ and let $D_3'^3(e) = \phi$.
If we let $e = uv$ be any edge of G and $|D_1'^2(e)| = |D_2'^1(e)| = 2$, then the graph is not regular. Also, we can assume that $|D_2'^3(e)| = |D_3'^2(e)| = 1$ and $|D_1'^2(e)| = |D_2'^1(e)| = 1$. Then we get multiple edge, which is a contradiction.
Subcase 3: Let $|D_2'^2(e)| = 3$ and $D_3'^3(e) = \phi$. Then, for any arbitrary edge $e = uv$, let $|D_1'^2(e)| = |D_2'^1(e)| = 2$, then we have two different edges in G say e_1, e_2 adjacent to u and two other edges in G say e_3, e_4 adjacent to v. Hence $deg(u) = deg(v) = 3$. Since $|D_2'^2(e)| = 3$, there are 3 other edges say e_1, e_2, e_3 which e_1 is adjacent to e'_1, e'_3 and e_2 is adjacent to e'_2 and e'_4 and e_3 must be a multiple edges. The regularity is not satisfied. So G is not NEDB which is not a contradiction.
Subcase 4: Assume $|D_2'^2(e)| = 4$ and $D_3'^3(e) = \phi$. Then add one edge between e_2 and e_3. Thus, all the vertices have degree 3 in this graph, and so the regularity condition is satisfied. Hence, we have 9 edges and 6 nodes such that we can divide them to 2 equal part, 3 by 3, in such away none of them in one part are adjacent, so it meas $G = K_{3,3}$.
Now, consider $D_3'^3(e) \neq \phi$. Suppose that $|D_3'^3(e)| = 1$, the only possible case may occur for $GP(4,1)$, which has $|D_1'^2(e)| = |D_2'^1(e)| = 2$ and $|D_2'^3(e)| = |D_3'^2(e)| = 2$ and $|D_2'^2(e)| = 2$. But it is contradiction to Propositions 2.4 and 3.5.
If $d = 3$, then again we must assume different cases:
Case 1: $D_2'^2(e) = \phi$.
Since $\gamma'_G = 2$, we can assume that $D_3'^3(e) = \phi$. So $\Sigma_{i=2}^{d+1} |D_i'^i(e)| = 0$. Since $\gamma'_G = 2$, then two subcases may occur.
Subcase 1: $|D_1'^2(e)| = |D_2'^1(e)| = 2$. Then these two edges must be adjacent,

which make an irregular graph which is contradiction to NEDB assumption.
Subcase 2: $|D_1'^2(e)| = |D_2'^1(e)| = 1$ and $|D_2'^3(e)| = |D_3'^2(e)| = 1$, we have a cycle on 6 nodes which is C_6.
Now, consider $D_3'^3(e) \neq \phi$, suppose $|D_3'^3(e)| = 1$, by above condition we get C_6 which is a cycle on 6 nodes.
Case 2: $|D_2'^2(e)| = 1$ and $D_3'^3(e) = \phi$.
First assumption, $|D_1'^2(e)| = |D_2'^1(e)| = 2$, again we have an irregular graph.
Second assumption, $|D_1'^2(e)| = |D_2'^1(e)| = 1$ and $|D_2'^3(e)| = |D_3'^2(e)| = 1$, so we get two vertices on degree 3 and two vertices on degree 2 and two vertices on degree 1. Hence the graph is not NEDB.
The proof is complete.

□

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