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Subclasses of P-valent Functions within Integral Operators

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Abstract

We introduce some generalizations of subclasses for analytic functions f within the linear integral operator $\Omega_w^k f(z) = I^{(k-1)}f(z) + \lambda\, I^{(k)}f(z)$ in the open disc $\mathcal{U}_w = \{z : |z-w| < 1\}$. We calculate coefficient bounds of subclasses of w-p-valently uniformly starlike functions. We otain some properties of the growth, distortion, extreme points. Finally, we estimate results of fractional Integrals bounds.

1 Introduction

For a fixed positive integer p, let $\mathcal{A}_p$ denote the class of functions

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}\ z^{n+p},\ for\ a_{n+p} \in \mathcal{C}$$

which are analytic in the open disc $\mathcal{U} = \{z : |z| < 1\}$. There are many papers written on the subject some of the latest being [10, 11, 12, 13, 15, 16, 17, 18]. Several authors have obtained valuable and interesting results of univalent functions, even with various generalizations, as we can see from the literature.

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For a fixed point w in the unit disc $\mathcal{U}$, Kanas and Ronning [8] introduced a more generalization form of analytic functions in the unit disc $\mathcal{U}$ of the form

$$f(z) = (z-w) + \sum_{n=2}^{\infty} a_n (z-w)^n, \ a_n \in \mathcal{C}$$

which are denoted by $\Gamma(w)$, $ST(w)$, and $CV(w)$, and they obtained some results related to the other univalent functions. Acu and Owa [10] introduced bounds for classes of ω-close-to-convex functions, ω-α-convex functions and other further studies of these classes. Al-Kasasbeh and Darus [11, 12] introduced classes of analytic univalent functions that are defined in the open disc $\mathcal{U}_w = \{z : |z-w| < 1\}$, and they proved corresponding results to these classes. The concept of uniformly starlikeness for analytic and univalent functions was introduced by Goodman [2, 3] and investigated by several authors (e.g see Ma and Minda [4], Ronning [5, 6], and Bharati et al. [7]). Shams et al. [9] defined the class of uniformly starlike functions by
$\mathcal{SD}(\delta, \gamma) = \{ f \in A(p) : \ \Re(\frac{zf'(z)}{f(z)}) > \delta \left| \frac{zf'(z)}{f(z)} - 1 \right| + \gamma, \ z \in \mathcal{U}, \ \delta \geq 0, \ \gamma \in [0, 1)\}$,
and lately, Nishiwaki and Owa [13] studied the class of p-valently uniformly starlike functions which is defined on the unit disc $\mathcal{U}$ by

$$\mathcal{SD}_p(\delta, \gamma) = \{ f \in A(p) : \ \Re\left(\frac{zf'(z)}{f(z)}\right) > \delta \left| \frac{zf'(z)}{f(z)} - p \right| + \gamma, \ \delta \geq 0, \ \gamma \in [0, p)\}.$$

In the open disc $\mathcal{U}_w = \{z : |z-w| < 1\}$(for a fixed complex number w), the class $\mathcal{A}_w(p)$ is defined to be the w-p-valent analytic function in the form

$$f(z) = (z-w)^p + \sum_{n=1}^{\infty} a_{n+p} \ (z-w)^{n+p}, \quad a_{n+p} \in \mathbb{C} \quad and \quad p\,, n \in \mathbb{N}.$$

In this article, the integral operator I_w^k is defined to be

$$I_w^k f(z) = I_w(I_w^{k-1} f(z)), \qquad for \quad k \in \mathbb{N}$$

where

$$I_w^0 f(z) = f(z), \qquad I_w^1 f(z) = I_w f(z) = \int_0^z \frac{f(t)}{t-w} dt,$$
$$I_w^2 f(z) = I_w(I_w f(z)) = I_w(\int_0^z \frac{f(t)}{t-w} dt) \ \text{ and } \ so \ on.$$

Also, for a nonnegative parameter λ and $n \in \mathbb{N}$, the integral operator Ω_w^k for an analytic function f in the open disc $\mathcal{U}_w = \{z : |z - w| < 1\}$ is

$$\Omega_w^k f(z) = I^{k-1} f(z) + \lambda\, I^k f(z)$$

and $\mathcal{SI}_p^w(\delta,\, \gamma)$ is a subclass of $\mathcal{A}_w(p)$ which is w-p-valently uniformly starlike functions

$$\mathcal{SI}_p^w(\delta,\, \gamma) = \{\, f \in \mathcal{A}_w(p) : \Re\left(\frac{f(z)}{I_w f(z)}\right) > \delta \left|\frac{f(z)}{I_w f(z)} - p\right| + \gamma,\ \delta \geq 0,\ \gamma \in [0,\, p)\}.$$

The Integral operator Ω_w^k within the class of w-p-valently uniformly starlike functions is introduced as follows.

Definition 1.1. . *Let $f(z) \in \mathcal{A}_w(p)$. Then $f(z) \in \mathcal{S}\Omega_w^k(p\,,\, \gamma,\, \delta,\, \lambda)$ if and only if*

$$\mathrm{Re}\left(\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)}\right) > \delta \left|\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)} - p\right| + \gamma,$$

for $z \in \mathcal{U}_w$, $\delta \geq 0$ and $0 \leq \gamma < p$.

The class $\mathcal{S}\Omega_w^k(p,\, \gamma,\, \delta,\, \lambda)$ is a generalization of various subclasses of univalent functions. It is easy to see that for the values $k = 1$, $w = 0$, and $p = 1$, $(z - w)f'(z) \in \mathcal{S}\Omega_0(1,\, \gamma,\, \delta,\, \lambda)$ if and only if $f(z) \in \mathcal{SD}(\delta, \gamma)$ [9] in the unit disc $\mathcal{U}$. Also, if $k = 1$, $w = 0$, and $p \in \mathbb{N}$, then $(z - w)f'(z) \in \mathcal{S}\Omega_0(1,\, \gamma,\, \delta,\, \lambda)$ if and only if $f(z) \in \mathcal{SD}_p(\delta, \gamma)$ [13] in the unit disc $\mathcal{U}$.

2 Main results

In this section, we estimate the coefficient bounds for a function f in the class $\mathcal{S}\Omega_w^k(p,\, \gamma,\, \delta,\, \lambda)$ which is analytic in the open disc $\mathcal{U}_w$.

Theorem 2.1. *A function $f(z) \in \mathcal{S}\Omega_w^k(p,\, \gamma,\, \delta,\, \lambda)$ if and only if for $\delta \geq 0$, $0 \leq \gamma < p$ and $k \geq 1$*

$$\sum_{n=1}^{\infty} |a_{n+p}| \leq \frac{p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}. \tag{2.1}$$

Proof. Since $f(z) \in \mathcal{S}\Omega_w^k(p,\, \gamma,\, \delta,\, \lambda)$, for $\delta \geq 0$, $0 \leq \gamma < p$, and $z \in \mathcal{U}_w$

$$\operatorname{Re}\left(\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)}\right) - \delta \left|\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)} - p\right| > \gamma,$$

and

$$\left|\Omega_w^{k-1} f(z)\right| - \delta \left|\Omega_w^{k-1} f(z) - p\,\Omega_w^k f(z)\right| > \gamma\, |\Omega_w^k f(z)|.$$

Therefore,

$$| \, p^{2-k}(z-w)^p + \sum_{n=1}^{\infty}(n+p)^{2-k}(z-w)^{n+p} + \lambda(p^{1-k}(z-w)^p + \sum_{n=1}^{\infty}(n+p)^{1-k}(z-w)^{n+p})| - \delta|p^{2-k}(z-w)^p + \sum_{n=1}^{\infty}(n+p)^{2-k}(z-w)^{n+p} + \lambda(p^{1-k}(z-w)^p + \sum_{n=1}^{\infty}(n+p)^{1-k}(z-w)^{n+p}) - p(|p^{1-k}(z-w)^p + \sum_{n=1}^{\infty}(n+p)^{1-k}(z-w)^{n+p} + \lambda(p^{-k}(z-w)^p + \sum_{n=1}^{\infty}(n+p)^{-k}(z-w)^{n+p})| - \gamma|p^{1-k}(z-w)^p + \sum_{n=1}^{\infty}(n+p)^{1-k}(z-w)^{n+p} + \lambda(p^{-k}(z-w)^p + \sum_{n=1}^{\infty}(n+p)^{-k}(z-w)^{n+p})| > 0.$$

Since $|z-w| < 1$, $(z-w)$ approaches 1. Hence

$$\sum_{n=1}^{\infty}|a_{n+p}|(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)] \leq\; p^{-k}(p+\lambda)(p\,\delta+p+\gamma)$$

For $0 \leq \gamma < p$, and $\delta \geq 0$, we have

$$\sum_{n=1}^{\infty}|a_{n+p}\,| \leq \frac{p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}.$$

Conversely, by definition,

$$\delta \left|\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)} - p\right| - \operatorname{Re}\left(\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)}\right) \leq -\gamma,$$

which is equivalent to

$$\delta \left|\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)} - p\right| - \operatorname{Re}\left(\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)} - 1\right) \leq 1-\gamma.$$

Therefore

$$\delta \left|\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)} - p\right| - \operatorname{Re}\left(\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)} - 1\right) \leq (\delta+1)\left|\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)} - p\right|$$

$$\leq (\delta+1)) \left| \frac{\sum_{n=1}^{\infty} a_{n+p}\,(n+p)^{-k}[n(n+p+\lambda)]}{p^{-k}(p+\lambda)+\sum_{n=1}^{\infty} a_{n+p}\,(n+p)^{-k}(n+p+\lambda)} \right|$$

$$\leq (\delta+1)) \left(\frac{\sum_{n=1}^{\infty}[n(n+p+\lambda)]\frac{p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{(n+p)[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}}{p^{-k}(p+\lambda)+\sum_{n=1}^{\infty}(n+p+\lambda)\frac{p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{(n+p)[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}} \right),$$

since there is $\gamma \in [0,\, p]$, and $\delta \geq 0$ such that

$$\left(\frac{\sum_{n=1}^{\infty}[n(n+p+\lambda)]\frac{(p+\lambda)(p\,\delta+p+\gamma)}{(n+p)[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}}{(p+\lambda)+\sum_{n=1}^{\infty}(n+p+\lambda)\frac{(p+\lambda)(p\,\delta+p+\gamma)}{(n+p)[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}} \right) \leq \left(\frac{1-\gamma}{\delta+1}\right),$$

then

$$\delta \left| \frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)} - p \right| - \mathrm{Re}\left(\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)} - 1 \right) \leq (\delta+1)\left(\frac{1-\gamma}{\delta+1}\right) = 1-\gamma.$$

which is equivalent to

$$\delta \left| \frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)} - p \right| - \mathrm{Re}\left(\frac{\Omega_w^{k-1} f(z)}{\Omega_w^k f(z)} \right) \leq (\delta+1)\left(\frac{1-\gamma}{\delta+1}\right) \leq -\gamma.$$

Thus $f(z) \in \mathcal{S}\Omega_w^k(p,\, \gamma,\, \delta,\, \lambda)$, and the result is sharp for
$f(z) = p^{-k}(z-w)^p + \frac{p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}(z-w)^{p+n}.\square$

Distortion and growth properties are discussed in the next Corollary.

Theorem 2.2. *Let $f(z) \in \mathcal{S}\Omega_w^k(p,\, \gamma,\, \delta,\, \lambda)$, for $z \in \mathcal{U}_w = \{z : r = |z-w| < 1\}$. Then*
$p^{-k}r^p - \frac{2^{k-1}(1+\lambda)(\,\delta+1+\gamma)}{[(2+\lambda)(1-\delta)-(1+\lambda)(1+\gamma)]}\, r^{p+1} \leq |f(z)| \leq p^{-k}r^p + \frac{2^{k-1}(1+\lambda)(\delta+1+\gamma)}{[(2+\lambda)(1-\delta)-(1+\lambda)(1+\gamma)]}\, r^{p+1}$
and
$p^{1-k}\, r^{p-1} - \frac{2^{k}(1+\lambda)(\,\delta+1+\gamma)}{[(2+\lambda)(1-\delta)-(1+\lambda)(1+\gamma)]} r^p \leq |f'(z)| \leq p^{1-k}\, r^{p-1} + \frac{2^{k}(1+\lambda)(\,\delta+1+\gamma)}{[(2+\lambda)(1-\delta)-(1+\lambda)(1+\gamma)]} r^p$

with equality for $f(z) = p^{-k}(z-w)^p + \frac{2^{k-1}(1+\lambda)(\,\delta+1+\gamma)}{[(2+\lambda)(1-\delta)-(1+\lambda)(1+\gamma)]}(z-w)^{p+1}$.

Proof. For $p = n = 1$ in (2.1), we have

$$\sum_{n=1}^{\infty} |a_{n+p}| \leq \frac{2^{k-1}(1+\lambda)(\,\delta+1+\gamma)}{[(2+\lambda)(1-\delta)-(1+\lambda)(1+\gamma)]}.$$

Thus,

$$|f(z)| \leq p^{-k}r^p + \sum_{n=1}^{\infty} |a_{n+p}|\, r^{p+1} \leq p^{-k}r^p + r^{p+1} \sum_{n=1}^{\infty} |a_{n+p}| \leq p^{-k}r^p + \frac{2^{k-1}(1+\lambda)(\,\delta+1+\gamma)}{[(2+\lambda)(1-\delta)-(1+\lambda)(1+\gamma)]} r^{p+1},$$

and

$$|f(z)| \geq p^{-k}r^p - \sum_{n=1}^{\infty} |a_{n+p}|\, r^{p+1} \geq p^{-k}r^p - r^{p+1} \sum_{n=1}^{\infty} |a_{n+p}| \geq p^{-k}r^p - \frac{2^{k-1}(1+\lambda)(\,\delta+1+\gamma)}{[(2+\lambda)(1-\delta)-(1+\lambda)(1+\gamma)]} r^{p+1}.$$

Also, from (2.1) and Theorem 2.1, it follows that

$$\sum_{n=1}^{\infty} (p+n)\,|a_{n+p}| \le \frac{2^k(1+\lambda)(\delta+1+\gamma)}{[(2+\lambda)(1-\delta)-(1+\lambda)(1+\gamma)]}.$$

For $r = |z-w| < 1$, we have

$$|f'(z)| \le p^{1-k}\,|z-w|^{p-1} + \sum_{n=1}^{\infty} (p+1)\,|a_{n+p}|\,|z-w|^{p} \le p^{1-k}\,r^{p-1} + (p+1)\,r^{p+1} \sum_{n=1}^{\infty} |a_{n+p}|$$

$$\le p^{1-k}\,r^{p-1} + \frac{2^k(1+\lambda)(\delta+1+\gamma)}{[(2+\lambda)(1-\delta)-(1+\lambda)(1+\gamma)]} r^p,$$

and

$$|f'(z)| \ge p^{1-k}\,|z-w|^{p-1} - \sum_{n=1}^{\infty} (p+1)\,|a_{n+p}|\,|z-w|^{p} \ge p^{1-k}\,r^{p-1} - (p+1)\,r^{p} \sum_{n=1}^{\infty} |a_{n+p}|$$

$$\ge p^{1-k}\,r^{p-1} - \frac{2^k(1+\lambda)(\delta+1+\gamma)}{[(2+\lambda)(1-\delta)-(1+\lambda)(1+\gamma)]} r^p. \square$$

Theorem 2.3. *Suppose that* $f_p(z) = p^{-k}(z-w)^p$ *and for each positive integer* $n \ge 1$,
$f_{p+n}(z) = p^{-k}(z-w)^p + \frac{p^{-k}(p+\lambda)(p\delta+p+\gamma)}{(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}(z-w)^{n+p}$, *for* $z \in \mathcal{U}_w$.
Then $f(z) \in \mathcal{S}\Omega_w^k(p,\gamma,\delta,\lambda)$ *if and only if* f *can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \mu_{p+n}\, f_{n+p}(z) \text{ where } \mu_{n+p} \ge 0, \text{ and } \sum_{n=0}^{\infty} \mu_{p+n} = 1.$$

Proof. Assume that

$$f(z) = \sum_{n=0}^{\infty} \mu_{p+n}\, f_{n+p}(z) = p^{-k}(z-w)^p + \sum_{n=1}^{\infty} \mu_{n+p}\, a_{n+p}(z-w)^{n+p}$$

Since

$$\sum_{n=1}^{\infty} \mu_{n+p} \left(\frac{(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}{p^{-k}(p+\lambda)(p\delta+p+\gamma)}\right)\left(\frac{p^{-k}(p+\lambda)(p\delta+p+\gamma)}{(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}\right)$$

$$= \sum_{n=1}^{\infty} \mu_{n+p} = 1 - \mu_p \le 1.$$

By Theorem 2.1,$f(z) \in \mathcal{S}\Omega_w^k(p,\, \gamma,\, \delta,\, \lambda)$.

Conversely, let $f(z) \in \mathcal{S}\Omega_w^k(p,\, \gamma,\, \delta,\, \lambda)$. Then

$$|a_{n+p}| \leq \tfrac{p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}, \quad for \quad n \geq 1.$$

Without loss of generality, assume that

$$\mu_{n+p} = \tfrac{(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}{p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}\, a_{n+p}\,, \quad for \quad n \geq 1,$$

and $\mu_p = 1 - \sum\limits_{n=1}^{\infty} \mu_{n+p}$. Then

$$\begin{aligned}
f(z) &= p^{-k}(z-w)^p + \sum_{n=1}^{\infty} \mu_{n+p}\, a_{n+p}(z-w)^{n+p} \\
&= p^{-k}(z-w)^p + \sum_{n=1}^{\infty} \mu_{n+p}\, \tfrac{p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}(z-w)^{n+p} \\
&= p^{-k}(z-w)^p + \sum_{n=1}^{\infty} \mu_{n+p}\, [f_{n+p}(z) - (z-w)^p] \\
&= (\, 1 - \sum_{n=1}^{\infty} \mu_{n+p}\,)\, (z-w)^p + \sum_{n=1}^{\infty} \mu_{n+p}\, f_{n+p}(z) \\
&= \mu_p(p^{-k}(z-w)^p) + \sum_{n=1}^{\infty} \mu_{n+p}\, f_{n+p}(z) \\
&= \mu_p\, f_p(z) + \sum_{n=1}^{\infty} \mu_{n+p}\, f_{n+p}(z) = \sum_{n=0}^{\infty} \mu_{n+p}\, f_{n+p}(z). \;\square
\end{aligned}$$

We recall here the definition of fractional integral due to Owa[1].

Definition 2.4. [1]. *The fractional integral of order ν is defined by*

$$I_z^{-\nu} f(z) = \frac{1}{\Gamma(\nu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\nu}} d\zeta$$

where $0 < \nu$, f is an analytic function in a simply connected region of the z-plane containing the origin and the mutiplicity of $(z-\zeta)^{\nu-1}$ is removed by requiring $\log(z-\zeta)$ be real when $z > \zeta$.

After some calculations for a function $f(z) \in \mathcal{A}_w(p)$, we find that

$$I_z^{-\nu} f(z) = \tfrac{p^{-k}(z-w)^{\nu+p}}{\Gamma(\nu+p+1)} + \sum_{n=1}^{\infty} \tfrac{\Gamma(n+p+1)}{\Gamma(n+p+\nu+1)}\, a_n(z-w)^{n+p+\nu}.$$

The fractional integral bounds estimation for $f(z) \in \mathcal{S}\Omega_w^k(p,\, \gamma,\, \delta,\, \lambda)$ appear in in the following corollary.

Corollary 2.5. *Suppose* $f(z) \in \mathcal{S}\Omega_w^k(p,\, \gamma,\, \delta,\, \lambda)$ *for* $z \in \mathcal{U}_w$. *Then*

$$|I_z^{-\nu}(f(z))| \leq \frac{|z-w|^{p+\nu}}{\Gamma(p+\nu+1)}(p^{-k} + \frac{\Gamma(n+p+1)^2\, p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{\Gamma(n+p+\nu+1)(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}\,|\, z-w|) \quad (2.2)$$

and

$$|I_z^{-\nu}(f(z))| \geq \frac{|z-a|^{p+\nu}}{\Gamma(p+\nu+1)}(p^{-k} - \frac{\Gamma(n+p+1)^2\, p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{\Gamma(n+p+\nu+1)(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}\,|\, z-w|). \quad (2.3)$$

With the above lower and upper sharp bounds in (2.2)*and* (2.3)*respectively equality occurs for the function given by;*

$$f(z) = p^{-k}(z-w)^p + \frac{p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}\,(z-w)^{n+p}.$$

Proof. Since

$$I_z^{-\nu} f(z) = \frac{(z-w)^{p+\nu}}{p^k\Gamma(p+\nu+1)} + \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)}{\Gamma(n+p+\nu+1)}\, a_{n+p}(z-w)^{n+p+\nu},$$

$$(I_z^{-\nu} f(z)\,(z-w)^{-\nu}\Gamma(p+\nu+1)) = p^{-k}(z-w)^p + \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+\nu+1)}{\Gamma(n+p+\nu+1)}\, a_{n+p}\,(z-w)^{n+p},$$

Also,

$$|I_z^{-\nu} f(z)\,(z-w)^{-\nu}\Gamma(p+\nu+1)| \geq \left|p^{-k}(z-w)\right|^p - \sum_{n=1}^{\infty} |a_{n+p}| \frac{\Gamma(n+p+1)\Gamma(p+\nu+1)}{\Gamma(n+p+\nu+1)}\; |z-w|^{n+p}.$$

By Theorem 2.1,

$$|a_{n+p}\,| \leq \frac{p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}.$$

and,

$$|I_z^{-\nu} f(z)\,(z-w)^{-\nu}\Gamma(p+\nu+1)| \geq \left|p^{-k}(z-w)\right|^p - \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+\nu+1)p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{\Gamma(n+p+\nu+1)(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}\; |z-w|^{n+p}.$$

The sequence with general term $\phi(n) = \frac{\Gamma(p+\nu+1)\Gamma(n+p+1)p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{\Gamma(n+p+\nu+1)(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}$ is decreasing, since $0 < \phi(n) < 1$.

So,

$$|I_z^{-\nu} f(z)\,(z-w)^{-\nu}\Gamma(p+\nu+1)| \geq \left|p^{-k}(z-w)\right|^p - \frac{\Gamma(n+p+1)\Gamma(p+\nu+1)p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{\Gamma(n+p+\nu+1)(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}\; |z-w|^{n+1}.$$

thus,

$$|I_z^{-\nu} f(z)| \geq \frac{|z-w|^{p+\nu}}{\Gamma(p+\nu+1)} \left(1 - \frac{\Gamma(n+p+1)\Gamma(p+\nu+1)p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{\Gamma(n+p+\nu+1)(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}\,|z-w|\right).$$

Also,

$$|I_z^{-\nu} f(z)\,(z-w)^{-\nu}\Gamma(p+\nu+1)| \leq \left|p^{-k}(z-w)\right|^p + \sum_{n=1}^{\infty} |a_{n+p}| \frac{\Gamma(n+p+1)\Gamma(p+\nu+1)}{\Gamma(n+p+\nu+1)}\ |z-w|^{n+p}.$$

and

$$|I_z^{-\nu} f(z)\,(z-w)^{-\nu}\Gamma(p+\nu+1)| \\ \leq \left|p^{-k}(z-w)\right|^p + \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+\nu+1)p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{\Gamma(n+p+\nu+1)(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}\ |z-w|^{n+p}.$$

Hence

$$|I_z^{-\nu} f(z)| \leq \frac{|z-w|^{p+\nu}}{\Gamma(p+\nu+1)} \left(1 + \frac{\Gamma(n+p+1)\Gamma(p+\nu+1)p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{\Gamma(n+p+\nu+1)(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}\,|z-w|\right).$$

□

Under the hypotheses of Definition 2.4, the fractional derivative of order $k-\nu$ is defined by

$$I_z^{k-\nu} f(z) = \frac{d^k}{d^k z}\, I_z^{-\nu} f(z) = \frac{d^{-\nu}}{dz^{-\nu}}\, I_z^{k} f(z) = I_z^{k-\nu} f(z)$$

where $0 \leq \nu < 1$, and $k \in \mathbb{N}^* = \{0, 1, 2, ...\}$.

For example the fractional derivative operator $I_z^{k-\nu}$ for a real number ϵ the function $f(z) = \frac{(z-\epsilon i)^p}{1-z+\epsilon i}$ is defined in open disc $\mathcal{U}_{\epsilon i} = \{z : r = |z-\epsilon i| < 1\}$ is

$$I_z^{k-\nu} f(z) = \frac{d^k}{d^k z}\left(I_z^{-\nu}\left(\frac{(z-\epsilon i)^p}{1-z+\epsilon i}\right)\right) = \frac{d^n}{d^n z}\, I_z^{-\nu}\left((z-\epsilon i)^p + \sum_{n=1}^{\infty} a_{n+p}(z-\epsilon i)^{n+p}\right)$$

$$= \frac{d^k}{d^k z}\Big(\frac{\Gamma(p+1)}{\Gamma(p+\nu+1)}(z-\epsilon i)^{p+\nu} \\ + \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+\nu+1)p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{\Gamma(n+p+\nu+1)(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}(z-\epsilon i)^{n+p+\nu}\Big)$$

$$= \frac{\Gamma(p+1)}{\Gamma(p+\nu+1-k)}(z-\epsilon i)^{p+\nu-k} \\ + \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+\nu+1)p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{\Gamma(n+p+\nu+1)(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}(z-\epsilon i)^{n+p-k+\nu}$$

which implies

$$\left(\frac{\Gamma(p+\nu+1-k)}{\Gamma(p+1)}(z-\epsilon i)^{\nu+k} I_z^{k-\nu} f(z)\right) = (z-\epsilon i)^p +$$

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+\nu+1)p^{-k}(p+\lambda)(p\,\delta+p+\gamma)}{\Gamma(n+p+\nu+1)(n+p)^{1-k}[(n+p+\lambda)(1-\delta)-(1+\lambda)(p+\gamma)]}(z-\epsilon i)^{n+p}.$$

We conclude this article by stating that the functions within differential operators that arise in physical problems are generally nonlinear. Therefore the geometric function theory and conformal mappings provide a powerful tool to obtain solutions of these problems which were difficult to solve otherwise.

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