Finite groups in which nearly $S$-permutability is a transitive relation

Khaled Mustafa Aljamal$^1$, Ahmad Termimi Ab Ghani$^1$,
Khaled A. Al-Sharo$^2$

$^1$School of Informatics and Applied Mathematics
Universiti Malaysia Terengganu
21030 Kuala Nerus, Terengganu, Malaysia

$^2$Department of Mathematics
Faculty of Science
Al al-Bayt University
Al-Mafraq, Jordan

e-mail: khaled_aljammal@yahoo.com, termimi@umt.edu.my,
sharo_kh@yahoo.com

(Received February 3, 2019, Accepted March 1, 2019)

Abstract

A subgroup $H$ of $G$ is called nearly $S$-permutable in $G$ if for every prime $p$ such that $(p, |H|) = 1$ and for every subgroup $K$ of $G$ containing $H$ the normalizer $N_K(H)$ contains some Sylow $p$-subgroup of $K$. A group $G$ is called an NSPT-group if nearly $S$-permutability is a transitive relation in $G$. A number of new characterizations of finite solvable NSPT-groups are given.

1 Introduction

Throughout this paper, we assume that all groups considered are finite. Our notation is standard and consistent with [3]. We introduce and study the class NSPT-group; that is, the class of all finite groups in which nearly $S$-permutability is a transitive relation. More specifically, we are interested in

Key words and phrases: solvable group, Sylow subgroup, permutable subgroup, nearly $S$-permutable subgroup.

AMS (MOS) Subject Classifications: 20D10, 20D20, 20D35.

ISSN 1814-0432, 2019, http://ijmcs.future-in-tech.net
subgroups, homomorphic images, and direct product of NSPT-groups and other related algebraic properties.

A subgroup $H$ of a group $G$ is said to permute with a subgroup $K$ if $HK$ is a subgroup of $G$. $H$ is said to be permutable in $G$ (or $S$-permutable) if it permutes with all the (Sylow) subgroups of $G$. One of the earliest results about permutable subgroups [7] states that every permutable subgroup... In [6] Kegel proved that $S$-permutable subgroups are necessarily subnormal. Actually a result stronger than permutable subgroups is that of subnormal subgroups. For a subgroup $H$ of $G$, it is enough to know that $H$ permutes with all of its conjugates to deduce that $H$ is subnormal (see [5], p. 50). Nearly $S$-permutability have been introduced and studied in [2], while permutable and $S$-permutable subgroups are subnormal while a nearly $S$-permutable subgroup need not be subnormal in general. As an example one might consider $D_{18}$ the dihedral group of order 18 which has three subgroups of order 6 and each of these subgroups will be nearly $S$-permutable but not subnormal. There has been several approaches to apply normality, permutability and $S$-permutability in the study of some finite group classes. To make our point clear we list some helping notations $H_{s-per}G$ denotes $H$ is $S$-permutable in $G$, $H_{per}G$ denotes $H$ is permutable in $G$, and $H_{nsp}G$ denotes $H$ is nearly $S$-permutable in $G$. Let $\rho$ be any of the properties {normality, permutability, Sylow permutability}. The property $\rho$ is said to be transitive in a group $G$ if for any two subgroups $A$ and $B$ in $G$ the relations $A \rho B$, and $B \rho G$ always implies $A \rho G$. By $T$-groups, $PT$-groups, and $PST$-groups we denote the class of groups in which normality, (rept. permutability, and $S$-permutability) is transitive relation.

One of the generalizations for $S$-permutable subgroups we are interested in is the nearly $S$-permutable subgroups which was introduced in [2]. In this paper we introduce and study finite groups in which nearly $S$-permutability is transitive relation. Our main object is to determine some properties of finite solvable NSPT-groups.

We first consider the following definition:

**Definition 1.1.** (Al-Sharo [2] ) A subgroup $H$ of $G$ is called nearly $S$-permutable in $G$ if for every prime $p$ such that $(p, |H|) = 1$ and for every subgroup $K$ of $G$ containing $H$ the normalizer $N_K(H)$ contains some Sylow $p$-subgroup of $K$. We shall write $H_{nsp}G$ to denote that $H$ is nearly $S$-permutable in $G$.

Motivated by the theory of $T$-groups (resp. $PT$-groups, $PST$-groups), we introduce the class of NSPT-groups.
Definition 1.2. A group $G$ is called NSPT-group if nearly $S$-permutability is a transitive relation in $G$. That is, $G$ is NSPT-group if for all subgroups $H$ and $K$ where $H \ nsp \ K \ nsp \ G$ we have $H \ nsp \ G$.

Since the concept of NSPT-groups relies on nearly $S$-permutability we fix some elementary properties of nearly $S$-permutable subgroups that follows directly from the definition.

Remark 1.3. Let $H$ be a nearly $S$-permutable subgroup of a group $G$. If $K$ is any subgroup of $G$ such that $H \leq K \leq G$. Then $H$ is nearly $S$-permutable in $K$.

Remark 1.4. If $G$ is $p$-group then every subgroup of $G$ is nearly $S$-permutable in $G$.

Remark 1.5. Let us denote by $\pi(G)$ the set of all prime divisors of the group $G$. If $H$ is subgroup of $G$ such that $\pi(H) = \pi(G)$ then $H$ is nearly $S$-permutable in $G$.

It is clear that the class $\mathfrak{A}$ of abelian groups is an example of NSPT-groups. Moreover, $\mathfrak{M}_p$-the class of all $p$-groups is another example of NSPT-group. One of the nice facts about the class of NSPT-groups is the following:

Theorem A. Every nilpotent group is NSPT-group.

As an example of non NSPT-group we have the following:

Example 1. Consider the alternating group $A_4 = \langle (1, 2, 3), (1, 2)(3, 4) \rangle$ of order 12. If we take $H = \langle (1, 3)(2, 4), (1, 2)(3, 4) \rangle$ to be the 2-Sylow subgroup of $A_4$ then $H$ is normal in $A_4$. Hence, $H$ is nearly $S$-permutable in $A_4$. Now $H$ being a group of order 4 then $H$ must abelian and every subgroup of $H$ would be norm in $H$. Let us take $K = \langle (1, 3)(2, 4) \rangle$. Then $K \ nsp \ H \ nsp \ A_4$ but $K$ is not $nsp \ A_4$. To see this note that $K \leq A_4$, and $g, c, d, (3, |K|) = 1$, and $N_{A_4}(K) = H$ which doesn’t contain any 3-Sylow subgroup of $A_4$. Therefore, $K$ is not $nsp \ A_4$ and $A_4$ is not NSPT-group.

In the years 1953, 1964, and 1975, Gaschütz, Zacher, and Agrawal, respectively, proved the following definitive results on solvable $T$-groups, $PT$-groups, and $PST$-groups.

Theorem 1. (Gaschütz [4], Zacher [10], Agrawal [1]) A solvable $T$-group ($PT$-group, $PST$-group) is supersolvable.
The next theorem gives a similar result for NSPT-groups.

**Theorem B.** A solvable NSPT-group is supersolvable.

**Theorem C.** If $G_1$ and $G_2$ are two NSPT-groups and $(|G_1|, |G_2|) = 1$, then $G = G_1 \times G_2$ is also a NSPT-group.

**Remark 1.6.** In Theorem C., the condition that $(|G_1|, |G_2|) = 1$ is necessary. The following example shows this.

**Example 2.** Let $C_3 = \langle c : c^3 = e \rangle$ be the cyclic group of order 3 and $S_3 = \langle a, b : a^3 = b^2 = (ba)^2 = 1 \rangle$ be the symmetric group on 3 letters. Then $C_3$ -being abelian group- must be NSPT-group. In the group $S_3$ the only nearly S-permutable subgroups are $\langle e \rangle$, $A_3 = \langle a \rangle$, and $S_3$ itself. Hence $S_3$ is also a NSPT-group. Let $D = S_3 \times C_3$. We show that $D$ is not NSPT-group. Let $B = \langle (a, e), (e, c) \rangle \cong A_3 \times C_3$, $B \in Syl_3(D)$. Then $B$ has order $3^2$ and index 2 in $D$. Therefore, $B$ is abelian normal subgroup in $D$. The normality of $B$ in $D$ implies that $B$ is nearly S-permutable in $D$. The fact that $B$ is abelian implies that every subgroup of $B$ is nearly $S$-permutable in $B$. In particular if we pick $A = \langle (a, c) \rangle$ then $A$ nsp $B$, and $B$ nsp $D$, but $A$ is not nsp $D$. To see this we consider $A \leq G$ with $p = 2$ for which $(2, |A|) = 1$. Then the 2-Sylow subgroup of $D$ has order 2 and is not contained in $N_D(A)$ and $A$ is not nsp $D$. Hence $D$ is not NSPT-group.

# 2 Preliminaries

In this section we list some results that are interesting in their own and some of which will be used in the proofs of the given theorem.

**Lemma 2.1.** (Frattini Argument, see [5, Lemma 1.13.]) Let $N \trianglelefteq G$ and suppose that $P \in Sul_p(N)$. Then $G = N_G(P)N$.

**Lemma 2.2.** Let $G_1$ and $G_2$ be two groups such that $(|G_1|, |G_2|) = 1$, and $G = G_1 \times G_2$. Then the following statements are true:

1) every $p$-Sylow subgroup of $G$ is isomorphic to a $p$-Sylow either of $G_1$ or $G_2$.

2) for any subgroup $H \leq G$ we have and $N_G(H) \cong N_{G_1}(H_1) \times N_{G_2}(H_2)$. Where $H_i \leq G_i$, for $i \in \{1, 2\}$.
Proof. 1) Since \( G_1 \cap G_2 = 1 \), and \( G_i \trianglelefteq G \). Then \( G \simeq G_1 G_2 \). Now if \( P \in \text{Syl}_p(G) \) and \( P_i \in \text{Syl}_p(G_i) \) then \( P_i = P \cap G_i \), for \( i \in \{1, 2\} \) and 1) follows.

2) The second part of this statement follows from the first, and the first follows from \((|G_1|, |G_2|) = 1\).

Lemma 2.3. Let \( H \) be a subgroup of \( G \). If \( H \) is \( S \)-permutable in \( G \) then \( H \) is nearly \( S \)-permutable in \( G \).

Proof. Let \( H \) be nearly \( S \)-permutable in \( G \). If \( \pi(G) = \pi(H) \) then by Remark 1.5 \( H \) will be nearly \( S \)-permutable in \( G \). So we may assume that \( \pi(G) \setminus \pi(H) \neq \emptyset \). Let \( p \) be any prime in \( \pi(G) \setminus \pi(H) \). For any subgroup \( K \) of \( G \) such that \( H \leq K \leq G \) we pick \( P_K \in \text{Syl}_p(K) \). Since \( H \) is \( S \)-per \( G \), then \( H \) is \( S \)-per \( K \) and therefore \( HP_K \) is a subgroup of \( K \). Therefor, \( H \) is \( S \)-permutable in \( HP_K \). So, \( H \) is subnormal in \( HP_K \). Note that \( g.c.d.(|H|, |HP_K : H|) = 1 \). That is \( H \) is a subnormal Hall subgroup of \( HP_K \). Hence, \( H \) is normal in \( HP_K \). Therefore, \( P_K \leq N_K(H) \) and \( H \) is nearly \( S \)-permutable in \( G \).

Lemma 2.4. ([2, Lemma 2.2]) Let \( G \) be a group, \( H \leq G \) and \( N \) be a normal subgroup of \( G \).

(1) If \( H \) is nearly \( S \)-permutable in \( G \), then \( HN \) is nearly \( S \)-permutable in \( G \).

(2) If \( H \) is nearly \( S \)-permutable in \( G \) and \( H \) is a group of prime power order, then \( H \cap N \) is nearly \( S \)-permutable in \( G \).

(3) If \( H \) is nearly \( S \)-permutable in \( G \) and \( H \) is a group of prime power order, then \( HN/N \) is nearly \( S \)-permutable in \( G/N \) for any normal subgroup \( N \) of \( G \).

(4) If \( H \) is nearly \( S \)-permutable in \( G \) and \( |H| = p^n \) for some prime \( p \), then \( H \leq O_p(G) \).

3 The Proofs

Proof of Theorem A. Let \( G \) be a nilpotent group then every \( p \)-Sylow subgroup of \( G \) is normal in \( G \). Therefore, every \( p \)-Sylow subgroup permutes with every subgroup of \( G \). That means every subgroup of \( G \) is \( S \)-permutable in \( G \). By Lemma 2.3 every subgroup of \( G \) is nearly \( s \)-permutable in \( G \). Hence, \( G \) is \( NSPT \)-group.
Proof of Theorem B. Let $G$ be a solvable group we prove that if $G$ is not supersolvable then $G$ is not $NSPT$-group. Since $G$ is solvable and not supersolvable group then $G$ has a chief factor $A/B$ which is elementary abelian of order $p^k$ for some prime $p$ and an integer $k > 1$. Let $P$ be a $p$-Sylow subgroup of $A$. From the Frattini argument (see Lemma 2.1) we get: $G = AN_G(P)$. Since $P/B$ is normal in $A/B$ then $P$ is normal in $A$, consequently $P$ is characteristic in $A$. Now $P$ char $A \trianglelefteq G$ implies $P$ normal in $G$. Hence $N_G(P) = G$. Let $Q$ be a $p$-subgroup such that $PB \leq Q \leq P$ such that $Q/B$ is in the center of a $p$-Sylow subgroup of $G/B$. First, we show that $Q$ is not nearly $S$-permutable in $G$.

Since $A/B$ is a chief factor of $G$, $Q$ can not be normal in $G$. Since, $N_G(Q) \geq P \in Syl_p(G)$ and $N_G(Q) \neq G$ then there is a prime $r \neq p$ and an $r$-element $g \in G$ such that $g \notin N_G(Q)$. Let $K = \langle P, g \rangle$. So, $|K| = |P|r^t$ where $g$ has order $r^t$. So, $N_K(Q)$ does not contain an $r$-Sylow subgroup $R$ of $K$. That is $Q$ is not nearly $S$-permutable in $G$. Hence, we have: $Q$ nsp $P$, $P$ nsp $G$, but $Q$ is not nsp $G$. Therefore $G$ is not $NSPT$-group. Contradiction, and the theorem follows.

Proof of Theorem C. Let $G_1$ and $G_2$ be two $NSPT$-groups with $(|G_1|, |G_2|) = 1$, and set $G = G_1 \times G_2$. Let $A \leq B$ be two subgroups of $G$ such that $A$ is nearly $S$-permutable in $B$, and $B$ is nearly $S$-permutable in $G$. Assume that $A$ is not nsp $G$. Then there exists a subgroup $K$ of $G$ such that $A \leq K \leq G$ and for some prime number $p$ with $(p, |A|) = 1$ some $p$-Sylow subgroup $P$ of $K$ is not in $N_K(A)$. From Lemma 2.2 (1) $P = P_1P_2$ where $P_i \leq G_i$ for $i \in \{1, 2\}$. From $(|G_1|, |G_2|) = 1$ one of the the $P_i$’s must be 1. So, we may assume that $p \in \pi(G_1)$. It is clear that all the proof’s assumptions may be translated to the group $G_1$. Again, by Lemma 2.2 (2) and the assumption $(|G_1|, |G_2|) = 1$ we get: $A = A_1A_2$ nsp $B = B_1B_2$ nsp $G = G_1 \times G_2$ implies $A_1$ nsp $B_1$ nsp $G_1$ and $A_1$ is not nsp $G_1$. Contradiction with the assumption that $G_1$ is $NSPT$-group and the theorem is proved.

\[\square\]
References


