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Roughness in Generalized (m, n) Bi-ideals in Ordered LA-Semigroups

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Abstract

In this paper, generalized rough (m, n) ordered ideals (resp., quasi-ideals, bi-ideals and interior ideals) have been defined in ordered LA-semigroups by means of a new type of relation called pseudoorder of relations. Properties based on them have been shown. It is proved that by using pseudoorder of relations, generalized m-left, n-right and (m, n) ordered (resp., quasi-, bi-, and interior)-ideals in ordered LA-semigroups S becomes generalized lower and upper rough m-left, n-right ordered ideals and generalized (m, n) ordered (resp., quasi-, bi-, and interior)-ideals of S.

1 Introduction

The notion of rough sets was introduced by Pawlak in [24]. The rough set theory has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy or incomplete information. In connection with algebraic structures, Biswas and Nanda [10] introduced the

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notion of rough subgroups, whereas Kuroki [20] introduced it for semigroups. Rough prime (m, n) bi-ideals in semigroups was investigated by Yaqoob et. al [31] and studied in case of rough fuzzy prime bi-ideals in semigroups [30]. Aslam et. al [9] presented some results on roughness in semigroups. Xiao and Zhang [29] studied rough prime ideals and rough fuzzy prime ideals in semigroups. Notes on (m, n) bi-Γ-ideals in Γ-semigroups was introduced by Moin and Rais in [4] where authors studied properties of (m-left, n-right, quasi and bi)-Γ-ideals in case of Γ-semigroups whereas rough (m, n) quasi-ideals in semigroups was introduced by Moin and Rais in [5]. Further Moin and Rais [6] defined rough (m, n) quasi-Γ-ideals in Γ-semigroups. generalized (m, n) bi-ideals in case of semigroups with involution was introduced by Moin et. al [7] whereas (m, n) quasi-ideals in semigroups was defined by Moin et. al [8].

The concept of an AG-groupoid was first given by Kazim and Naseeruddin [15] in 1972 and they called it left almost semigroups (LA-semigroups). Holgate [14] called LA-semigroup to left invertive groupoid. In some direction of fuzziness ordered AG-groupoids has been studied by Faisal et al.[12]. Ordered LA-semigroup has been taken under consideration in terms of interval valued fuzzy ideals by Asghar Khan et al.[16]. An LA-semigroup is a groupoid having the left invertive law

$$(ab)c = (cb)a, \text{ for all } a,\ b,\ c \in S.$$

In an LA-semigroup [15], the medial law holds

$$(ab)(cd) = (ac)(bd), \text{ for all } a,\ b,\ c,\ d \in S.$$

An LA-semigroup with right identity becomes a commutative monoid [22]. The connection of a commutative inverse semigroup with an LA-semigroup has been given in [23] as, a commutative inverse semigroup $(S, \circ)$ becomes an LA-semigroup $(S, \cdot)$ under $a \cdot b = b \circ a^{-1}$, for all $a, b \in S$. A commutative semigroup with identity comes from LA-semigroup by the use of a right identity. The concept of an ordered LA-semigroup was introduced by Shah et. al [28] and further it was extended to the theory of fuzzy sets in ordered LA-semigroups [18]. Generalized roughness in $(\in, \in \vee qk)$ have been studied by Muhammad et. al [1]. Recently, generalized roughness in LA-Semigroups was studied by Noor et. al [25]. Fuzzy (2, 2)-regular ordered Γ-AG∗∗-Groupoids is investigates and studied by Faisal et. al [13]. Generalized roughness in ordered semigroups is studied by Moin [2] recently whereas T-roughness and its ideals in ternary semigroups were introduced in [3].

We prove that generalized m-left, n-right, (m, n)-(quasi-, bi-, interior)-ordered ideals of ordered LA-semigroup S is the generalized rough m-left, n-right, (m, n)-(quasi-, bi-, interior)-ordered ideals. By using pseudoorder of relations, it is proved that generalized m-left, n-right ordered ideals and (m, n) ordered (resp., quasi-, bi-, and interior)-ideals in ordered LA-semigroups S becomes generalized lower and upper rough m-left, n-right ordered ideals and generalized (m, n) (resp., quasi-, bi-, and interior)-ideals of S.

2 Preliminaries and Basic Definitions

Definition 2.1. [18] *An ordered LA-semigroup (po-LA-semigroup) is a structure $(S, ., \leq)$ in which the following conditions hold:*
(i) $(S, .)$ *is an LA-semigroup.*
(ii) $(S, \leq)$ *is a poset (reflexive, anti-symmetric and transitive).*
(iii) *for all a, b and $x \in S$, $a \leq b$ implies $ax \leq bx$ and $xa \leq xb$.*

Example 2.2. [18] *Consider an open interval $\mathbb{R}_{\mathbb{O}} = (0, 1)$ of real numbers under the binary operation of multiplication. Define $a * b = ba^{-1}r^{-1}$, for all $a, b, r \in \mathbb{R}_{\mathbb{O}}$, then it is easy to see that $(\mathbb{R}_{\mathbb{O}}, *, \leq)$ is an ordered LA-semigroup under the usual order "$\leq$" and we have called it a real ordered LA-semigroup.*

Definition 2.3. *A non-empty subset A of an ordered LA-semigroup S, is called an LA-subsemigroup of S if $A^2 \subseteq A$.*

For a non-empty subset A of an ordered LA semigroup S, we define

$$(A] = \{t \in S \mid t \leq a, \text{ for some } a \in A\}.$$

For $A = \{a\}$, we shall write $(a]$.

Definition 2.4. *A non-empty subset A of an ordered LA semigroup S, is called m-left ordered generalized ideals of S (resp. n-right ordered generalized ideals of S) if*
(i) $A^m S \subseteq A$ (resp. $SA^n \subseteq A$)*;*
(ii) *$a \in A$ and $b \in S, b \leq a \Rightarrow b \in A$.*

Equivalently, $(A^m S] \subseteq A$ (resp.$\mathrm{A^n} \subseteq \mathrm{A}$]. Here m and n are non-negative integers.

Definition 2.5. *A non-empty subset A of an ordered LA semigroup S is called (m, n) ordered generalized quasi-ideal of S if*
(i) $A^m S \cap SA^n \subseteq A$*;*
(ii) *$a \in A$ and $b \in S, b \leq a \Rightarrow b \in A$.*

Definition 2.6. *Let A be a non-empty subset of an ordered LA semigroup S then A is called (m,n) ordered generalized bi-ideal of A if*
(i) $A^m S A^n \subseteq A$.
(ii) $a \in A$ *and* $b \in S, b \leq a \Rightarrow b \in A$.

Every m-left ordered generalized ideal and n-right ordered ideal in ordered semigroup S is an (m,n)-bi-ideal of S where A^0 is defined as $A^0 S A^n = S A^n = S$ when $m = 0$ and $A^m S A^0 = A^m S = S$ when $n = 0$.

Definition 2.7. *A non-empty subset A of an ordered LA-semigroup S is called an ordered generalized interior (m,n)-ideal of S if*
(i) $S^m A S^n \subseteq A$.
(ii) *If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.*

A becomes m-left or n-right ideals of S if it is a subsemigroup of S. The same is true for all kind of ideals (quasi-, bi-, interior)-ideals in S. For the sake of convenience we write ideals in lie of generalized ideals.

Definition 2.8. *Let S be an ordered LA-semigroup. A non-empty subset A of S is called a prime ideal if $xy \in A$ implies $x \in A$ or $y \in A$ for all $x, y \in S$. Let A be an ideal of S. If A is prime subset of S, then A is called prime-ideal.*

Definition 2.9. *A relation θ on an ordered LA-semigroup S is called a pseudoorder if*
(1) $\leq \subseteq \theta$
(2) *θ is transitive, that is $(a,b),(b,c) \in \theta$ implies $(a,c) \in \theta$ for all $a,b,c \in S$.*
(3) *θ is compatible, that is if $(a,b) \in \theta$ then $(ax,bx) \in \theta$ and $(xa,xb) \in \theta$ for all $a,b,x \in S$.*

An equivalence relation θ on an ordered LA-semigroup S is called a congruence relation if $(a,b) \in \theta$, then $(ax,bx) \in \theta$ and $(xa,xb) \in \theta$, for all $a,b,x \in S$.

A congruence θ on S is called complete if $[a]_\theta [b]_\theta = [ab]_\theta$ for all $a, b \in S$ and $[a]_\theta$ is the congruence class containing the element $a \in S$.

3 Generalized rough subsets in ordered LA-semigroups

Let X be a non-empty set and θ be a binary relation on X. By $\wp(X)$ we mean the power set of X. For all $A \subseteq X$, we define θ_- and $\theta_+ : \wp(X) \longrightarrow \wp(X)$ by

$$\theta_-(A) = \{x \in X : \forall\ y,\ x\theta y \Rightarrow y \in A\} = \{x \in X : \theta N(x) \subseteq A\},$$

and

$$\theta_+(A) = \{x \in X : \exists\ y \in A, \text{ such that } x\theta y\} = \{x \in X : \theta N(x) \cap A \neq \emptyset\}.$$

Where $\theta N(x) = \{y \in X : x\theta y\}$. $\theta_-(A)$ and $\theta_+(A)$ are called the lower approximation and the upper approximation operations, respectively [19].

Example 3.1. *Let* $X = \{a, b, c\}$ *and* $\theta = \{(a, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$. *Then* $\theta N(a) = \{a\}$; $\theta N(b) = \{b, c\}$; $\theta N(c) = \{a, b, c\}$; $\theta_-(\{a\}) = \{a\}$; $\theta_-(\{b\}) = \phi$; $\theta_-(\{c\}) = \phi$; $\theta_-(\{a, b\}) = \{a\}$; $\theta_-(\{a, c\}) = \{a\}$; $\theta_-(\{b, c\}) = \{b\}$; $\theta_-(\{a, b, c\}) = \{a, b, c\}$; $\theta_+(\{a\}) = \{a, c\}$; $\theta_+(\{b\}) = \{b, c\}$; $\theta_+(\{c\}) = \{b, c\}$; $\theta_+(\{a, b\}) = \{a, b, c\}$; $\theta_+(\{a, c\}) = \{a, b, c\}$; $\theta_+(\{b, c\}) = \{b, c\}$; $\theta_+(\{a, b, c\}) = \{a, b, c\}$.

Theorem 3.2. [24] *Let* θ *and* λ *be relations on* X. *If* A *and* B *are non-empty subsets of* S. *Then the following hold:*
(1) $\theta_-(X) = X = \theta_+(X)$;
(2) $\theta_-(\emptyset) = \emptyset = \theta_+(\emptyset)$;
(3) $\theta_-(A) \subseteq A \subseteq\ \theta_+(A)$;
(4) $\theta_+(A \cup B) = \theta_+(A) \cup \theta_+(B)$;
(5) $\theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B)$;
(6) $A \subseteq B$ *implies* $\theta_-(A) \subseteq \theta_-(B)$;
(7) $A \subseteq B$ *implies* $\theta_+(A) \subseteq \theta_+(B)$;
(8) $\theta_-(A \cup B) \supseteq \theta_-(A) \cup \theta_-(B)$;
(9) $\theta_+(A \cap B) \subseteq \theta_+(A) \cap \theta_+(B)$.

Definition 3.3. *Let* θ *be a pseudoorder on an ordered LA-semigroup* S *and* A *be a non-empty subset of* S. *Then the sets*

$$\theta_-(A) = \{x \in S : \forall\ y,\ x\theta y \Rightarrow y \in A\} = \{x \in S : \theta N(x) \subseteq A\},$$

and

$$\theta_+(A) = \{x \in S : \exists\ y \in A,\ \textit{such that } x\theta y\} = \{x \in S : \theta N(x) \cap A \neq \emptyset\}.$$

are called the θ*-lower approximation and the* θ*-upper approximation of* A.

For a non-empty subset A of S, $\theta(A) = (\theta_-(A), \theta_+(A))$ is called a rough set with respect to θ if $\theta_-(A)$ and $\theta_+(A)$ are not same.

Example 3.4. *Consider* $S = \{1, 2, 3, 4, 5\}$ *with the following operation* ”.” *and the order* ” $\leq$ ” :

.	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	4	5	3
4	1	2	5	3	4
5	1	2	3	4	5

$$\leq := \{(1,1),(2,2),(2,3),(2,4),(2,5),(3,3),(4,4),(5,5)\}.$$

We give the covering relation " $\prec$ " of S as follows:

$$\prec := \{(2,3),(2,4),(2,5)\}$$

Hence S is an ordered LA-semigroup because the elements of S satisfies left invertive law.

Now let

$$\theta = \{(1,1),(1,4),(2,2),(2,3),(2,4),(2,5),(3,3),(4,4),(5,3),(5,4),(5,5)\}$$

be a complete pseudoorder on S, such that

$$\theta N(1) = \{1,4\},\ \theta N(2) = \{2,3,4,5\} \text{ and } \theta N(3) = \{3\},\ \theta N(4) = \{4\},\ \theta N(5) = \{3,4,5\}.$$

Now for $A = \{1,2,4\} \subseteq S$,

$$\theta_-(\{1,2,4\}) = \{1,4\} \text{ and } \theta_+(\{1,2,4\}) = \{1,2,3,4,5\}.$$

So, $\theta_-(\{1,2,4\})$ is θ-lower approximation of A and $\theta_+(\{1,2,4\})$ is θ-upper approximation of A.

For a non-empty subset A of S, $\theta(A) = (\theta_-(A), \theta_+(A))$ is called a rough set with respect to θ if $\theta_-(A) \neq \theta_+(A)$.

Lemma 3.5. *If $A \subseteq B \subseteq S$, then $\theta_-(A) \subseteq \theta_-(B)$ and $\theta_+(A) \subseteq \theta_+(B)$.*

Proof. Let $x \in \theta_-(A)$. Then $\theta N(x) \subseteq A \subseteq B$. Thus $x \in \theta_-(B)$ and $\theta_-(A) \subseteq \theta_-(B)$. If $y \in \theta_+(A)$, then $\theta N(y) \cap A \neq \emptyset$. Since $A \subseteq B$, $\theta N(y) \cap B \neq \emptyset$ and so $y \in \theta_+(B)$.
Hence, $\theta_+(A) \subseteq \theta_+(B)$. $\square$

Theorem 3.6. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A and B are non-empty subsets of S, then $\theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B)$.*

Proof Let $a \in \theta_-\ (A \cap B)$. Then $\theta N(a) \subseteq A \cap B$. So

$\theta N(a) \subseteq A, \theta N(a) \subseteq B \Longleftrightarrow a \in \theta_-(A) \cap \theta_-(B)\theta_-(A \cap B) = \theta_-(A) \cap \theta_-(B).$

□

Theorem 3.7. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A and B are non-empty subsets of S. Then*

$$\theta_+(A)\theta_+(B) \subseteq \theta_+(AB).$$

Proof. Let z be any element of $\theta_+(A)\theta_+(B)$. Then $z = xy$ where $x \in \theta_+(A)$ and $y \in \theta_+(B)$. Thus there exist elements l, $m \in S$ such that

$$l \in A \text{ and } x\theta l \ ; m \in B \text{ and } y\theta m.$$

Since θ is a pseudoorder on S, so $xy\theta lm$. As $ab \in AB$, so we have

$$z = xy \in \theta_+(AB).$$

Thus $\theta_+(A)\theta_+(B) \subseteq \theta_+(AB)$. □

Definition 3.8. *Let θ be a pseudoorder on an ordered LA-semigroup S, then for each $x, y \in S$ $\theta N(x)\theta N(y) \subseteq \theta N(xy)$. If*

$$\theta N(x)\theta N(y) = \theta N(xy),$$

then θ is called complete pseudoorder.

Theorem 3.9. *Let θ be pseudoorder on an ordered LA-Γ-semigroup S. Then for a non-empty subset A of S*

(1) $(\theta_+(A))^n \subseteq \theta_+(A^n)\ \forall n \in N$.

(2) *If θ is complete, then $(\theta_-(A))^n \subseteq \theta_-(A^n)\ \forall n \in N$.*

Theorem 3.10. *Let θ be a complete pseudoorder on an ordered LA-semigroup S. If A and B are non-empty subsets of S. Then*

$$\theta_-(A)\theta_-(B) \subseteq \theta_-(AB).$$

Proof. Let z be any element of $\theta_-(A)\theta_-(B)$. Then $z = xy$ where $x \in \theta_-(A)$ and $y \in \theta_-(B)$. Thus we have $\theta N(x) \subseteq A$ and $\theta N(y) \subseteq B$. Since θ is complete pseudoorder on S, so we have

$$\theta N(xy) = \theta N(x)\theta N(y) \subseteq AB,$$

which implies that $xy \in \theta_-(AB)$. Thus $\theta_-(A)\theta_-(B) \subseteq \theta_-(AB)$. □

Theorem 3.11. *Let θ and λ be pseudoorders on an ordered LA-semigroup S and A be a non-empty subset of S. Then for any $m \in \mathbb{N}$*

$$(\theta \cap \lambda)_+(A^m) \subseteq \theta_+(A^m) \cap \lambda_+(A^m).$$

Proof. The proof is straightforward. □

Theorem 3.12. *Let θ and λ be pseudoorders on an ordered LA-semigroup S and A be a non-empty subset of S. Then for any $n \in \mathbb{N}$*

$$(\ \theta \cap \lambda)_-(A^n) = \theta_-(A^n) \cap \lambda_-(A^n).$$

Proof. The proof is straightforward. □

4 Generalized ordered rough (m, n)-(quasi-, bi-, interier)-ideals in ordered LA-semigroups

Definition 4.1. *Let θ be a pseudoorder on an ordered LA-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) rough LA-subsemigroup of S if $\theta_+(A)$ (resp., $\theta_-(A)$) is an LA-subsemigroup of S.*

Theorem 4.2. *Let θ be a pseudoorder on an ordered LA-semigroup S and A be an LA-subsemigroup of S. Then*

(1) *$\theta_+(A)$ is an LA-subsemigroup of S.*

(2) *If θ is complete, then $\theta_-(A)$ is, if it is non-empty, an LA-subsemigroup of S.*

Proof. (1) Let A be an LA-subsemigroup of S. Then by Theorem 3.2(3),

$$\emptyset \neq A \subseteq \theta_+(A).$$

By Theorem 3.2(7) and Theorem 3.7, we have

$$\theta_+(A)\theta_+(A) \subseteq \theta_+(A^2) \subseteq \theta_+(A).$$

Thus $\theta_+(A)$ is an LA-subsemigroup of S, that is, A is a θ-upper rough LA-subsemigroup of S.

(2) Let A be an LA-subsemigroup of S. Then by Theorem 3.2(6) and Theorem 3.10, we have

$$\theta_-(A)\theta_-(A) \subseteq \theta_-(A^2) \subseteq \theta_-(A).$$

Thus $\theta_-(A)$ is, if it is non-empty, an LA-subsemigroup of S, that is, A is a θ-lower rough LA-subsemigroup of S. $\square$

The following example shows that the converse of above theorem does not hold.

Example 4.3. *We consider a set $S = \{1, 2, 3, 4, 5\}$ with the following operation "." and the order "$\leq$" :*

.	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	4	5	3
4	1	2	3	4	5
5	1	2	5	3	4

$$\leq := \{(1, 1), (1, 2), (2, 2), (2, 4), (3, 3), (4, 4), (5, 5)\}.$$

We give the covering relation "$\prec$" of S as follows:

$$\prec := \{(1, 2)\}$$

Here S is not an ordered semigroup because $3 = 3 \cdot (4 \cdot 5) \neq (3 \cdot 4) \cdot 5 = 4$. But the elements of S satisfies left invertive law. Hence S is an ordered LA-semigroup.

Now let

$$\theta = \{1, 1), (1, 2), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$$

be a complete pseudoorder on S, such that

$$\theta N(1) = \{1, 2\},\ \theta N(2) = \{2\} \text{ and } \theta N(3) = \theta N(4) = \theta N(5) = \{3, 4, 5\}.$$

Now for $\{1, 2, 3\} \subseteq S$,

$$\theta_-(\{1, 2, 3\}) = \{1, 2\} \text{ and } \theta_+(\{1, 2, 3\}) = \{1, 2, 3, 4, 5\}.$$

It is clear that $\theta_-(\{1, 2, 3\})$ and $\theta_+(\{1, 2, 3\})$ are both LA-subsemigroups of S but $\{1, 2, 3\}$ is not an LA-subsemigroup of S.

Definition 4.4. *Let θ be a pseudoorder on an ordered LA-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) ordered rough m-left ideal of S if $\theta_+(A)$ (resp., $\theta_-(A)$) is an ordered m-left ideal of S.*

Similarly we can define θ-upper, θ-lower ordered rough n-right ideal and θ-upper, θ-lower ordered rough (m, n) ideals of S.

Theorem 4.5. *Let θ be a pseudoorder on an ordered LA-semigroup S and A be an ordered m-left* (n-right, (m, n)) *ideal of S. Then*

(1) $\theta_+(A)$ *is an ordered m-left* (n-right, (m, n)-bi)-ideals of S.

(2) If θ is complete, then $\theta_-(A)$ is, if it is non-empty, a ordered m-left (n-right, (m, n)-bi)-ideal of S.

Proof. (1) Let A be a ordered m-left ideal of S. By Theorem 3.2(1), $\theta_+(S) = S$.

(i) Now by Theorem 3.7, we have

$$S^m\theta_+(A) = \theta_+(S^m)\theta_+(A) \subseteq \theta_+(S^m A) \subseteq \theta_+(A).$$

(ii) Let $a \in \theta_+(A)$ and $b \in S$ such that $b \leq a$. Then there exist $y \in A$, such that $a\theta y$ and $b\theta a$. Since θ is transitive, so $b\theta y$ implies $b \in \theta_+(A)$.

This proves that $\theta_+(A)$ is an ordered m-left-ideal of S, that is, A is a generalized θ-upper ordered rough m-left-ideal of S. In the similar fashion we can show that generalized θ-upper approximation of an n-right ((m, n)-bi-)-ideal of S is an n-right ((m, n)-bi-)-ideal of S.

(2) Let A be a ordered m-left ideal of S. By Theorem 3.2(1), $\theta_-(S) = S$.

(i) Now by Theorem 3.10, we have

$$S^m\theta_-(A) = \theta_-(S^m)\theta_-(A) \subseteq \theta_-(S^m A) \subseteq \theta_-(A).$$

(ii) Let $a \in \theta_-(A)$ and $b \in S$ such that $b \leq a$. Then $[a]_\theta \subseteq A$ and $b\theta a$. This implies that $[a]_\theta = [b]_\theta$. Since $[a]_\theta \subseteq A$, so $[b]_\theta \subseteq A$. Thus $b \in \theta_-(A)$.

This proves that $\theta_-(A)$ is, if it is non-empty, an ordered m- left-ideal of S, that is, A is a generalized θ-lower ordered rough m-left, n-right ((m, n)-bi)-ideal of S. In the similar fashion it can be proved that generalized θ-lower approximation of an n-right ((m, n)-bi-)-ideal of S is an n-right((m, n)-bi-)-ideal of S. □

Definition 4.6. *Let θ be a pseudoorder on an ordered LA-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) ordered rough (m, n)-bi-ideal of S if $\theta_+(A)$ (resp., $\theta_-(A)$) is an ordered (m, n)-bi-ideal of S.*

Theorem 4.7. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A is an ordered (m, n)-bi-ideal of S, then it is a θ-upper ordered rough (m, n)-bi-ideal of S.*

Proof. Let A be an ordered (m, n)-bi-ideal of S.

(i) By Theorem 3.7, we have

$$(\theta_+(A))^m S(\theta_+(A))^n \subseteq (\theta_+(A^m)\theta_+(S))\theta_+(A^n) \subseteq \theta_+((A^m S)A^n) \subseteq \theta_+(A).$$

(ii) Let $a \in \theta_+(A)$ and $b \in S$ such that $b \leq a$. Then there exist $y \in A$, such that $a\theta y$ and $b\theta a$. Since θ is transitive, so $b\theta y$ implies $b \in \theta_+(A)$.

From this and Theorem 4.2(1), we have $\theta_+(A)$ is an ordered (m, n)-bi-ideal of S, that is, A is a θ-upper ordered rough (m, n)-bi-ideal of S. □

Theorem 4.8. *Let θ be a complete pseudoorder on an ordered LA-semigroup S. If A is an ordered (m, n)-bi-ideal of S, then $\theta_-(A)$ is, if it is non-empty, an ordered (m, n)-bi-ideal of S.*

Proof. Let A be an ordered (m, n)-bi-ideal of S.

(i) By Theorem 3.10, we have

$$(\theta_-(A))^m S(\theta_-(A))^n \subseteq (\theta_-(A^m))(\theta_-(S))(\theta_-(A^n)) \subseteq \theta_-((A^m S)A^n) \subseteq \theta_-(A).$$

(ii) Let $a \in \theta_-(A)$ and $b \in S$ such that $b \leq a$. Then $[a]_\theta \subseteq A$ and $b\theta a$. This implies that $[a]_\theta = [b]_\theta$. Since $[a]_\theta \subseteq A$, so $[b]_\theta \subseteq A$. Thus $b \in \theta_-(A)$.

From this and Theorem 4.2(2), we obtain that $\theta_-(A)$ is, if it is non-empty, an ordered (m, n)-bi-ideal of S. □

Theorem 4.9. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A and B are an ordered n-right and an ordered m-left ordered ideals of S respectively, then*

$$\theta_+(AB) \subseteq \theta_+(A) \cap \theta_+(B).$$

Proof. The proof is straightforward. □

Theorem 4.10. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A is an ordered n-right and B is an ordered m-left ideals of S, then*

$$\theta_-(AB) \subseteq \theta_-(A) \cap \theta_-(B).$$

Proof. The proof is straightforward. □

Definition 4.11. *Let θ be a pseudoorder on an ordered LA-semigroup S. Then a non-empty subset A of S is called a θ-upper (resp., θ-lower) ordered rough (m, n)-interior ideal of S if $\theta_+(A)$ (resp., $\theta_-(A)$) is an ordered (m, n)-interior ideal of S.*

Theorem 4.12. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A is an ordered interior (m,n)-ideal of S, then A is a θ-upper ordered rough (m,n)-interior ideal of S.*

Proof. The proof of this theorem is similar to the Theorem 4.7. □

Theorem 4.13. *Let θ be a pseudoorder on an ordered LA-semigroup S. If A is an ordered interior (m,n)-ideal of S, then $\theta_-(A)$ is, if it is non-empty, an ordered interior (m,n)-ideal of S.*

Proof. The proof of this theorem is similar to the Theorem 4.8. □

We call A an ordered rough (m,n)-interior ideal of S if it is both a θ-lower and θ-upper ordered rough (m,n)-interior ideal of S.

Definition 4.14. *Let θ be a pseudoorder on an ordered LA-semigroup S. Then a non-empty subset Q of S is called a θ-upper (resp., θ-lower) ordered rough (m,n)-quasi-ideal of S if $\theta_+(Q)$* (resp., *$\theta_-(Q)$) is an ordered (m,n)-quasi-ideal of S.*

Theorem 4.15. *Let θ be a complete pseudoorder on an ordered LA-semigroup S. If Q is an ordered (m,n)-quasi-ideal of S, then Q is a θ-lower ordered rough (m,n)-quasi-ideal of S.*

Proof. Let Q be an ordered (m,n)-quasi-ideal of S.

(i) Now by Theorem 3.2(5) and Theorem 3.10, we get

$$\begin{aligned} \theta_-(Q^m)S \cap S\theta_-(Q^n) &= \theta_-(Q^m)\theta_-(S) \cap \theta_-(S)\theta_-(Q^n) \\ &\subseteq \theta_-(Q^mS) \cap \theta_-(SQ^n) \\ &= \theta_-(Q^mS \cap SQ^n) \\ &\subseteq \theta_-(Q). \end{aligned}$$

(ii) Let $a \in \theta_-(Q)$ and $b \in S$ such that $b \leq a$. Then $[a]_\theta \subseteq Q$ and $b\theta a$. This implies that $[a]_\theta = [b]_\theta$. Since $[a]_\theta \subseteq Q$, so $[b]_\theta \subseteq Q$. Thus $b \in \theta_-(Q)$.

Thus we obtain that $\theta_-(Q)$ is an ordered (m,n)-quasi-ideal of S, that is, Q is a θ-lower ordered rough (m,n)-quasi-ideal of S. □

Theorem 4.16. *Let θ be a complete pseudoorder on an ordered LA-semigroup S. If Q is an ordered (m,n)-quasi-ideal of S, then Q is a θ-upper ordered rough (m,n)-quasi-ideal of S.*

Proof. Let Q be an ordered (m, n)-quasi-ideal of S.
(i) Now by Theorem 3.2(9) and Theorem 3.7, we get

$$\begin{aligned}\theta_+(Q^m)S \cap S\theta_+(Q^n) &= \theta_+(Q^m)\theta_+(S) \cap \theta_+(S)\theta_+(Q^n) \\ &\subseteq \theta_+(Q^m S) \cap \theta_+(SQ^n) \\ &= \theta_+(Q^m S \cap SQ^n) \\ &\subseteq \theta_+(Q).\end{aligned}$$

(ii) Let $a \in \theta_+(Q)$ and $b \in S$ such that $b \leq a$. Then $[a]_\theta \subseteq Q$ and $b\theta a$. This implies that $[a]_\theta = [b]_\theta$. Since $[a]_\theta \subseteq Q$, so $[b]_\theta \subseteq Q$. Thus $b \in \theta_+(Q)$.

Thus we obtain that $\theta_+(Q)$ is an ordered (m, n)-quasi-ideal of S, that is, Q is a θ-upper ordered rough (m, n)-quasi-ideal of S. □

Theorem 4.17. *Let θ be a complete pseudoorder on an ordered LA-semigroup S. Let L and R be a θ-lower ordered rough m-left ideal and a θ-lower ordered rough n-right ideal of S, respectively. Then $L \cap R$ is a θ-lower ordered rough (m, n)-quasi-ideal of S.*

Proof. The proof is straightforward. □

5 Conclusion

The properties of generalized m-left, n-right, (m, n)-(quasi-, bi-, interior)-ideals of ordered LA-semigroups in terms of rough sets precisely generalized rough m-left, n-right, (m, n)-(quasi-, bi-, interior)-ideals of ordered LA-semigroups have been discussed and studied. Through pseudoorders of relations, it is proved that generalized two-sided ideals and generalized (m, n) (resp., quasi-, bi-, and interior)-ideals in ordered LA-semigroups becomes generalized lower and upper rough two-sided ideals and generalized (m, n) (resp., quasi-, bi-, and interior)-ideals in ordered LA-semigroups.

In our future studies, following topics may be considered:

1. Rough fuzzy generalized prime and semiprime (m, n) bi-ideals of ordered LA-semigroups.
2. Rough fuzzy (m, n)-ideals (resp. interior ideals) in ordered LA-semigroups.
3. Rough fuzzy (m, n)-quasi-ideals of ordered LA-semigroups.

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