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A first order convergent numerical method for solving the delay differential problem

Erkan Cimen

Department of Mathematics
Van Yuzuncu Yil University
65080, Van, Turkey

email: cimenerkan@hotmail.com

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Abstract

In this paper, the boundary-value problem for a parameter dependent linear first order delay differential equation is analyzed. A finite difference method for approximate solution of this problem is presented. The method is based on fitted difference scheme on a uniform mesh which is achieved by using the method of integral identities which includes the exponential basis functions and applying interpolating quadrature formulas which contain the remainder term in integral form. Also, the method is proved first-order convergent in the discrete maximum norm. Moreover, a numerical example is solved using both the presented method and the Euler method and compared the computed results.

1 Introduction

Many physical and biological processes can be modeled using delay differential equations (DDEs). They arise in the study of dynamical diseases, models of epidemiology, climate systems, economics and control theory [3, 8, 9, 10, 11, 16, 17, 19]. Especially, the models of dynamical systems usually depend

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on parameters [14, 22, 23]. For instance, parameter dependent DDEs typically appear in the theory of epidemiology. From this point of view, let us examine following model:

$$x'(t) = f(t, x(t), x(t-\tau)) + \lambda,\ t \in (a, b)$$

$$x(t) = g(t),\ t \in [a-\tau, a];\ x(b) = \gamma$$

where g is the state of population, γ is the state that should have the population and λ is a control parameter. Also $x'(t)$ is the speed of growth of the population with the law $x' = f + \lambda$. If $\lambda > 0$, the population will grow, while if $\lambda < 0$, the population will eventually become extinct. The case $\lambda = 0$ represents stasis, that is, the population size on average will not change [6, 9]. As an example, in time-delayed logistic model:

$$\frac{dN}{dt} = R(N(t-\tau)) - \lambda N(t)$$

can be given, where $N(t)$ is a population size, R is the birth function which involves maturation delay τ, λ is death rate of the current population [12].

Motivated by the above models, we are interested in the following delay differential problem in the interval $\overline{I} = [0, T]$:

$$Lu \equiv u'(t) + a(t)u(t) + b(t)u(t-r) = \lambda f(t) + g(t),\ t \in I, \tag{1.1}$$

$$u(t) = \psi(t),\ t \in I_0;\ u(T) = A \tag{1.2}$$

where $I = (0, T] = \cup_{p=1}^{m} I_p$, $I_p = \{t : r_{p-1} < t \leq r_p\}$, $1 \leq p \leq m$ and $r_s = sr$, $0 \leq s \leq m$ and, $I_0 = [-r, 0]$ (without loss of generality we assume that T/r is integer). $a(t)$, $b(t)$, $f(t)$, $g(t)$ and $\psi(t)$ are given sufficiently smooth functions in $\overline{I}$ and I_0, r is a constant delay, which is independent of λ is real parameter and A is a given constant. However, we assume

$$a(t) \geq \alpha > 0,\ \ |f(t)| \geq m_1 > 0,$$

and $\{u(t), \lambda\}$ is a solution of (1.1)-(1.2).

In recent years, it seems that the studies on these problems are being taken more and more into consideration, both in terms of modeling and solution of them. Existence and uniqueness of solution to DDEs with parameter are discussed in [13, 15, 18] and the references therein. Furthermore, there are many researchers who have investigated oscillation, Hopf bifurcation and numerical approaches for this type equations [1, 20, 21] and the references therein. The authors in [4, 5] are studied the numerical solutions of DDEs without parameter by using standard methods, such as Euler, Runge-Kutta methods. Nevertheless, authors in [7] are suggested numerical solutions of neutral DDEs without parameter by using finite difference method.

On the other hand, since these equations involve not only a delay term but also a parameter, it is very difficult to solve explicitly; therefore, it is important to develop effective numerical methods to solve them.

In this study, for the solution of (1.1)-(1.2), we present a powerful approximate method which includes a finite-difference scheme on a uniform mesh. Our approach constructs a difference scheme that is based on the method of integral identities by using exponetial basis functions and interpolating quadrature rules with the weight and remainder terms in integral form. As a result of this, a local truncation error includes only first derivative of exact solution and hence facilitates analysis of the convergence. The remainder paper is organized as follows. We put forward some important properties of the exact solution of (1.1)-(1.2) in Section 2. In Section 3, we introduce the finite difference discretization. In Section 4, after providing error analysis for the approximate solution, we have proved the convergence of the presented method. We formulate the iterative algorithm for solving the discrete problem and present numerical results which validate the theoretical analysis computationally in Section 5.

Notation. Henceforth, C denotes a generic positive constant. Some specific, fixed constants of this kind are indicated by subscripting C and $\overline{C}$. For any continuous function $w(t)$, $\|w\|_\infty$ denotes a continuous maximum norm on the corresponding interval, in particular we shall use $\|w\|_{\infty,I_p} = \|w\|_{\infty,p} = \max_{t\in I_p}|w(t)|\,,\ 0 \leq p \leq m$.

2 Analytical results

In this section, we give a priori estimates for the exact solution of (1.1)-(1.2), which are needed in later sections for the analysis of appropriate numerical solution.

Lemma 2.1. *Assume that a, b, f, $g \in C(\overline{I})$, $\psi \in C(I_0)$ and*

$$\delta := m_1^{-1}\,\|f\|_\infty\,[(1+\alpha^{-1}\,\|b\|_\infty)^{m-1} - 1] < 1.$$

Then for the solution $\{u(t), \lambda\}$ of the problem (1.1)-(1.2) the following estimates hold:

$$|\lambda| \leq C_0, \tag{2.3}$$

$$\|u\|_{\infty,p} \leq C_p,\ 1 \leq p \leq m \tag{2.4}$$

where

$$\begin{aligned} C_0 &= [(\frac{\|a\|_\infty m_1^{-1}}{1-e^{-\|a\|_\infty T}} + m_1^{-1}\|b\|_\infty (1+\alpha^{-1}\|b\|_\infty)^{m-1})\|\psi\|_{\infty,0} \\ &+ \frac{\|a\|_\infty m_1^{-1}}{1-e^{-\|a\|_\infty T}}|A| + (m_1^{-1} + \frac{(1+\alpha^{-1}\|b\|_\infty)^{m-1}-1}{\|b\|_\infty})\|g\|_\infty](1-\delta)^{-1}, \end{aligned}$$

$$C_p = \|\psi\|_\infty (1+\alpha^{-1}\|b\|_\infty)^p + \alpha^{-1}\sum_{k=1}^{p}(1+\alpha^{-1}\|b\|_\infty)^{p-k}\|g\|_\infty,\ 1 \leq p \leq m$$

and

$$\|u'\|_{\infty,p} \leq \overline{C}_p,\ 1 \leq p \leq m, \tag{2.5}$$

$$\overline{C}_1 = \|a\|_{\infty,1} C_1 + \|b\|_{\infty,1}\|\psi\|_{\infty,0} + C_0\|f\|_{\infty,1} + \|g\|_{\infty,1},$$

$$\overline{C}_p = \|a\|_{\infty,p} C_p + \|b\|_{\infty,p} C_{p-1} + C_0\|f\|_{\infty,p} + \|g\|_{\infty,p},\ p = 2,3,...m.$$

Proof. From (1.1) we have

$$u(t) = \psi(0)e^{-\int_0^t a(\eta)d\eta} + \int_0^t [\lambda f(\xi) + g(\xi) - b(\xi)u(\xi - r)]e^{-\int_\xi^t a(\eta)d\eta} d\xi. \tag{2.6}$$

from which, by using the condition $u(T) = A$, we have

$$|\lambda| = \frac{|A - \psi(0)e^{-\int_0^T a(\eta)d\eta} - \int_0^T [g(\xi) - b(\xi)u(\xi - r)]e^{-\int_\xi^T a(\eta)d\eta} d\xi|}{|\int_0^T f(\xi)e^{-\int_\xi^T a(\eta)d\eta} d\xi|}. \tag{2.7}$$

Now, if we apply the mean value theorem for integrals, we can write

$$\frac{\left|\int_0^T [g(\xi) - b(\xi)u(\xi - r)]e^{-\int_\xi^T a(\eta)d\eta} d\xi\right|}{\left|\int_0^T f(\xi)e^{-\int_\xi^T a(\eta)d\eta} d\xi\right|} \leq m_1^{-1}(\|g\|_\infty + \|b\|_\infty \max_{0\leq p\leq m-1}\|u\|_{\infty,p}).$$

Since

$$\int_0^T |f(\xi)|\, e^{-\int_\xi^T a(\eta)d\eta} d\xi \geq \int_0^T m_1 e^{-\|a\|_\infty (T-\xi)} d\xi = \frac{m_1}{\|a\|_\infty}(1-e^{-\|a\|_\infty T}),$$

it follows from (2.7) that

$$|\lambda| \leq \frac{\|a\|_\infty m_1^{-1}(|A| + |\psi(0)|)}{1-e^{-\|a\|_\infty T}} + m_1^{-1}(\|g\|_\infty + \|b\|_\infty \max_{0\leq p\leq m-1} \|u\|_{\infty,p}). \quad (2.8)$$

Next, from (2.6),

$$|u(t)| \leq |\psi(0)|\, e^{-\int_0^t a(\eta)d\eta} + \int_0^t [|\lambda|\, |f(\xi)| + |g(\xi)| + |b(\xi)|\, |u(\xi - r)|\,] e^{-\int_\xi^t a(\eta)d\eta} d\xi$$

$$\leq |\psi(0)|\, e^{-\alpha t} + \int_0^t [|\lambda|\, |f(\xi)| + |g(\xi)| + |b(\xi)|\, |u(\xi - r)|\,] e^{-\alpha(t-\xi)} d\xi$$

for $p = 1$ $(t \in I_1)$

$$\begin{aligned}\|u\|_{\infty,1} &\leq |\psi(0)| + \alpha^{-1}[|\lambda|\, \|f\|_{\infty,1} + \|g\|_{\infty,1} + \|b\|_{\infty,1} \|\psi\|_{\infty,0}](1-e^{-\alpha t}) \\ &\leq |\psi(0)| + \alpha^{-1}[|\lambda|\, \|f\|_{\infty,1} + \|g\|_{\infty,1} + \|b\|_{\infty,1} \|\psi\|_{\infty,0}].\end{aligned}$$

Also, for $p = 2$ $(t \in I_2)$, from (1.1) we get

$$|u(t)| \leq |u(r)|\, e^{-\int_r^t a(\eta)d\eta} + \int_r^t [|\lambda|\, |f(\xi)| + |g(\xi)| + |b(\xi)|\, |u(\xi - r)|\,] e^{-\int_\xi^t a(\eta)d\eta} d\xi$$

$$\leq |u(r)|\, e^{-\alpha(t-r)} + \int_r^t [|\lambda|\, |f(\xi)| + |g(\xi)| + |b(\xi)|\, |u(\xi - r)|\,] e^{-\alpha(t-\xi)} d\xi$$

and

$$\begin{aligned}\|u\|_{\infty,2} &\leq |u(r)| + \alpha^{-1}[|\lambda|\, \|f\|_{\infty,2} + \|g\|_{\infty,2} + \|b\|_{\infty,2} \|u\|_{\infty,1}](1-e^{-\alpha(t-r)}) \\ &\leq (1 + \alpha^{-1} \|b\|_{\infty,2}) \|u\|_{\infty,1} + \alpha^{-1}(|\lambda|\, \|f\|_{\infty,2} + \|g\|_{\infty,2}).\end{aligned}$$

Thus, for $t \in I_p$, from (1.1) we can write

$$|u(t)| \leq |u(r_{p-1})|\, e^{-\int_{r_{p-1}}^t a(\eta)d\eta} + \int_{r_{p-1}}^t [|\lambda|\, |f(\xi)| + |g(\xi)| + |b(\xi)|\, |u(\xi - r)|\,] e^{-\int_\xi^t a(\eta)d\eta} d\xi$$

$$\leq |u(r_{p-1})| e^{-\alpha(t-r_{p-1})} + \int_{r_{p-1}}^{t} [|\lambda| |f(\xi)| + |g(\xi)| + |b(\xi)| |u(\xi - r)|] e^{-\alpha(t-\xi)} d\xi$$

and

$$\begin{aligned} \|u\|_{\infty,p} &\leq |u(r_{p-1})| + \alpha^{-1}[|\lambda| \|f\|_{\infty,p} + \|g\|_{\infty,p} + \|b\|_{\infty,p} \|u\|_{\infty,p-1}](1 - e^{-\alpha(t-r_{p-1})}) \\ &\leq (1 + \alpha^{-1} \|b\|_{\infty,p}) \|u\|_{\infty,p-1} + \alpha^{-1}(|\lambda| \|f\|_{\infty,p} + \|g\|_{\infty,p}) \end{aligned}$$

herefrom, with $\|u\|_{\infty,p} = v_p$ we get following first order difference inequality

$$v_p \leq \kappa v_{p-1} + \phi_p,$$

where

$$\kappa = (1 + \alpha^{-1} \|b\|_{\infty}), \ \phi_p = \alpha^{-1}(|\lambda| \|f\|_{\infty} + \|g\|_{\infty,p}).$$

From this inequality it follows that

$$v_p \leq v_0 \kappa^p + \sum_{j=1}^{p} \kappa^{p-j} \phi_j.$$

The last inequality together with (2.8) leads to (2.3), (2.4).

The proof of (2.5) is by induction in p. Firstly, from (1.1) we have

$$|u'(t)| \leq |a(t)| |u(t)| + |b(t)| |u(t-r)| + |\lambda| |f(t)| + |g(t)| . \qquad (2.9)$$

So,

$$\|u'(t)\|_{\infty,1} \leq \|a\|_{\infty,1} C_1 + \|b\|_{\infty,1} \|\psi\|_{\infty,0} + |\lambda| \|f\|_{\infty,1} + \|g\|_{\infty,1}$$

for $p = 1$ ($t \in I_1$) and therefore the inequality (2.5) hold for $p = 1$. Now, let the inequality (2.5) be true for $p = k$. That is

$$\overline{C}_k = \|a\|_{\infty,k} C_k + \|b\|_{\infty,k} C_{k-1} + C_0 \|f\|_{\infty,k} + \|g\|_{\infty,k} .$$

For $t \in I_{k+1}$ because of (2.9) we get

$$\begin{aligned} |u'(t)| &\leq |a(t)| |u(t)| + |b(t)| |u(t-r)| + |\lambda| |f(t)| + |g(t)| \\ &\leq \|a\|_{\infty,k+1} \|u\|_{\infty,k+1} + \|b\|_{\infty,k+1} \|u\|_{\infty,k} + |\lambda| \|f\|_{\infty,k+1} + \|g\|_{\infty,k+1} \end{aligned}$$

and hence this inequality (2.5) holds for $p = k + 1$. □

3 Construction of the difference scheme

In what follows, we denote by ω_{N_0} be a uniform mesh on $\overline{I}$:

$$\omega_{N_0} = \{t_i = i\tau,\ i = 1, 2, ...N_0,\ \tau = T/N_0 = r/N\}$$

which contains by N mesh points at each subinterval I_p $(1 \leq p \leq m)$:

$$\omega_{N_p} = \{t_i : (p-1)N + 1 \leq i \leq pN\},\ 1 \leq p \leq m,$$

and consequently

$$\omega_{N_0} = \cup_{p=1}^{m} \omega_{N_p}.$$

y_i denotes an approximation of $u(t)$ at t_i and moreover for any mesh function $w(t)$, we use $w_i = w(t_i)$ and

$$w_{\bar{t},i} = (w_i - w_{i-1})/\tau,\ \|w\|_{\infty,N,p} = \|w\|_{\infty,\omega_{N,p}} := \max_{1\leq i\leq N} |w_i|\,.$$

Firstly, for the difference approximation the problem (1.1), we are using the following identity

$$\tau^{-1}\int_{t_{i-1}}^{t_i} Lu(t)\varphi_i(t)dt = \tau^{-1}\int_{t_{i-1}}^{t_i} [\lambda f(t) + g(t)]\varphi_i(t)dt,\ 1 \leq i \leq N_0, \quad (3.10)$$

with basis function

$$\varphi_i(t) = e^{-\int_t^{t_i} a(s)ds},\ t_{i-1} \leq t \leq t_i$$

which is the solution of the following initial value problem

$$-\varphi_i{}'(t) + a(t)\varphi_i(t) = 0,\ t_{i-1} < t \leq t_i,\ \varphi_i(t_i) = 1.$$

Now, the Eq. (3.10) is rewritten as

$$\tau^{-1}\int_{t_{i-1}}^{t_i} u'(t)\varphi_i(t)dt + \tau^{-1}\int_{t_{i-1}}^{t_i} a(t)u(t)\varphi_i(t)dt + \tau^{-1}\int_{t_{i-1}}^{t_i} b(t)u(t-r)\varphi_i(t)dt$$

$$= \lambda\tau^{-1}\int_{t_{i-1}}^{t_i} f(t)\varphi_i(t)dt + \tau^{-1}\int_{t_{i-1}}^{t_i} g(t)\varphi_i(t)dt,\ 1 \leq i \leq N_0. \quad (3.11)$$

Next, using the formulas (2.1) and (2.2) from [2] on interval (t_{i-1}, t_i) taking into consideration (3.11) we have following precise relation

$$\ell u_i \equiv A_i u_{\bar{t},i} + B_i u_{\bar{t},i-N} + D_i u_i + E_i u_{i-N} = \lambda F_i + G_i + R_i,\ 1 \leq i \leq N_0,\ (3.12)$$

where

$$A_i = \tau^{-1} \int_{t_{i-1}}^{t_i} \varphi_i(t)dt + \tau^{-1} \int_{t_{i-1}}^{t_i} (t - t_i)a(t)\varphi_i(t)dt,$$

$$B_i = \tau^{-1} \int_{t_{i-1}}^{t_i} (t - t_i)b(t)\varphi_i(t)dt,$$

$$D_i = \tau^{-1} \int_{t_{i-1}}^{t_i} a(t)\varphi_i(t)dt, \ E_i = \tau^{-1} \int_{t_{i-1}}^{t_i} b(t)\varphi_i(t)dt,$$

$$F_i = \tau^{-1} \int_{t_{i-1}}^{t_i} f(t)\varphi_i(t)dt, \ G_i = \tau^{-1} \int_{t_{i-1}}^{t_i} g(t)\varphi_i(t)dt,$$

$$R_i = \tau^{-1} \int_{t_{i-1}}^{t_i} dtb(t)\varphi_i(t) \int_{t_{i-1}}^{t_i} u'(\xi - r)[T_0(t - \xi) - \tau^{-1}(t - t_{i-1})]d\xi, \quad (3.13)$$

$$T_k(x) = x^k/k!, \ x \geq 0; \ T_k(x) = 0, \ x < 0.$$

As a consequence of (3.12), we propose the following difference scheme for approximating (1.1)-(1.2):

$$\ell y_i \equiv A_i y_{\bar{t},i} + B_i y_{\bar{t},i-N} + D_i y_i + E_i y_{i-N} = \overline{\lambda} F_i + G_i, \ 1 \leq i \leq N_0, \quad (3.14)$$

$$y_i = \psi_i, \ \ -N \leq i \leq 0; \ y_{N_0} = A. \quad (3.15)$$

Secondly, if we use implicit Euler method to solve the problem (1.1)-(1.2), we can easily obtain following scheme

$$\ell y_i \equiv y_{\bar{t},i} + a_i y_i + b_i y_{i-N} = \overline{\lambda} f_i + g_i, \ 1 \leq i \leq N_0, \quad (3.16)$$

$$y_i = \psi_i, \ \ -N \leq i \leq 0; \ y_{N_0} = A. \quad (3.17)$$

4 Error analysis

In order to investigate the convergence of this method, note that the error function $z_i = y_i - u_i$, $0 \leq i \leq N_0$, $\lambda^N = \overline{\lambda} - \lambda$ are the solution of the following discrete problem

$$\ell z_i = R_i, \ 1 \leq i \leq N_0, \quad (4.18)$$

$$z_i = 0, \ -N \leq i \leq 0; \ z_{N_0} = 0 \quad (4.19)$$

where the truncation error R_i is given by (3.13).

Lemma 4.1. *If a, b, f, $g \in C(\overline{I})$ and $\psi \in C^1(I_0)$, then for the truncation error R_i we have*

$$\|R\|_{\infty,N,p} \leq CN^{-1}. \tag{4.20}$$

Proof. From (3.13), we present

$$|R_i| \leq \tau^{-1} \int_{t_{i-1}}^{t_i} dt\, |b(t)|\, \varphi_i(t) \int_{t_{i-1}}^{t_i} |u'(\xi - r)|\, d\xi$$

$$\leq C\tau^{-1} \int_{t_{i-1}}^{t_i} dt\varphi_i(t) \int_{t_{i-1}}^{t_i} |u'(\xi - r)|\, d\xi$$

and owing to Lemma 2.1 and $0 < \varphi_i(t) \leq 1$

$$|R_i| \leq C\tau.$$

□

Lemma 4.2. *For the pair $\left\{z_i, \lambda^N\right\}$*

$$\|z\|_{\infty,N,p} \leq C \sum_{k=1}^{p} \|R\|_{\infty,\omega_{N,k}}, \tag{4.21}$$

$$\left|\lambda^N\right| \leq C\, \|R\|_{\infty,\omega_{N_0}} \tag{4.22}$$

estimates are hold.

Proof. From the (4.18), we can write

$$A_i z_{\overline{t},i} + B_i z_{\overline{t},i-N} + D_i z_i + E_i z_{i-N} = \lambda^N F_i + R_i,\ 1 \leq i \leq N_0.$$

Hence

$$z_i = \frac{A_i}{A_i + \tau D_i} z_{i-1} - \frac{B_i + \tau E_i}{A_i + \tau D_i} z_{i-N} + \frac{B_i}{A_i + \tau D_i} z_{i-N-1} + \frac{\tau F_i}{A_i + \tau D_i}\lambda^N + \frac{\tau R_i}{A_i + \tau D_i}.$$

Solving this difference equation with respect to z_i and take into account $z_0 = 0$, we have

$$\begin{aligned} z_i \quad = \quad & -\sum_{k=1}^{i} \frac{(B_k + \tau E_k) z_{k-N}}{A_k + \tau D_k} Q_{ik} + \sum_{k=1}^{i} \frac{B_k z_{k-N-1}}{A_k + \tau D_k} Q_{ik} \\ & + \lambda^N \sum_{k=1}^{i} \frac{\tau F_k}{A_k + \tau D_k} Q_{ik} + \sum_{k=1}^{i} \frac{\tau R_k}{A_k + \tau D_k} Q_{ik} \end{aligned}$$

where

$$Q_{ik}=\begin{cases}1, & \text{for } k=i\\ \prod_{j=k+1}^{i}\frac{A_j}{A_j+\tau D_j}, & \text{for } 0\le k\le i-1.\end{cases}$$

For $i=N_0$, take into account that $z_{N_0}=0$, we get

$$\lambda^N=\frac{\sum_{k=1}^{i}\frac{(B_k+\tau E_k)z_{k-N}}{A_k+\tau D_k}Q_{ik}-\sum_{k=1}^{i}\frac{B_k z_{k-N-1}}{A_k+\tau D_k}Q_{ik}-\sum_{k=1}^{i}\frac{\tau R_k}{A_k+\tau D_k}Q_{ik}}{\sum_{k=1}^{i}\frac{\tau F_k}{A_k+\tau D_k}Q_{ik}}.$$

Further, since $A_i>0$ and $D_i>0$ $(1\le i\le N_0)$, similar to the proof of Lemma 2.1, we get

$$\left|\lambda^N\right|\le m_1^{-1}(\|R\|_{\infty,\omega_{N_0}}+\|b\|_\infty\max_{1\le p\le m-1}\|z\|_{\infty,N,p}) \tag{4.23}$$

and

$$\begin{aligned}\|z\|_{\infty,p} &\le |z_{p-1}|+\alpha^{-1}(\left|\lambda^N\right|\|f\|_{\infty,p}+\|R\|_{\infty,p}+\|b\|_{\infty,p}\|z\|_{\infty,p-1})\\ &\le (1+\alpha^{-1}\|b\|_\infty)\|z\|_{\infty,p-1}+\alpha^{-1}(\left|\lambda^N\right|\|f\|_\infty+\|R\|_{\infty,p})\end{aligned}$$

therefore

$$\|z\|_{\infty,p}\le\alpha^{-1}\sum_{k=1}^{p}(1+\alpha^{-1}\|b\|_\infty)^{p-k}(\left|\lambda^N\right|\|f\|_\infty+\|R\|_{\infty,p}),\ 1\le p\le m. \tag{4.24}$$

When considering (4.23) and (4.24), (3.14)-(3.15) can be easily obtain. □

Finally, we give the main result of this paper.

Theorem 4.3. *Suppose $\{u,\lambda\}$ is the solution of (1.1)-(1.2) and $\{y,\overline{\lambda}\}$ is the solution (3.14)-(3.15). Then, we have*

$$\begin{aligned}\|y-u\|_{\infty,\omega_{N_0}} &\le CN^{-1},\\ \left|\overline{\lambda}-\lambda\right| &\le CN^{-1}.\end{aligned}$$

Proof. This follows immediately by combining the Lemmas 4.1 and 4.2. □

5 Algorithm and numerical results

In this section, we present computational results obtained by applying the numerical methods (3.14)-(3.15) and (3.16)-(3.17) to the particular problem.

We rewritten difference scheme (3.14),

$$y_i = \frac{A_i}{A_i+\tau D_i}y_{i-1} - \frac{B_i+\tau E_i}{A_i+\tau D_i}y_{i-N} + \frac{B_i}{A_i+\tau D_i}y_{i-N-1} + \frac{\tau F_i}{A_i+\tau D_i}\overline{\lambda} + \frac{\tau G_i}{A_i+\tau D_i}$$

from hence, for $1 \le i \le N_0 - 1$ with together $y_0 = \psi_0$, by using Bellman's steps method, we get

$$\left\{y_i, \overline{\lambda}\right\} = \begin{cases} \left\{y_i^{(1)}, \overline{\lambda}\right\}, & 1 \le i \le N, \\ \left\{y_i^{(2)}, \overline{\lambda}\right\}, & N \le i \le 2N, \\ \vdots & \vdots \\ \left\{y_i^{(m)}, \overline{\lambda}\right\}, & (m-1)N \le i \le mN \end{cases}$$

and then in the last step, in $y_i^{(m)}$ taking into account the $y_{N_0} = A$, $\overline{\lambda}$ is obtained. Analysis of the difference scheme (3.16)-(3.17) can be found in similar way.

Example 5.1. *Now, we look at the particular problem:*

$$u'(t) + 2u(t) + u(t-1) = \lambda + e^{t-1}, \ 0 < t < 2$$

subject to the interval and boundary conditions

$$u(t) = e^t, \ \ -1 \leqslant t \leqslant 0; \ u(2) = 1.$$

of which the exact solution for $0 \leqslant t \leqslant 2$ *is given by*

$$u(t) = \begin{cases} \frac{\lambda}{2} + (1-\frac{\lambda}{2})e^{-2t}, & t \in [0,1), \\ (1-\frac{\lambda}{2})(e^{-2}+1-t)e^{-2(t-1)} + \\ +(\frac{\lambda}{4}-\frac{1}{3})(1+e^{-2(t-1)}) + \frac{1}{3}(1+e^{t-1}), & t \in [1,2], \end{cases}$$

and

$$\lambda = \frac{4(3-e+4e^{-2}-3e^{-4})}{3(1+3e^{-2}-2e^{-4})}.$$

We define the exact error e_i^N, the computed maximum pointwise error e^N and the exact error for λ as follows, respectively:

$$e_i^N = |y_i - u_i|, \ e^N = \max_{0 \leqslant i \leqslant N} e_i^N, \ e_\lambda^N = \left|\lambda - \overline{\lambda}\right|$$

where y is the numerical approximation to u for various values of N. The computational results obtained by both present method (PM) and implicit Euler method (EM) are given in the Tables 1-3.

Table 1: The numerical results on $(0,2]$ (PM)

Nodes t_i	Exact solution	Numerical solution $N=64$	Pointwise error $\|y-u\|$	Numerical solution $N=128$	Pointwise error $\|y-u\|$
0.125	0.8615178	0.8615188	$1.041E-6$	0.8615181	$2.601E-7$
0.250	0.7536677	0.7536695	$1.733E-6$	0.7536682	$4.333E-7$
0.375	0.6696740	0.6696762	$2.139E-6$	0.6696746	$5.348E-7$
0.500	0.6042597	0.6042620	$2.304E-6$	0.6042603	$5.760E-7$
0.625	0.5533149	0.5533172	$2.262E-6$	0.5533155	$5.653E-7$
0.750	0.5136391	0.5136411	$2.034E-6$	0.5136396	$5.085E-7$
0.875	0.4827396	0.4827412	$1.637E-6$	0.4827400	$4.093E-7$
1.000	0.4586750	0.4586760	$1.079E-6$	0.4586752	$2.697E-7$
1.125	0.4557446	0.4557423	$2.258E-6$	0.4557440	$5.646E-7$
1.250	0.4826706	0.4826668	$3.855E-6$	0.4826697	$9.639E-7$
1.375	0.5319608	0.5319565	$4.304E-6$	0.5319597	$1.076E-6$
1.500	0.5987187	0.5987146	$4.018E-6$	0.5987177	$1.005E-6$
1.625	0.6799604	0.6799571	$3.284E-6$	0.6799596	$8.211E-7$
1.750	0.7741203	0.7741180	$2.293E-6$	0.7741198	$5.731E-7$
1.875	0.8806964	0.8806953	$1.171E-6$	0.8806961	$2.929E-7$

$e_\lambda^N = 1.740E-5$ when $N=64$, $e_\lambda^N = 4.350E-6$ when $N=128$.

Table 2: The numerical results on $(0, 2]$ (EM)

Nodes t_i	Exact solution	Numerical solution $N = 64$	Pointwise error $\|y - u\|$	Numerical solution $N = 128$	Pointwise error $\|y - u\|$
0.125	0.8615178	0.8626377	$1.120E-3$	0.8620849	$5.671E-4$
0.250	0.7536677	0.7552500	$1.582E-3$	0.7544687	$8.009E-4$
0.375	0.6696740	0.6712955	$1.621E-3$	0.6704949	$8.209E-4$
0.500	0.6042597	0.6056611	$1.401E-3$	0.6049695	$7.098E-4$
0.625	0.5533149	0.5543489	$1.034E-3$	0.5538396	$5.246E-4$
0.750	0.5136391	0.5142337	$5.946E-4$	0.5139425	$3.034E-4$
0.875	0.4827396	0.4828722	$1.327E-4$	0.4828105	$7.092E-5$
1.000	0.4586750	0.4583542	$3.207E-4$	0.4585179	$1.570E-4$
1.125	0.4557446	0.4565342	$7.896E-4$	0.4561510	$4.064E-4$
1.250	0.4826706	0.4837630	$1.092E-3$	0.4832296	$5.590E-4$
1.375	0.5319608	0.5329431	$9.823E-4$	0.5324627	$5.019E-4$
1.500	0.5987187	0.5994228	$7.042E-4$	0.5990784	$3.597E-4$
1.625	0.6799604	0.6803641	$4.037E-4$	0.6801670	$2.066E-4$
1.750	0.7741203	0.7742830	$1.627E-4$	0.7742041	$8.373E-5$
1.875	0.8806964	0.8807189	$2.242E-5$	0.8807085	$1.204E-5$

$e_\lambda^N = 6.868E-3$ when $N = 64$, $e_\lambda^N = 3.426E-3$ for $N = 128$.

Table 3: Comparison of e^N and e_λ^N for both methods on $(0, 2]$

	(EM)		(PM)	
N	e^N	e_λ^N	e^N	e_λ^N
32	3.208E-3	1.380E-2	1.721E-5	6.961E-5
64	1.645E-3	6.868E-3	4.304E-6	1.740E-5
128	8.325E-4	3.426E-3	1.076E-6	4.350E-6
256	4.189E-4	1.711E-3	2.695E-7	1.088E-6
512	2.101E-4	8.549E-4	1.284E-7	2.718E-7
1024	1.052E-4	4.273E-4	6.199E-8	6.853E-8

6 Conclusion

In this paper, we have proposed an approximate method for solving the boundary-value problem for a linear first order delay differential equation depending on a parameter. The method was based on an appropriate scheme which has been described by an exponentially fitted difference scheme on an equidistant mesh on each time subinterval. As results from the method, first order convergence in the discrete maximum norm was obtained. An example was solved using both the presented method and implicit Euler method. Moreover, the computational results for $N = 64, 128$ were displayed in Tables 1-3. The results showed that the presented method was more effective and accurate than the other method. Theoretical results represented undergoing research more complicated delay problems, such as nonlinear delay or neutral delay problem including parameter.

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