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Further Remarks on b-Metrics, Metric-Preserving Functions, and other Related Metrics

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Abstract

Previously, we investigated some relations between b-metrics and metric-preserving functions. In this article, we continue the investigation by giving a solution to a problem we left open in the previous article. In addition, there are some results in the literature which involve the concept of b-metric and inframetric (or weak-ultrametric). We show that they are actually the same.

1 Introduction

Previously, we investigated some relations between b-metrics and metric-preserving functions and left an open problem for future research. After more careful analysis, we can give a solution to that problem in this article. This leads to a complete description for the relations between the functions which are considered in [12]. The definitions of b-metrics and metric-preserving functions are as follows:

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Definition 1.1. *Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a b-metric if it satisfies the following three conditions:*

(B1) *for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$,*

(B2) *for all $x, y \in X$, $d(x, y) = d(y, x)$,*

(B3) *there exists $s \geq 1$ such that*

$$d(x, y) \leq s(d(x, z) + d(z, y)) \quad \text{for all } x, y, z \in X.$$

Definition 1.2. *The function $f : [0, \infty) \to [0, \infty)$ is called metric preserving if for all metric spaces (X, d), $f \circ d$ is a metric on X.*

The concept of b-metrics is introduced by Bakhtin [1] and appears in many articles, see for example in [5, 7, 12, 22]. We also refer the reader to [2, 3, 4, 6, 8, 15, 16, 18, 20, 21] for more information on metric-preserving functions and to [17] for applications in fixed point theory. In connection with metric-preserving functions and b-metrics, the first and second authors [12] define the following notions.

Definition 1.3. *Let $f : [0, \infty) \to [0, \infty)$. We say that*

(i) *f is b-metric-preserving if for all b-metric spaces (X, d), $f \circ d$ is a b-metric on X,*

(ii) *f is metric-b-metric-preserving if for all metric spaces (X, d), $f \circ d$ is a b-metric on X, and*

(iii) *f is b-metric-metric-preserving if for all b-metric spaces (X, d), $f \circ d$ is a metric on X.*

We let $\mathcal{M}$ be the set of all metric-preserving functions, $\mathcal{B}$ the set of all b-metric-preserving functions, $\mathcal{MB}$ the set of all metric-b-metric-preserving functions, and $\mathcal{BM}$ the set of all b-metric-metric-preserving functions.

Previously, Khemaratchatakumthorn and Pongsriiam [12, Theorem 15 and Example 16] obtain the following result.

Theorem 1.4. *[12] We have $\mathcal{BM} \subseteq \mathcal{M} \subseteq \mathcal{B} \subseteq \mathcal{MB}$, $\mathcal{M} \not\subseteq \mathcal{BM}$, and $\mathcal{B} \not\subseteq \mathcal{M}$.*

From Theorem 1.4, we have an almost complete picture on the subset relations between $\mathcal{BM}$, $\mathcal{M}$, $\mathcal{B}$, and $\mathcal{MB}$ except that we do not know if $\mathcal{MB} \subseteq \mathcal{B}$ or not. We thought that $\mathcal{MB} \nsubseteq \mathcal{B}$, but we could not find a function f in $\mathcal{MB}$ which is not in $\mathcal{B}$. In this article, we show that, in fact, such a function does not exist. That is $\mathcal{MB} = \mathcal{B}$ (see Theorem 3.1).

Some metrics have different names but they actually are the same. For example, b-metric is also called near-metric in [7]. Inframetric (or weak-ultrametric) is used by some researchers [7, 9, 10] and seems to be different from b-metric. The definition of inframetric is as follows.

Definition 1.5. *Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called an inframetric (or weak ultrametric, or pseudo-distance) if it satisfies the following three conditions:*

(I1) *for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$,*

(I2) *for all $x, y \in X$, $d(x, y) = d(y, x)$,*

(I3) *there exists $C \geq 1$ such that*

$$d(x, y) \leq C \max\{d(x, z), d(z, y)\} \quad \text{for all } x, y, z \in X.$$

In this article, after proving $\mathcal{MB} = \mathcal{B}$, we also show that b-metrics and inframetrics are equivalent concepts.

2 Preliminaries and Lemmas

In order to prove our main theorem, we need to recall some basic definitions and results in [12].

Definition 2.1. *Let $f : [0, \infty) \to [0, \infty)$. Then f is said to be amenable if $f^{-1}(\{0\}) = \{0\}$. In addition, we say that f is quasi-subadditive if there exists $s \geq 1$ such that $f(a + b) \leq s(f(a) + f(b))$ for all $a, b \in [0, \infty)$.*

Definition 2.2. *A triangle triplet is a triple (a, b, c) of nonnegative real numbers for which*

$$a \leq b + c, \; b \leq a + c, \text{ and } c \leq a + b,$$

or equivalently,

$$|a - b| \leq c \leq a + b.$$

Let $s \geq 1$ and $a, b, c \geq 0$. A triple (a, b, c) is said to be an s-triangle triplet if

$$a \leq s(b+c),\ b \leq s(a+c),\ \text{and } c \leq s(a+b).$$

We let Δ and Δ_s be the set of all triangle triplets and s-triangle triplets, respectively.

Theorem 2.3. *[12, Theorem 17] Suppose $f : [0, \infty) \to [0, \infty)$ is amenable. Then the following statements are equivalent.*

(i) *$f \in \mathcal{MB}$.*

(ii) *There exists $s \geq 1$ such that $(f(a), f(b), f(c)) \in \Delta_s$ for all $(a, b, c) \in \Delta$.*

Theorem 2.4. *[12, Theorem 20] Let $f : [0, \infty) \to [0, \infty)$. If $f \in \mathcal{MB}$, then f is amenable and quasi-subadditive.*

3 Main Results

Theorem 3.1. *We have $\mathcal{MB} = \mathcal{B}$. That is for any $f : [0, \infty) \to [0, \infty)$, f is metric-b-metric-preserving functions if and only if f is b-metric-preserving functions.*

Proof. Since it is already proved in [12, Theorem 15] that $\mathcal{B} \subseteq \mathcal{MB}$, we only need to show that $\mathcal{MB} \subseteq \mathcal{B}$. Let $f \in \mathcal{MB}$ and let (X, d) be a b-metric space. By Theorem 2.4, f is amenable and quasi-subadditive. Then the condition (B1) is satisfied by $f \circ d$ since f is amenable. In addition, $f \circ d$ also satisfies the condition (B2) because $d(x, y) = d(y, x)$. So it only remains to show that (B3) holds for $f \circ d$. Since f is quasi-subadditive, there exists $t \geq 1$ such that

$$f(a+b) \leq t(f(a) + f(b)) \text{ for all } a, b \in [0, \infty). \tag{3.1}$$

Since d is a b-metric, there exists $s_1 \geq 1$ such that

$$d(x, y) \leq s_1(d(x, z) + d(z, y)) \quad \text{for all } x, y, z \in X.$$

We can choose $n \in \mathbb{N}$ such that $n > s_1$, and therefore

$$d(x, y) \leq n(d(x, z) + d(z, y)) \quad \text{for all } x, y, z \in X. \tag{3.2}$$

Since $f \in \mathcal{MB}$, we obtain by Theorem 2.3 that there exists $s_2 \geq 1$,

$$(f(a), f(b), f(c)) \in \Delta_{s_2} \quad \text{for any } (a, b, c) \in \Delta. \tag{3.3}$$

Let $s = 2s_2nt^n$. Let $x, y, z \in X$ and let $a = d(x, y)$, $b = d(x, z)$, and $c = d(z, y)$. By (3.2), we have

$$a \leq nb + nc.$$

Then $(a, nb+nc, nb+nc) \in \Delta$. By (3.3), $(f(a), f(nb+nc), f(nb+nc)) \in \Delta_{s_2}$. We obtain

$$(f \circ d)(x, y) = f(a) \leq s_2(f(nb + nc) + f(nb + nc)) = 2s_2 f(n(b + c)). \quad (3.4)$$

Next we will show that

$$f(mx) \leq mt^{m-1} f(x) \text{ for all } x \in [0, \infty) \text{ and } m \in \mathbb{N}. \quad (3.5)$$

We let $x \in [0, \infty)$ and prove (3.5) by induction on m. The result is clear when $m = 1$. So let $m \geq 1$ and assume that (3.5) holds for m. Since $t \geq 1$, we see that

$$mt^{m-1} + 1 \leq (m + 1)t^{m-1}.$$

Then we obtain by (3.1) and the induction hypothesis that

$$\begin{aligned} f((m + 1)x) &\leq t(f(mx) + f(x)) \\ &\leq t\left(mt^{m-1} f(x) + f(x)\right) \\ &= t\left(mt^{m-1} + 1\right) f(x) \\ &\leq t(m + 1)t^{m-1} f(x) = (m + 1)t^m f(x). \end{aligned}$$

This proves (3.5). Then by (3.4), (3.5), and (3.1), we obtain

$$\begin{aligned} (f \circ d)(x, y) &\leq 2s_2 nt^{n-1} f(b + c) \\ &\leq 2s_2 nt^n (f(b) + f(c)) \\ &= s((f \circ d)(x, z) + (f \circ d)(z, y)), \end{aligned}$$

as required. This shows that $f \circ d$ is a b-metric and the proof is complete. □

Corollary 3.2. *Let $f : [0, \infty) \to [0, \infty)$ be amenable. Then the following statements are equivalent.*

(i) $f \in \mathcal{B}$.

(ii) $f \in \mathcal{MB}$.

(iii) *There exists $s \geq 1$ such that $(f(a), f(b), f(c)) \in \Delta_s$ for all $(a, b, c) \in \Delta$.*

Proof. This follows from Theorems 2.3 and 3.1. □

As mentioned in the introduction, there are some metrics with different names but they are actually equivalent concepts.

Theorem 3.3. *Suppose X is a nonempty set and $d : X \times X \to \mathbb{R}$. Then d is a b-metric if and only if d is a weak ultrametric (or inframetric).*

Proof. Assume that d is a b-metric. Then there exists $s \geq 1$ such that

$$d(x, y) \leq s(d(x, z) + d(z, y)) \text{ for all } x, y, z \in X.$$

Since the conditions (I1) and (I2) are the same as (B1) and (B2), we only need to consider (I3). We have

$$\begin{aligned} d(x, y) &\leq s(d(x, z) + d(z, y)) \\ &\leq s(\max\{d(x, z), d(z, y)\} + \max\{d(x, z), d(z, y)\}) \\ &= 2s \max\{d(x, z), d(z, y)\}, \quad \text{for all } x, y, z \in X. \end{aligned}$$

Therefore d is a weak ultrametric. For the converse, assume that d is a weak ultrametric. Then there exists $C \geq 1$ such that

$$d(x, y) \leq C \max\{d(x, z), d(z, y)\} \quad \text{for all } x, y, z \in X.$$

But $\max\{d(x, z), d(z, y)\} \leq d(x, z) + d(z, y)$, the desired result follows easily. This completes the proof. □

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