Further Remarks on b-Metrics, Metric-Preserving Functions, and other Related Metrics

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Abstract

Previously, we investigated some relations between b-metrics and metric-preserving functions. In this article, we continue the investigation by giving a solution to a problem we left open in the previous article. In addition, there are some results in the literature which involve the concept of b-metric and inframetric (or weak-ultrametric). We show that they are actually the same.

1 Introduction

Previously, we investigated some relations between b-metrics and metric-preserving functions and left an open problem for future research. After more careful analysis, we can give a solution to that problem in this article. This leads to a complete description for the relations between the functions which are considered in [12]. The definitions of b-metrics and metric-preserving functions are as follows:

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Definition 1.1. Let $X$ be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a b-metric if it satisfies the following three conditions:

(B1) for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$, 

(B2) for all $x, y \in X$, $d(x, y) = d(y, x)$, 

(B3) there exists $s \geq 1$ such that 

$$d(x, y) \leq s(d(x, z) + d(z, y)) \quad \text{for all } x, y, z \in X.$$ 

Definition 1.2. The function $f : [0, \infty) \to [0, \infty)$ is called metric preserving if for all metric spaces $(X, d)$, $f \circ d$ is a metric on $X$.

The concept of b-metrics is introduced by Bakhtin [1] and appears in many articles, see for example in [5, 7, 12, 22]. We also refer the reader to [2, 3, 4, 6, 8, 15, 16, 18, 20, 21] for more information on metric-preserving functions and to [17] for applications in fixed point theory. In connection with metric-preserving functions and b-metrics, the first and second authors [12] define the following notions.

Definition 1.3. Let $f : [0, \infty) \to [0, \infty)$. We say that

(i) $f$ is b-metric-preserving if for all b-metric spaces $(X, d)$, $f \circ d$ is a b-metric on $X$,

(ii) $f$ is metric-b-metric-preserving if for all metric spaces $(X, d)$, $f \circ d$ is a b-metric on $X$, and

(iii) $f$ is b-metric-metric-preserving if for all b-metric spaces $(X, d)$, $f \circ d$ is a metric on $X$.

We let $\mathcal{M}$ be the set of all metric-preserving functions, $\mathcal{B}$ the set of all b-metric-preserving functions, $\mathcal{MB}$ the set of all metric-b-metric-preserving functions, and $\mathcal{BM}$ the set of all b-metric-metric-preserving functions.

Previously, Khemaratchatakumthorn and Pongsriiam [12, Theorem 15 and Example 16] obtain the following result.

Theorem 1.4. [12] We have $\mathcal{BM} \subseteq \mathcal{M} \subseteq \mathcal{B} \subseteq \mathcal{MB}$, $\mathcal{M} \not\subseteq \mathcal{BM}$, and $\mathcal{B} \not\subseteq \mathcal{M}$.
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From Theorem 1.4, we have an almost complete picture on the subset relations between \(B, M, \mathcal{M}, \mathcal{B}\) except that we do not know if \(\mathcal{MB} \subseteq \mathcal{B}\) or not. We thought that \(\mathcal{MB} \not\subseteq \mathcal{B}\), but we could not find a function \(f\) in \(\mathcal{MB}\) which is not in \(\mathcal{B}\). In this article, we show that, in fact, such a function does not exist. That is \(\mathcal{MB} = \mathcal{B}\) (see Theorem 3.1).

Some metrics have different names but they actually are the same. For example, b-metric is also called near-metric in [7]. Inframetric (or weak-ultrametric) is used by some researchers [7, 9, 10] and seems to be different from b-metric. The definition of inframetric is as follows.

**Definition 1.5.** Let \(X\) be a nonempty set. A function \(d : X \times X \to [0, \infty)\) is called an inframetric (or weak ultrametric, or pseudo-distance) if it satisfies the following three conditions:

1. (I1) for all \(x, y \in X\), \(d(x, y) = 0\) if and only if \(x = y\),
2. (I2) for all \(x, y \in X\), \(d(x, y) = d(y, x)\),
3. (I3) there exists \(C \geq 1\) such that
   \[
   d(x, y) \leq C \max\{d(x, z), d(z, y)\} \quad \text{for all } x, y, z \in X.
   \]

In this article, after proving \(\mathcal{MB} = \mathcal{B}\), we also show that b-metrics and inframetrics are equivalent concepts.

## 2 Preliminaries and Lemmas

In order to prove our main theorem, we need to recall some basic definitions and results in [12].

**Definition 2.1.** Let \(f : [0, \infty) \to [0, \infty)\). Then \(f\) is said to be amenable if \(f^{-1}(\{0\}) = \{0\}\). In addition, we say that \(f\) is quasi-subadditive if there exists \(s \geq 1\) such that \(f(a + b) \leq s(f(a) + f(b))\) for all \(a, b \in [0, \infty)\).

**Definition 2.2.** A triangle triplet is a triple \((a, b, c)\) of nonnegative real numbers for which

\[
 a \leq b + c, \quad b \leq a + c, \quad \text{and} \quad c \leq a + b,
\]

or equivalently,

\[
|a - b| \leq c \leq a + b.
\]
Let \( s \geq 1 \) and \( a, b, c \geq 0 \). A triple \((a, b, c)\) is said to be an \( s \)-triangle triplet if
\[
a \leq s(b + c), \quad b \leq s(a + c), \quad \text{and} \quad c \leq s(a + b).
\]

We let \( \Delta \) and \( \Delta_s \) be the set of all triangle triplets and \( s \)-triangle triplets, respectively.

**Theorem 2.3.** [12, Theorem 17] Suppose \( f : [0, \infty) \to [0, \infty) \) is amenable. Then the following statements are equivalent.

(i) \( f \in MB \).

(ii) There exists \( s \geq 1 \) such that \((f(a), f(b), f(c)) \in \Delta_s \) for all \((a, b, c) \in \Delta\).

**Theorem 2.4.** [12, Theorem 20] Let \( f : [0, \infty) \to [0, \infty) \). If \( f \in MB \), then \( f \) is amenable and quasi-subadditive.

### 3 Main Results

**Theorem 3.1.** We have \( MB = B \). That is for any \( f : [0, \infty) \to [0, \infty) \), \( f \) is metric-b-metric-preserving functions if and only if \( f \) is b-metric-preserving functions.

**Proof.** Since it is already proved in [12, Theorem 15] that \( B \subseteq MB \), we only need to show that \( MB \subseteq B \). Let \( f \in MB \) and let \((X, d)\) be a b-metric space. By Theorem 2.4, \( f \) is amenable and quasi-subadditive. Then the condition (B1) is satisfied by \( f \circ d \) since \( f \) is amenable. In addition, \( f \circ d \) also satisfies the condition (B2) because \( d(x, y) = d(y, x) \). So it only remains to show that (B3) holds for \( f \circ d \). Since \( f \) is quasi-subadditive, there exists \( t \geq 1 \) such that
\[
f(a + b) \leq t(f(a) + f(b)) \quad \text{for all} \quad a, b \in [0, \infty).
\]

Since \( d \) is a b-metric, there exists \( s_1 \geq 1 \) such that
\[
d(x, y) \leq s_1(d(x, z) + d(z, y)) \quad \text{for all} \quad x, y, z \in X.
\]

We can choose \( n \in \mathbb{N} \) such that \( n > s_1 \), and therefore
\[
d(x, y) \leq n(d(x, z) + d(z, y)) \quad \text{for all} \quad x, y, z \in X.
\]

Since \( f \in MB \), we obtain by Theorem 2.3 that there exists \( s_2 \geq 1 \),
\[
(f(a), f(b), f(c)) \in \Delta_{s_2} \quad \text{for any} \quad (a, b, c) \in \Delta.
\]
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Let \( s = 2s_2n^t \). Let \( x, y, z \in X \) and let \( a = d(x, y) \), \( b = d(x, z) \), and \( c = d(z, y) \). By (3.2), we have

\[
a \leq nb + nc.
\]

Then \((a, nb+nc, nb+nc) \in \Delta\). By (3.3), \((f(a), f(nb+nc), f(nb+nc)) \in \Delta_{s_2}\). We obtain

\[
(f \circ d)(x, y) = f(a) \leq s_2(f(nb + nc) + f(nb + nc)) = 2s_2f(n(b + c)).
\] (3.4)

Next we will show that

\[
f(mx) \leq mt^{m-1}f(x) \text{ for all } x \in [0, \infty) \text{ and } m \in \mathbb{N}.
\] (3.5)

We let \( x \in [0, \infty) \) and prove (3.5) by induction on \( m \). The result is clear when \( m = 1 \). So let \( m \geq 1 \) and assume that (3.5) holds for \( m \). Since \( t \geq 1 \), we see that

\[
mt^{m-1} + 1 \leq (m + 1)t^{m-1}.
\]

Then we obtain by (3.1) and the induction hypothesis that

\[
f((m+1)x) \leq t(f(mx) + f(x))
\leq t(\left(mt^{m-1}f(x) + f(x)\right))
= t\left(mt^{m-1} + 1\right)f(x)
\leq t(m + 1)t^{m-1}f(x) = (m + 1)t^m f(x).
\]

This proves (3.5). Then by (3.4), (3.5), and (3.1), we obtain

\[
(f \circ d)(x, y) \leq 2s_2nt^{n-1}f(b + c)
\leq 2s_2nt^n(f(b) + f(c))
= s((f \circ d)(x, z) + (f \circ d)(z, y)),
\]
as required. This shows that \( f \circ d \) is a b-metric and the proof is complete. \( \square \)

**Corollary 3.2.** Let \( f : [0, \infty) \to [0, \infty) \) be amenable. Then the following statements are equivalent.

(i) \( f \in \mathcal{B} \).

(ii) \( f \in \mathcal{M}\mathcal{B} \).

(iii) There exists \( s \geq 1 \) such that \((f(a), f(b), f(c)) \in \Delta_s \) for all \((a, b, c) \in \Delta\).
Proof. This follows from Theorems 2.3 and 3.1. □

As mentioned in the introduction, there are some metrics with different names but they are actually equivalent concepts.

**Theorem 3.3.** Suppose $X$ is a nonempty set and $d : X \times X \rightarrow \mathbb{R}$. Then $d$ is a $b$-metric if and only if $d$ is a weak ultrametric (or inframetric).

**Proof.** Assume that $d$ is a $b$-metric. Then there exists $s \geq 1$ such that

$$d(x, y) \leq s(d(x, z) + d(z, y))$$

for all $x, y, z \in X$.

Since the conditions (I1) and (I2) are the same as (B1) and (B2), we only need to consider (I3). We have

$$d(x, y) \leq s(d(x, z) + d(z, y))$$

$$\leq s(\max\{d(x, z), d(z, y)\} + \max\{d(x, z), d(z, y)\})$$

$$= 2s \max\{d(x, z), d(z, y)\}, \text{ for all } x, y, z \in X.$$

Therefore $d$ is a weak ultrametric. For the converse, assume that $d$ is a weak ultrametric. Then there exists $C \geq 1$ such that

$$d(x, y) \leq C \max\{d(x, z), d(z, y)\} \text{ for all } x, y, z \in X.$$

But $\max\{d(x, z), d(z, y)\} \leq d(x, z) + d(z, y)$, the desired result follows easily. This completes the proof. □

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References


